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THE SECANT CONJECTURE IN THE REAL SCHUBERT CALCULUS

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Abstract. We formulate the Secant Conjecture, which is a generalization of the Shapiro Conjecture for Grassmannians. It asserts that an intersection of Schubert varieties in a Grassmannian is transverse with all points real if the flags defining the Schubert varieties are secant along disjoint intervals of a rational normal curve. We present theoretical evidence for this conjecture as well as computational evidence obtained in over one terahertz-year of computing, and we discuss some of the phenomena we observed in our data.

1. Introduction

Some solutions to a system of real polynomial equations are real and the rest occur in complex conjugate pairs. While the total number of solutions is determined by the structure of the equations, the number of real solutions depends rather subtly on the coefficients. Sometimes there is finer information available in terms of upper bounds [19, 2] or lower bounds [7, 31] on the number of real solutions. The Shapiro and Secant Conjectures assert the extreme situation of having only real solutions.

The Shapiro Conjecture for Grassmannians posits that if the Wronskian of a vector space of univariate *complex* polynomials has only real roots, then that space is spanned by *real* polynomials. This striking instance of unexpected reality was proven by Eremenko and Gabrielov for two-dimensional spaces of polynomials [8, 9], and the general case was established by Mukhin, Tarasov, and Varchenko [23, 25]. While the statement concerns spaces of polynomials, or more generally the Schubert calculus on Grassmannians, its proofs complex analysis [8, 9] and mathematical physics [23, 25]. This story was described in the AMS Bulletin [35].

The Shapiro conjecture first gained attention through partial results and computations [33, 38], and further work [34] led to an extension that appears to hold for flag manifolds, the Monotone Conjecture. This extension was made in [27], which also reported on partial results and experimental evidence. The Monotone Conjecture for a certain family of two-step flag manifolds was proved by Eremenko, Gabrielov, Shapiro, and Vainshtein [10].

The result of [10] was in fact a proof of reality in the Grassmannian of codimension-two planes for intersections of Schubert varieties defined with respect to certain *disjoint secant flags*. The Secant Conjecture postulates an extension of this result to all Grassmannians. We give the

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simplest open instance of the Secant Conjecture. Let $x_1,\dots,x_6$ be indeterminates and consider the polynomial

$$f(s,t,u;x) \ := \ \det\begin{pmatrix} 1 & 0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 & x_6 \\ 1 & s & s^2 & s^3 & s^4 \\ 1 & t & t^2 & t^3 & t^4 \\ 1 & u & u^2 & u^3 & u^4 \end{pmatrix}, \tag{1.1}$$

which depends upon parameters s, t, and u.

Conjecture 1.1. *Let $s_1 < t_1 < u_1 < s_2 < t_2 < \cdots < u_5 < s_6 < t_6 < u_6$ be real numbers. Then the system of polynomial equations*

$$f(s_i,t_i,u_i;x) \ = \ 0 \qquad i = 1,\dots,6 \tag{1.2}$$

has five distinct solutions, and all of them are real.

Geometrically, the equation $f(s,t,u;x) = 0$ says that the 2-plane (spanned by the first two rows of the matrix in (1.1)) meets the 3-plane which is secant to the rational curve $\gamma\colon y \mapsto (1,y,y^2,y^3,y^4)$ at the points $\gamma(s),\gamma(t),\gamma(u)$. The hypotheses imply that each of the six 3-planes is secant to γ along an interval $[s_i,u_i]$, and these six intervals are pairwise disjoint. The conjecture asserts that all of the 2-planes meeting six 3-planes are real when the 3-planes are secant to the rational normal curve along disjoint intervals. This statement was true in each of the 285,502 instances we tested.

The purpose of this paper is to explain the Secant Conjecture and its relation to the other reality conjectures, to describe the data supporting it from a large computational experiment, and to highlight some other features in our data beyond the Secant Conjecture. These data may be viewed online [40]. We will assume some background on the Shapiro Conjecture as described in the survey [35] and paper [27], and we will not describe the execution of the experiment, as the methods paper [14] presented the software framework we have developed for such distributed computational experiments.

This paper is organized as follows. In Section 2, we present the full Secant Conjecture, giving a history of its formulation. Section 3 presents some theoretical justification for the Secant Conjecture as well as a generalization based on limiting cases. In Section 4 we analyze the problem of lines meeting all possible configurations of four secant lines, giving conditions on the secant lines that imply that both solutions are real. Section 5 describes a statistic, the *overlap number*, which measures the extent of overlap among intervals of secancy. In Section 6 we explain the data from our experiment. About 3/4 of our over 2 billion computations did not directly test the Secant Conjecture, but rather tested geometric configurations that were close to those of the conjecture. Consequently, our data contain much more information than that in support of the Secant Conjecture, and we explore that information in the remaining sections. Section 7 discusses the lower bounds on the numbers of real solutions we typically observed for small overlap number, producing a striking *inner border* in the tabulation of our data. Finally, in Section 8, we discuss Schubert problems with provable lower bounds and gaps in their numbers of real solutions, a phenomenon we first noticed while trying to understand our data.

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2. Schubert Calculus and the Secant Conjecture

We give background from the Schubert Calculus necessary to state the Secant Conjecture, and then we state the equivalent dual Cosecant Conjecture.

2.1. **Schubert Calculus.**

The Schubert Calculus [11, 12] involves problems of determining the linear spaces that have specified positions with respect to other, fixed (flags of) linear spaces. For example, what are the 3-planes in $\mathbb{C}^7$ meeting 12 given 4-planes non-trivially? (There are 462 [28].) The specified positions are a *Schubert problem*, which determines the number of solutions. The actual solutions depend upon the linear spaces imposing the conditions, or *instance* of the Schubert problem.

The *Grassmannian $G(k,n)$* is the set of all k-dimensional linear subspaces of $\mathbb{C}^n$, which is an algebraic manifold of dimension $k(n-k)$. A *flag $F_\bullet$* is a sequence of linear subspaces

$$F_\bullet \;:\; F_1 \subset F_2 \subset \cdots \subset F_n\,,$$

where $\dim F_i = i$. A *partition* $\lambda\colon (n-k) \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$ is a weakly decreasing sequence of integers. A fixed flag $F_\bullet$ and a partition λ define a *Schubert variety $X_\lambda F_\bullet$*,

$$X_\lambda F_\bullet \;:=\; \{H \in G(k,n) \mid \dim H \cap F_{n-k+i-\lambda_i} \geq i \quad \text{for } i = 1,\ldots,k\}\,,$$

which is a subvariety of codimension $|\lambda| := \lambda_1 + \cdots + \lambda_k$. Not every element of the flag is needed to define the Schubert variety.

A *Schubert problem* is a list $\lambda^1,\ldots,\lambda^m$ of partitions with $|\lambda^1| + \cdots + |\lambda^m| = k(n-k)$. For sufficiently general flags $F^1_\bullet,\ldots,F^m_\bullet$, the intersection

$$X_{\lambda^1}F^1_\bullet \cap X_{\lambda^2}F^2_\bullet \cap \cdots \cap X_{\lambda^m}F^m_\bullet$$

is transverse [20] and consists of a certain number, $d(\lambda^1,\ldots,\lambda^m)$, of points, which may be computed using algorithms in the Schubert Calculus (see [11, 21]). (*Transverse* means that at each point of the intersection, the annihilators of the tangent spaces to the Schubert varieties are in direct sum.) We write a Schubert problem multiplicatively, $\lambda^1 \cdots \lambda^m = d(\lambda^1,\ldots,\lambda^m)$. For example, writing $\square$ for the partition (1,0) with $|\square| = 1$, we have $\square\cdot\square\cdot\square\cdot\square\cdot\square\cdot\square = \square^6 = 5$ for the Schubert problem on $G(2,5)$ involving six partitions, each equal to $\square$. In this notation, Schubert's problem that we mentioned above is $\square^{12} = 462$ on $G(3,7)$.

A rational normal curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is affinely equivalent to the *moment curve*

$$\gamma \;:\; t \;\longmapsto\; (1, t, t^2, \ldots, t^{n-1})\,.$$

The *osculating flag $F_\bullet(t)$* has i-dimensional subspace the span of the first i derivatives $\gamma(t),\gamma'(t),\ldots,\gamma^{(i-1)}(t)$ of γ at t. We state the Theorem of Mukhin, et al. [23, 25].

Theorem 2.1 (The Shapiro Conjecture)**.** *For any Schubert problem* $\lambda^1,\dotsc,\lambda^m$ *on a Grassmannian* $G(k,n)$ *and any distinct real numbers* $t_1,\dotsc,t_m$*, the intersection*

$$X_{\lambda^1}F_\bullet(t_1)\cap X_{\lambda^2}F_\bullet(t_2)\cap\dotsb\cap X_{\lambda^m}F_\bullet(t_m)$$

is transverse and consists of $d(\lambda^1,\dotsc,\lambda^m)$ *real points.*

Transversality is unexpected as osculating flags are not general.

The Shapiro Conjecture concerns intersections of Schubert varieties given by flags osculating a rational normal curve, and in this form it makes sense for every flag manifold G/P. Purbhoo showed that it holds for the orthogonal Grassmannians [26], but counterexamples are known for other flag manifolds. There is an appealing version of it—the Monotone Conjecture—that appears to hold for the classical flag variety [27].

2.2. **The Secant Conjecture.** Eremenko, et al. [10] proved a generalization of the Monotone Conjecture for flags consisting of a codimension-two plane lying on a hyperplane, where it becomes a statement about real rational functions. Their theorem asserts that a Schubert problem on $G(n{-}2,n)$ has only real solutions if the flags satisfy a special property that we now describe. A flag $F_\bullet$ of linear subspaces is *secant along an interval* I of a rational normal curve γ if every subspace in the flag is spanned by its intersection with I. This means that there are distinct points $t_1,\dotsc,t_{n-1}\in I$ such that for each $i=1,\dotsc,n{-}1$, the subspace F_i of the flag $F_\bullet$ is spanned by $\gamma(t_1),\dotsc,\gamma(t_i)$.

Secant Conjecture 2.2. *For any Schubert problem* $\lambda^1,\dotsc,\lambda^m$ *on a Grassmannian* $G(k,n)$ *and any flags* $F^1_\bullet,\dotsc,F^m_\bullet$ *that are secant to a rational normal curve* γ *along disjoint intervals, the intersection*

$$X_{\lambda^1}F^1_\bullet\cap X_{\lambda^2}F^2_\bullet\cap\dotsb\cap X_{\lambda^m}F^m_\bullet$$

is transverse and consists of $d(\lambda^1,\dotsc,\lambda^m)$ *real points.*

Conjecture 1.1 is the case of this Secant Conjecture for the Schubert problem $\square^6=5$ on $G(2,5)$. The Schubert variety $X_{\square}F_\bullet$ is

$$X_{\square}F_\bullet \;=\; \{H\in G(2,5)\mid \dim H\cap F_3\geq 1\}\,;$$

that is, the set of 2-planes meeting a fixed 3-plane non-trivially. Since F_4 and F_5 are irrelevant we drop them from the flag and refer to F_3 and $X_{\square}F_3$. For every Schubert condition, there is a largest element of the flag imposing a relevant condition; call this the *relevant subspace*. The relevant subspace in this example is F_3.

For $s,t,u\in\mathbb{R}$, let $F_3(s,t,u)$ be the linear span of $\gamma(s)$, $\gamma(t)$, and $\gamma(u)$, a 3-plane secant to γ with points $\gamma(s)$, $\gamma(t)$, and $\gamma(u)$ of secancy. Thus, the condition $f(s,t,u;x)=0$ of Conjecture 1.1 implies that the linear span H of the first two rows of the matrix in (1.1)—a general 2-plane in 5-space—meets the linear span $F_3(s,t,u)$ of the last three rows. Thus

$$f(s,t,u;x)\;=\;0\quad\Longleftrightarrow\quad H\in X_{\square}F_3(s,t,u)\,.$$

Lastly, the condition on the ordering of the points s_i,t_i,u_i in Conjecture 1.1 implies that the six flags $F_3(s_i,t_i,u_i)$ are secant along disjoint intervals.

2.3. **Grassmann Duality and the Cosecant Conjecture.** Associating a linear subspace H of a vector space $V \simeq \mathbb{C}^n$ to its annihilator $\delta(H) := H^\perp \subset V^*$ induces an isomorphism $\delta\colon G(k,n) \to G(n-k,n)$ called *Grassmann duality*. This notion extends to flags and the dual of an osculating flag is an osculating flag. Secancy is not preserved under duality. We next formulate the (equivalent) dual statement to the Secant Conjecture, which we call the Cosecant Conjecture.

Grassmann duality respects Schubert varieties. Given a flag $F_\bullet \subset \mathbb{C}^n$, let $F_\bullet^\perp$ be the flag whose i-dimensional subspace is $F_i^\perp := (F_{n-i})^\perp$. Then

$$\delta(X_\lambda F_\bullet) \;=\; X_{\lambda^T} F_\bullet^\perp\,,$$

where λ^T is the *conjugate partition* to λ. For example,

$$\ydiagram{2}^{\,T} \;=\; \ydiagram{1,1}\,, \qquad \ydiagram{2,1}^{\,T} \;=\; \ydiagram{2,1}\,, \qquad \text{and} \qquad \ydiagram{3,1}^{\,T} \;=\; \ydiagram{2,1,1}\,.$$

That is, if we represent λ by its *Young diagram*—a left-justified array of boxes with λ_i boxes in row i—then the diagram of λ^T is the matrix-transpose of the diagram of λ.

If $\gamma(t) = (1,t,t^2,\dotsc,t^{n-1})$ is the rational normal curve, then the dual of the family $F_{n-1}(t)$ of its osculating $(n-1)$-planes is a curve $\gamma^\perp(t) := (F_{n-1}(t))^\perp$, which is

$$\gamma^\perp(t) \;=\; \left(\tbinom{n-1}{n-1}(-t)^{n-1}\,,\ \dotsc\,,\ -\tbinom{n-1}{3}t^3\,,\ \tbinom{n-1}{2}t^2\,,\ -(n-1)t\,,1\right)\,,$$

in the basis dual to the standard basis. Moreover, $(F_{n-k}(t))^\perp$ is the osculating k-plane to this dual rational normal curve $\gamma^\perp$ at the point $\gamma^\perp(t)$. Thus Grassmann duality preserves Schubert varieties given by flags osculating the rational normal curve, and the dual statement to Theorem 2.1 is simply itself.

This is however not the case for secant flags. The general secant $(n-1)$-plane

$$F_{n-1}(s_1,s_2,\dotsc,s_{n-1}) = \ \operatorname{span}\{\gamma(s_1)\,,\gamma(s_2)\,,\ \dotsc\ ,\gamma(s_{n-1})\}\,,$$

secant to γ at the points $\gamma(s_1),\dotsc,\gamma(s_{n-1})$, has dual space spanned by the vector

$$\left((-1)^{n-1}e_{n-1}\,,\ \dotsc\ ,\ -e_3\,,e_2\,,\ -e_1\,,1\right)\,,$$

where e_i is the ith elementary symmetric function in the parameters $s_1,\dotsc,s_{n-1}$. This dual space is not secant to the dual rational normal curve $\gamma^\perp$.

In general, a *cosecant subspace* is a subspace that is dual to a secant subspace. If

$$F_k(s_1,s_2,\dotsc,s_k) \;=\; \operatorname{span}\{\gamma(s_1)\,,\gamma(s_2)\,,\ \dotsc\ ,\gamma(s_k)\}\,,$$

then the corresponding cosecant subspace is

$$F_{n-1}^\perp(s_1) \,\cap\, F_{n-1}^\perp(s_2) \,\cap\, \cdots \,\cap\, F_{n-1}^\perp(s_k)\,,$$

the intersection of k hyperplanes osculating the rational normal curve $\gamma^\perp$. A *cosecant flag* is a flag whose subspaces are cut out by hyperplanes osculating γ. It is *cosecant along an interval* of γ if these hyperplanes osculate γ at points of the interval.

Thus, under Grassmann duality the Secant Conjecture for $G(n-k,n)$ becomes the following equivalent *Cosecant Conjecture* for $G(k,n)$.

Conjecture 2.3 (Cosecant Conjecture)**.** *For any Schubert problem $\lambda^1,\ldots,\lambda^m$ on a Grassmannian $G(k,n)$ and any flags $F_\bullet^1,\ldots,F_\bullet^m$ that are cosecant to a rational normal curve γ along disjoint intervals, the intersection*

$$X_{\lambda^1}F_\bullet^1 \cap X_{\lambda^2}F_\bullet^2 \cap \cdots \cap X_{\lambda^m}F_\bullet^m$$

is transverse and consists of $d(\lambda^1,\ldots,\lambda^m)$ real points.

3. Some special cases of the Secant Conjecture

A degree of justification for posing the Secant Conjecture is provided by the history of its development from the Shapiro and Monotone Conjectures, as this shows its connection to proven results and established conjectures, and its validity for $G(n{-}2,n)$ [10]. Here, we give more concrete justifications, which include proofs in some special cases.

3.1. **Arithmetic progressions of secancy.**

Fix a parametrization $\gamma\colon \mathbb{R}\to\mathbb{R}^n$ of a rational normal curve. For $t\in\mathbb{R}$ and $h>0$, let $F_\bullet^h(t)$ be the flag whose i-dimensional subspace is

$$F_i^h(t) \ :=\ \operatorname{span}\{\gamma(t),\gamma(t{+}h),\ \ldots,\gamma(t{+}(i{-}1)h)\}\,,$$

which is spanned by an arithmetic progression of length i with step size h. Work of Mukhin, et al. [24] implies the Secant Conjecture for the Schubert problem

$$\square^{k(n-k)} \ =\ [k(n{-}k)]!\frac{1!2!\cdots(k{-}1)!}{(n{-}k)!\cdots(n{-}2)!(n{-}1)!} \tag{3.1}$$

for such secant flags.

Let $\mathbb{C}_{n-1}[t]$ be the space of polynomials of degree at most $n{-}1$. The discrete Wronskian with step size h of polynomials $f_1,\ldots,f_k$ is the determinant

$$\mathrm{W}_h(f_1,f_2,\ldots,f_k) \ :=\ \det\begin{pmatrix} f_1(t) & f_1(t+h) & \cdots & f_1(t+(k{-}1)h)\\ f_2(t) & f_2(t+h) & \cdots & f_2(t+(k{-}1)h)\\ \vdots & \vdots & \ddots & \vdots\\ f_k(t) & f_k(t+h) & \cdots & f_k(t+(k{-}1)h)\end{pmatrix}\,. \tag{3.2}$$

For general $f_1,\ldots,f_k\in\mathbb{C}_{n-1}[t]$, this polynomial has degree $k(n{-}k)$. Up to a scalar, the polynomial W_h depends only on the linear span of the polynomials $f_1,\ldots,f_k$, giving a map

$$\mathrm{W}_h\ :\ G(k,\mathbb{C}_{n-1}[t])\ \longrightarrow\ \mathbb{P}^{k(n-k)}\,,$$

where $\mathbb{P}^{k(n-k)}$ is the projective space of polynomials of degree at most $k(n-k)$. Mukhin, et al. [24] show that W_h is a finite map. It is a linear projection of the Grassmannian in its Plücker embedding, so the fiber over a general polynomial $w(t)\in\mathbb{P}^{k(n-k)}$ consists of $d(\square^{k(n-k)})$ reduced points, each of which is a space V of polynomials with discrete Wronskian $w(t)$. As a special case of Theorem 2.1 in [24], we have the following statement.

Proposition 3.1. *Let $V\subset\mathbb{C}_{n-1}[t]$ be a k-dimensional space of polynomials whose discrete Wronskian $\mathrm{W}_h(V)$ has distinct real roots $z_1,\ldots,z_N$, each of multiplicity 1. If for all $i\neq j$, we have $|z_i-z_j|\geq h$, then the space V has a basis of real polynomials.*

Corollary 3.2. *Set* $N := k(n{-}k)$ *and suppose that* $F^h_\bullet(z_1),\dots,F^h_\bullet(z_N)$ *are disjoint secant flags with* $z_i + (n{-}1)h < z_{i+1}$ *for each* $i = 1,\dots,N-1$*. Then the intersection*

$$X_{\square}F^h_\bullet(z_1) \,\cap\, X_{\square}F^h_\bullet(z_2) \,\cap\, \cdots \,\cap\, X_{\square}F^h_\bullet(z_N) \tag{3.3}$$

in $G(n{-}k,n)$ *is transverse with all points real.*

Proof. We identify points in the intersection (3.3) with the fibers of the discrete Wronski map W_h over the polynomial $(t{-}z_1)\cdots(t{-}z_{k(n-k)})$, which will prove reality. Transversality follows by an argument of Eremenko and Gabrielov given in [36, Ch. 13]: a finite analytic map between complex manifolds that has only real points in its fibers above an open set of real points is necessarily unramified over those points.

A polynomial of degree $n{-}1$ is the composition of the parametrization $\gamma\colon \mathbb{C} \to \mathbb{C}^n$ of the rational normal curve with a linear form $\mathbb{C}^n \to \mathbb{C}$. In this way, a subspace V of polynomials of dimension k corresponds to a surjective map $V\colon \mathbb{C}^n \to \mathbb{C}^k$. We will identify such a map with its kernel H, which is a point in $G(n{-}k,n)$.

The column space of the matrix in (3.2) is the image under V of the linearly independent vectors $\gamma(t),\gamma(t{+}h),\dots,\gamma(t{+}(k{-}1)h)$. These vectors span $F^h_k(t)$. Thus the determinant $W_h(V)$ vanishes at a point t exactly when the map

$$V \;:\; F^h_k(t) \;\longrightarrow\; \mathbb{C}^k$$

does not have full rank; that is, when

$$\dim H \cap\; F^h_k(t) \;\geq\; 1\,,$$

which is equivalent to $H \in X_{\square}F^h_\bullet(t) \subset G(n{-}k,n)$.

It follows that points in the intersection (3.3) correspond to k-dimensional spaces of polynomials V with discrete Wronskian $(t-z_1)\cdots(t-z_{k(n-k)})$, and each of these are real, by Proposition 3.1. ☐

3.2. **The Shapiro Conjecture is the limit of the Secant Conjecture.** The osculating plane $F_i(s)$ is the unique i-dimensional plane having maximal order of contact with the rational normal curve γ at the point $\gamma(s)$. This implies that it is a limit of secant planes, and in fact every limit of secant planes in which the points come together is an osculating plane.

Lemma 3.3. *Let* $\{s^{(j)}_1,\dots,s^{(j)}_i\}$ *for* $j = 1,2,\dots$ *be a sequence of lists of* i *distinct complex numbers that all converge to the same number,* $\lim_{j\to\infty} s^{(j)}_p = s$*, for each* $p = 1,\dots,i$ *and for some number* s*. Then*

$$\lim_{j\to\infty} \operatorname{span}\{\gamma(s^{(j)}_1),\gamma(s^{(j)}_2),\dots,\gamma(s^{(j)}_i)\} \;=\; F_i(s)\,.$$

As transversality and reality are preserved under perturbation, we conclude that Theorem 2.1 is a limiting case of the Secant Conjecture. Conversely, Theorem 2.1 implies the following.

Theorem 3.4. *Let* $\lambda^1,\dots,\lambda^m$ *be a Schubert problem and* $t_1,\dots,t_m$ *be distinct points of the rational normal curve* γ*. Then there exists an* $\epsilon > 0$ *such that if for each* $i = 1,\dots,m$*,* $F^i_\bullet$ *is a flag secant to* γ *along an interval of length* ϵ *containing* t_i*, then the intersection*

$$X_{\lambda^1}F^1_\bullet \,\cap\, X_{\lambda^2}F^2_\bullet \,\cap\, \cdots \,\cap\, X_{\lambda^m}F^m_\bullet \tag{3.4}$$

is transverse with all points real.

This implies that for generic secant flags $F_\bullet^1,\dots,F_\bullet^m$, the intersection (3.4) is transverse, which implies that secant flags are sufficiently general for the Schubert Calculus. Furthermore, Theorem 3.4 reduces the Secant Conjecture 2.2 to its transversality statement.

3.3. **Generalized Secant Conjecture.** Theorem 3.4 suggests a conjecture involving flags that are intermediate between secant and osculating, and which includes the Secant Conjecture and Theorem 2.1 as special cases.

A *generalized secant subspace* to the rational normal curve γ is spanned by osculating subspaces of γ. This notion includes secant subspaces, for a one-dimensional subspace that osculates γ is simply one that is spanned by a point of γ. A flag $F_\bullet$ is *generalized secant* to γ if each of the linear spaces in $F_\bullet$ are generalized secant subspaces. A generalized secant flag is *secant along an interval* of γ if the osculating subspaces that span its linear spaces osculate γ at points of the interval.

Conjecture 3.5 (Generalized Secant Conjecture)**.** *For any Schubert problem $\lambda^1,\dots,\lambda^m$ on a Grassmannian $G(k,n)$ and any generalized secant flags $F_\bullet^1,\dots,F_\bullet^m$ that are secant to a rational normal curve γ along disjoint intervals, the intersection*

$$X_{\lambda^1}F_\bullet^1 \cap X_{\lambda^2}F_\bullet^2 \cap \cdots \cap X_{\lambda^m}F_\bullet^m$$

is transverse and consists of $d(\lambda^1,\dots,\lambda^m)$ real points.

This includes the Secant Conjecture as the case when all of the flags are secant flags, but it also includes Theorem 2.1, which is when all flags are osculating. Many of the computations in our experiment tested instances of this conjecture where one or two flags were osculating while the rest were secant flags. This choice was made to make the computation feasible for some Schubert problems.

There is also a Generalized Cosecant Conjecture and a corresponding version of Theorem 3.4, which we do not formulate.

4. The problem of four secant lines

We give an in-depth look at the Schubert problem $\square^4 = 2$ on $G(2,4)$ where $\square$ denotes the Schubert condition that a two-plane in $\mathbb{C}^4$ meets a fixed two-plane nontrivially. Equivalently, $\square^4 = 2$ is the Schubert problem of lines in $\mathbb{P}^3$ that meet four fixed lines. Let $\gamma : \mathbb{R} \to \mathbb{P}^3$ be a rational normal curve. We consider the lines in $\mathbb{P}^3$ that meet four lines $\ell_1,\ell_2,\ell_3,\ell_4$ which are secant to γ.

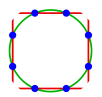
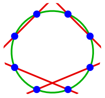
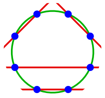
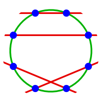
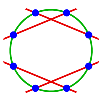
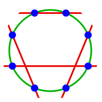
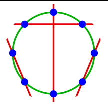
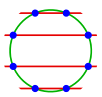
For $s_1, s_2 \in \mathbb{R}$ let $\ell(s_1,s_2)$ denote the secant line to γ through $\gamma(s_1)$, $\gamma(s_2)$. Given $s_1 < \cdots < s_8$, the Secant Conjecture (which is in this case a theorem of Eremenko, et al. [10]) asserts that both lines meeting the four fixed lines

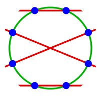
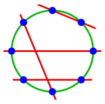
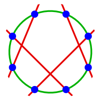
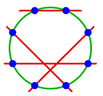
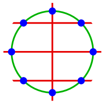
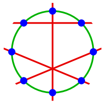
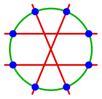
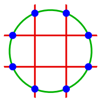
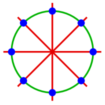
$$\ell(s_1,s_2)\,,\ \ \ell(s_3,s_4)\,,\ \ \ell(s_5,s_6)\,,\ \ \ell(s_7,s_8)$$

are real. We investigate phenomena beyond the Secant Conjecture by letting ρ be a permutation of $\{1,\dots,8\}$ and taking $\ell_1,\ell_2,\ell_3,\ell_4$ to be $\ell(s_{\rho(1)}, s_{\rho(2)}),\dots,\ell(s_{\rho(7)}, s_{\rho(8)})$.

There are 17 combinatorial configurations of four secant lines along $\gamma \simeq S^1$. These are indicated by the chord diagrams in Table 1, which shows the number of real solutions found when we computed 100,000 instances of each configuration. For most configurations, we only observed real solutions, and in only four configurations did we find any non-real solutions. We will give a simple explanation of this observation.

Table 1. Configurations of four secant lines with results of an experiment.

									
real	0			23723					29398
roots	2	100000	100000	76277	100000	100000	100000	100000	70602

									
0				52395				65783	
2	100000	100000	100000	47605	100000	100000	100000	34217	100000

Counting constants shows there is a unique doubly-ruled quadric surface Q that contains the lines ℓ_1, ℓ_2, and ℓ_3 in one ruling, as shown in Figure 1. The two lines of the second ruling of Q through the two points of intersection of ℓ_4 with Q are the solutions to the Schubert problem $\square^4 = 2$ for these four secant lines.

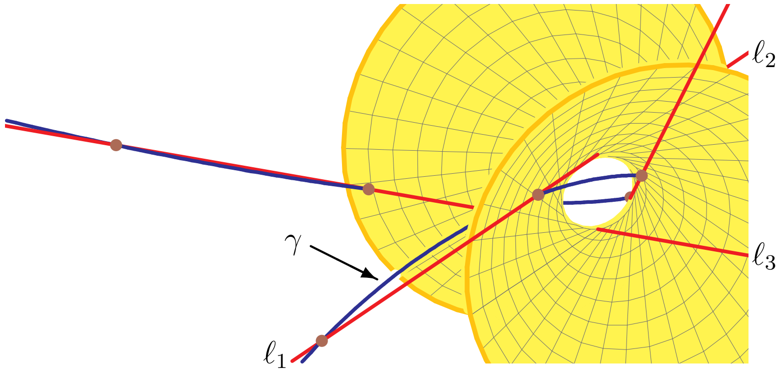


Figure 1. Quadric through three secant lines.

The quadric Q divides its complement in $\mathbb{RP}^3$ into two connected components (the domains where the quadratic form is positive or negative), called the *sides* of Q. Three lines ℓ_1, ℓ_2, ℓ_3 give six points of secancy which are the intersections of γ with Q and which divide γ into six segments that alternate between the two sides of Q. If the fourth secant line ℓ_4 has its two points of secancy lying on opposite sides of Q, then ℓ_4 has a real intersection with Q, so that the Schubert problem has one (and hence two) real solutions. The points of secancy of ℓ_4 lie on opposite sides of Q if in the interval between the two points of secancy, the curve γ crosses Q an odd number of times. That is, the interval contains an odd number of points of secancy of the lines ℓ_1, ℓ_2, and ℓ_3.

This simple topological argument shows that if at least one of the four secant lines has such an odd interval of secancy, then the Schubert problem will have only real solutions, independently of the actual positions of the secant lines. Twelve of the 17 configurations have at least one odd interval of secancy, and therefore will always give two real solutions. Four configurations with only even intervals of secancy were observed to have either zero or two real solutions. Only the configuration with disjoint intervals of secancy has even intervals of secancy, and yet has only real solutions. This deeper fact was proven in [10].

5. Overlap number

For most Schubert problems, the number of different configurations of secant flags is astronomical. Consider the problem $\square^4 \cdot \ydiagram{3,1}^2 = 12$ on the Grassmannian of 3-planes in 7-space. The condition $\square$ has relevant subspace F_4 and the condition $\ydiagram{3,1}$ has relevant subspace F_5. The resulting 26 points of secancy have at least

$$\left\lceil \binom{26}{4,4,4,4,5,5} \cdot \frac{1}{4!} \cdot \frac{1}{2!} \cdot \frac{1}{26} \cdot \frac{1}{2} \right\rceil \;=\; 3{,}381{,}948{,}761{,}563$$

combinatorially different configurations. To cope with this complexity, we introduce a statistic on these configurations—the overlap number—which is zero if and only if the flags are disjoint, and we tabulate the results of our experiment using this statistic.

In an instance of a Schubert problem $\lambda^1, \ldots, \lambda^m$ with relevant subspaces of respective dimensions $i_1, \ldots, i_m$, to define the relevant subspaces of the jth secant flag,

$$F_1^j \subsetneq F_2^j \subsetneq \cdots \subsetneq F_{i_j}^j,$$

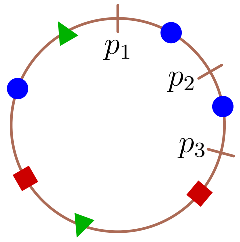
we need a choice of an ordered set T_j of i_j points of γ. The overlap number measures how much these sets of points $T_1, \ldots, T_m \subset \gamma$ overlap.

Let T be their union. Since γ is topologically a circle, removing a point $p \in \gamma \setminus T$, we may assume that $T_1, \ldots, T_m \subset \mathbb{R}$. Each set T_j defines an interval I_j of $\mathbb{R}$ and we let o_j be the number of points of $T \setminus T_j$ lying in I_j. This sum $\Sigma := o_1 + \cdots + o_m$ depends upon $p \in \gamma \setminus T$, and the *overlap number* is the minimum of these sums as p varies.

For example, consider a Schubert problem with relevant subspaces of dimensions 3, 2, and 2. Suppose that we have chosen seven points on γ in groups of 3, 2, and 2. This is represented schematically on the left in Figure 2, in which γ is a circle, and the points in the sets T_1, T_2, and T_3 are represented by circles (●), squares (■), and triangles (▲), respectively. For each of three points p_1, p_2, and p_3 of γ, we compute the number o_i and their sum Σ, displaying the results in the table on the right-hand side of Figure 2. The minimum of the sum Σ for all choices of points is achieved by p_3.

If one (or more) of the flags are osculating, we compute the overlap number by treating the point of osculation as a point with multiplicity equal to the dimension of the relevant subspace.

Figure 2. Computation of overlap number.



		●	■	▲	Σ
p_1	●—●—■—▲—■—●—▲	3	1	2	6
p_2	●—■—▲—■—●—▲—●	4	1	2	7
p_3	■—▲—■—●—▲—●—●	1	1	2	4

6. Experimental evidence for the Secant Conjecture

We tested the Secant Conjecture by conducting a massive experiment whose data are available on-line [40]. This experiment used symbolic exact arithmetic to compute the number of real solutions for specific instances of Schubert problems. These computations are possible because Schubert problems are readily modeled on a computer, and for those of moderate size, we may algorithmically determine the number of real solutions with software tools. Our experiment primarily used the mathematical software Singular [6] and Maple (see [14] for further details about the implementation of the computations, including a comprehensive list of software tools used). If the software is reliably implemented, which we believe, then this computation provides a proof that the given instance has the computed number of real solutions. This procedure may be semi-automated and run on supercomputers (as described in [14]), which allows us to amass the considerable evidence we have collected in support of the Secant Conjecture.

6.1. **Experimental data.**

Table 2 shows how many Schubert problems on each Grassmannian of k-planes in n-space had been studied when we halted the experiment on 26 May 2010. Our experiment not only tested the Secant Conjecture but also studied the relationship between the overlap number and the number of real solutions for many Schubert problems on small Grassmannians. We computed 2,058,810,000 instances of 703 Schubert problems. About one-fourth of these (498,737,669) were instances of the (Generalized) Secant Conjecture, and the rest involved non-disjoint secant flags. The Generalized Secant Conjecture held in every computed instance. The remaining 1,560,072,331 instances involved secant flags with some overlap in their intervals of secancy, measured by the overlap number.

Table 2. Schubert problems studied

$k \backslash n-k$	2	3	4	5	6
2	1	5	22	81	55
3	5	64	114	79	
4	22	107	67		
5	81				

The experiment computed Schubert problems using either 0, 1, or 2 osculating flags, with the rest secant flags. In the on-line database [40], this number of osculating flags determines the *computation type* which is 1, 2, or 3 for 0, 1, or 2 osculating flags. The experiment used randomly chosen flags, which were generated using random generator seeds that are stored in our database, so that all computations are reproducible.

Table 3 shows part of the data we obtained testing the full Secant Conjecture for the Schubert problem $\ydiagram{1}^4 \cdot \ydiagram{3,1}^2 = 12$ on $G(3{,}7)$. We used 7.52 gigahertz-years to compute 10,000,000 instances of this Schubert problem, all involving secant flags. The rows are labeled with the even integers from 0 to 12, as the number of real solutions has the same parity as the number of complex solutions. The first column with overlap number 0 represents tests of the Secant Conjecture. Since the only entry is in the row for 12 real solutions, the Secant Conjecture was verified in 2,320,873 instances. The column labeled overlap number 1 is empty because flags for this problem cannot have overlap number 1. Perhaps the most interesting feature is that for overlap number 2, all computed solutions were real, while for overlap number 3 at least six solutions were real, and for overlap number 4, at least four were real. It is only with overlap number 9 and above that we computed an instance with no real solutions.

Table 3. Experimental data for $\ydiagram{1}^4 \cdot \ydiagram{3,1}^2 = 12$ with all secant flags.

		Overlap Number										
	\	0	1	2	3	4	5	6	···	9	···	Total
Real Solutions	0								···	1	···	691
	2						9	7	···	8	···	72857
	4					79	917	1990	···	524	···	523362
	6				814	5713	12550	18330	···	4531	···	1418911
	8				635	4646	15947	17180	···	6055	···	1983639
	10				1226	6912	18403	17236	···	6801	···	1649923
	12	2320873		51120	99413	206398	203426	179955	···	42883	···	4350617
	Total	2320873		51120	102088	223748	251252	234698	···	60803	···	10000000

We also computed 200,000,000 instances of this same Schubert problem with four secant flags (for the Schubert variety $X_{\ydiagram{1}}$) and two osculating flags (for the Schubert variety $X_{\ydiagram{3,1}}$). These data are compiled in Table 4. This computation took 261 gigahertz-days—twenty times as many instances as Table 3 in about one-tenth of the time. This speed-up occurs because using two osculating flags gives a formulation with only four variables instead of 12. This computation tested the Generalized Secant Conjecture; its computed instances form the first column. As the only entry in that column is in the row for 12 real solutions, the Generalized Secant Conjecture was verified in 49,743,228 instances. As with Table 3 there is visibly an inner border to these data, but for this computation there are instances with no real solutions starting with overlap number eight.

6.2. **Computing Schubert problems.** A $k \times (n-k)$ matrix $X \in \mathbb{C}^{k\times(n-k)}$ determines a general point in $G(k{,}n)$, namely the row space H of the $k \times n$ matrix (also written H)

$$H \;:=\; \bigl(I_k \;:\; X\bigr)\,. \tag{6.1}$$

TABLE 4. Experimental data for $\ydiagram{1}^6 \cdot \ydiagram{3,1}^2 = 12$ with two osculating flags.

Overlap Number

Real Solutions

\	0	1	2	3	4	5	6	$\cdots$	Total
0								$\cdots$	13894
2						3799	19	$\cdots$	1357929
4					24756	93214	186521	$\cdots$	12146335
6					210843	495977	731938	$\cdots$	29925437
8					254875	640663	508884	$\cdots$	36708450
10					153520	442928	229530	$\cdots$	26500908
12	49743228		1171814	2324847	5900258	5944524	3971316	$\cdots$	93347047
Total	49743228		1171814	2324847	6544252	7621105	5628208	$\cdots$	200000000

If we represent an i-plane F_i as the row space of an $i \times n$ matrix F_i of full rank, then

$$\dim H \cap F_i \;\geq\; j \iff \operatorname{rank}\begin{pmatrix} H \\ F_i \end{pmatrix} \;\leq\; k + i - j\,, \tag{6.2}$$

which is given by the vanishing of all $(k{+}i{-}j{+}1) \times (k{+}i{-}j{+}1)$ subdeterminants. We represent a flag $F_\bullet$ by a full rank $n \times n$ matrix whose first i rows span F_i. Then (6.2) leads to equations for the Schubert variety $X_\lambda F_\bullet$ in the coordinate patch (6.1). In practice, we only need an $(n{-}k{+}i{-}\lambda_i) \times n$ matrix, where λ_i is the last nonzero part of λ.

To represent a secant i-plane, we use an $i \times n$ matrix $F_i(t_1, \ldots, t_i)$ whose jth row is the vector $\gamma(t_j)$, where $t_1, \ldots, t_i \in \mathbb{R}$, and $\gamma(t) = (1, t, \ldots, t^{n-1})$ is the rational normal curve. Similarly, the i-plane $F_i(t)$ osculating γ at the point $\gamma(t)$ is represented by the $i \times n$ matrix whose jth row is $\gamma^{(j-1)}(t)$.

For example, Conjecture 1.1 involves the Schubert problem $\ydiagram{1}^6 = 5$ on $G(2,5)$ where $\ydiagram{1}$ is the Schubert condition of a 2-plane meeting a 3-plane. The solutions are 2-planes spanned by the first two rows of the matrix in (1.1). The last three rows in the matrix are the points $\gamma(s_i)$, $\gamma(t_i)$, $\gamma(u_i)$ that span the 3-plane of a secant flag.

We use the computer algebra system Singular [6] to compute an *eliminant* of the polynomial system modeling a given instance of the Schubert problem $\lambda^1, \ldots, \lambda^m$. This is a univariate polynomial $f(x)$ whose roots are all the x-coordinates of solutions to the Schubert problem in the patch (6.1). (See, for example, [5, Chap. 2].) By the Shape Lemma [4], when the eliminant $f(x)$ has degree equal to $d(\lambda^1, \ldots, \lambda^m)$ and is square-free, then the solutions to the Schubert problem are in one-to-one correspondence with the roots of the eliminant $f(x)$, with real roots corresponding to real solutions. We use the `realroot` command of the mathematical software Maple to compute the number of real roots of the eliminant $f(x)$.

If the eliminant does not satisfy these hypotheses, then we compute an eliminant with respect to a different coordinate of the patch (6.1). It is sometimes the case that no coordinate provides a satisfactory eliminant. This will occur if there is a solution with multiplicity (the Schubert varieties do not meet transversally) or if the coordinate patch does not contain all solutions. In general it will occur when the computed instance lies in a *discriminant hypersurface* in the space of all instances. When developing and testing our software for this experiment, we observed that this situation was extremely rare, and it only occurred when the overlap number was positive and there were multiple solutions, which agrees with the transversality assertion in the Secant Conjecture. When our software detects that no coordinate provides a satisfactory eliminant, it deterministically perturbs the points of secancy, preserving the overlap number, and repeats this elimination procedure. This has always worked to give an eliminant satisfying the hypotheses.

As with Tables 3 and 4, working in a different set of local coordinates enables us to efficiently compute instances of the Generalized Secant Conjecture 3.5 for one (and sometimes two) osculating flags. With one flag osculating at $\gamma(\infty)$, we may use local coordinates as described in [27].

With two osculating flags, there is a smaller choice of local coordinates available. Suppose that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors corresponding to columns of our matrices. Then the flag $F_\bullet(\infty)$ osculating the rational normal curve γ at $\gamma(\infty) = \mathbf{e}_n$ and the flag $F_\bullet(0)$ osculating at $\gamma(0) = \mathbf{e}_1$ have

$$F_i(\infty) \;=\; \operatorname{span}\{\mathbf{e}_{n+1-i}, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_n\} \qquad \text{and} \qquad F_i(0) \;=\; \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_i\}\,.$$

General points in $X_\lambda F_\bullet(\infty) \cap X_\mu F_\bullet(0)$ are represented by $k \times n$ matrices where row i has a 1 in column $\lambda_{k+1-i} + i$, arbitrary entries in subsequent columns up to column $n-k-1+i-\mu_i$, and 0's elsewhere. Here is such a matrix with $k = 3$, $n = 8$, $\lambda = \ydiagram{2,1}$, and $\mu = \ydiagram{3,1}$:

$$\begin{pmatrix} 1 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & * \end{pmatrix}.$$

6.3. **Numerical experimentation.** In [13], 25,000 instances of the Shapiro Conjecture for the Schubert problem $\ydiagram{2}^8 = 126$ were computed, and for each instance the software alphaCertified used 256-bit precision to softly certify that all solutions were real. (A *soft certificate* is one computed with floating point arithmetic that would be rigorous if computed with exact rational arithmetic.) The solutions were computed using the software package Bertini [3], which is based on numerical homotopy continuation [30]. Given a system of n polynomial equations in n unknowns, Smale's α-theory [29] gives algorithms for certifying that Newton iterations applied to an approximate solution will converge to a solution, and also may be used to certify that the solution is real. As explained in [13], this Schubert problem has such a formulation. These algorithms are implemented in the software alphaCertified [13].

7. Lower bounds and inner borders

The most ubiquitous and enigmatic phenomenon that we have observed in our data is the apparent "inner border" in many of the tables. Typically, we do not observe instances with zero or few real solutions when the overlap number is small. This is manifested by a prominent staircase separating observed pairs of (real solutions, overlap number) from unobserved pairs. This feature is clearly visible in Tables 3 and 4, and in Table 5 for the problem $\ydiagram{1}^8 = 14$ in $G(2,6)$. There, it is only with overlap number 8 or larger that we observe instances with two

Table 5. Real solutions vs. overlap number for $\square^8 = 14$.

\	0	1	2	3	4	5	6	⋯	Total
0								⋯	4272
2								⋯	127217
4					693	1481	6660	⋯	879658
6					224	510	2541	⋯	2304233
8					526	939	3561	⋯	2914837
10					1052	2074	6985	⋯	2205198
12					1556	2595	7300	⋯	1224667
14	3328772		60860	120625	310819	246910	237704	⋯	5339918
Total	3328772		60860	120625	305870	254509	264751	⋯	15000000

real solutions; and with overlap number 16 or larger, instances with no real solutions. (These columns are not displayed for reasons of space.)

This problem involves 2-planes meeting eight secant 4-planes. There are over 10^{18} configurations of eight secant 4-planes, and hence it is impossible to systematically study all configurations as in Section 4. This is the case for most of the problems we studied. Because of the coarseness of our measure of overlap, we doubt it is possible to formulate a meaningful conjecture about this inner border based on our data. Nevertheless, we believe that this problem, like the problem of four lines, contains rich geometry, with certain configurations having a lower bound on the number of real solutions.

There are many meaningful polynomial systems or geometric problems having a non-zero lower bound on their number of real solutions. These include rational curves interpolating points on toric del Pezzo surfaces [15, 16, 17, 22, 39], sparse polynomial systems from posets [18, 31], and some lower bounds in the Schubert calculus [1, 7].

Lower bounds and inner borders were also observed studying the Monotone Conjecture [27, § 3.2.2]. The original example of a lower bound was due to Eremenko and Gabrielov [7]. The Wronskian of linearly independent polynomials $f_1(t), f_2(t), \ldots, f_k(t)$ of degree $n-1$,

$$\mathrm{W}(f_1, f_2, \ldots, f_k) \ := \ \det \begin{pmatrix} f_1(t) & f_1'(t) & f_1''(t) & \cdots & f_1^{(k-1)}(t) \\ f_2(t) & f_2'(t) & f_2''(t) & \cdots & f_2^{(k-1)}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_k(t) & f_k'(t) & f_k''(t) & \cdots & f_k^{(k-1)}(t) \end{pmatrix},$$

has degree $k(n-k)$, which gives a finite map $\mathrm{W}\colon G(k, \mathbb{C}_{n-1}[t]) \longrightarrow \mathbb{P}^{k(n-k)}$ with the general fiber consisting of $d(\square^{k(n-k)})$ (see (3.1)) linear spaces of polynomials. Theorem 2.1 implies that if $w(t)$ is a polynomial with $k(n-k)$ distinct real roots then each of the $d(\square^{k(n-k)})$ points in the fiber of W over $w(t)$ is real. Eremenko and Gabrielov showed that if n is odd, there is a non-trivial lower bound on the number of real spaces of polynomials in the fiber of W over *any* polynomial $w(t)$ with real coefficients.

Azar and Gabrielov [1] studied the problem $\square^{2n-4}$ in $G(n-2,n)$ of $(n-2)$-planes in $\mathbb{C}^n$ which meet one secant line and $2n-5$ tangent lines. When the interval of secancy contains no tangent points, this is an instance of the Generalized Secant Conjecture 3.5. They establish lower bounds on the number of real solutions which depend upon the configuration of the points of secancy and tangency.

8. Gaps

The Schubert problem $\boxplus^4 = 6$ on $G(4,8)$ involves 4-planes whose intersection with each of four general 4-planes is at least two-dimensional. We computed 1,000,000 instances of this problem, obtaining the results in Table 6. A system of real polynomial equations with 6 solutions can, *a priori*, have 0, 2, 4, or 6 real solutions; yet, strikingly, this Schubert problem only has 2 or 6 real solutions, never 0 or 4.

Table 6. Real solutions vs. overlap number for $W_{\boxplus}^4 = 6$.

	0	1	2	3	4	5	6	···	Total
0									0
2				1441	7730	14277	16636	···	147326
4									0
6	280304		13131	25708	62833	55919	57719	···	852674
Total	280304	0	13131	27149	70563	70196	74355	···	1000000

Although our observations involved only secant flags, this phenomenon holds for any real flags. As we describe below, this follows from ideas in Vakil's discussion of this Schubert problem in [37, §3.13]. (Vakil's discussion, however, focuses on explaining a different phenomenon, namely, Derksen's observation that the Galois group of this Schubert problem is *deficient*, i.e., smaller than the symmetric or alternating group.)

We consider the *auxiliary Schubert problem* $\square\!\square\!\square^4 = 4$ on $G(2,8)$, counting 2-planes which meet four general 4-planes. Given 4-planes $W_1,\dots,W_4$, let $P_1,\dots,P_4$ be the 2-planes which meet them. Then the solutions to the original Schubert problem $W_{\boxplus}^4 = 6$ are precisely the 6 sums of the form $P_i + P_j$. Such a sum is real if and only if P_i and P_j are each real or if P_i and P_j are a pair of complex conjugate subspaces.

If the W_i are real, then there can be 0, 1, or 2 complex conjugate pairs among the P_i. Then the number of solutions $P_i + P_j$ which are real is, respectively, 6, 2, and 2. This explains the observations in Table 6.

This is the first in a family of Schubert problems in $G(4,2n)$ for $n \geq 4$ with such gaps in their numbers of real solutions. These involve enumerating the 4-planes which have at least a two-dimensional intersection with each of four general n-planes in $\mathbb{C}^{2n}$. For each, there is an auxiliary Schubert problem on $G(2,2n)$ of 2-planes meeting four general n-planes. This will have n solutions, and the solutions to the original problem are 4-planes spanned by pairs of solutions to the auxiliary problem. The original problem will have $\binom{n}{2}$ solutions, corresponding to pairs of solutions to the auxiliary problem. A solution is real either when both elements of the pair are real or when the pair consists of complex conjugate solutions. We remark that the auxiliary problem may have any number r of real solutions, where $0 \leq r \leq n$ and $n-r$ is even—this may be deduced from the description of the Schubert problem in terms of elementary geometry given, for example, in [32, § 8.1]. These restrictions are identical to restrictions on the number of real quadratic factors of a general real polynomial of degree n, as in [31, Theorem 7.8]. We summarize this discussion.

Theorem 8.1. *The Schubert problem of 4-planes that have at least a two-dimensional intersection with each of four general real n-planes in $\mathbb{C}^{2n}$ has $\binom{n}{2}$ solutions. The number of real solutions is*

$$\binom{r}{2} \;+\; c\,,$$

where the auxiliary problem of 2-planes meeting each of four general real n-planes in $\mathbb{C}^{2n}$ has r real solutions and c pairs of complex conjugate solutions and $r + 2c = n$.

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