Enumerative geometry for real varieties

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1. Introduction

Of the geometric figures in a given family satisfying real conditions, some figures are real while the rest occur in complex conjugate pairs, and the distribution of the two types depends subtly upon the configuration of the conditions. Despite this difficulty, applications ([**7, 35, 40**]) may demand real solutions. Fulton [**12**] asked how many solutions of an enumerative problem can be real. We consider a special case of his question: Given a problem of enumerative geometry, are there real conditions such that every figure satisfying them is real? Such an enumerative problem is *fully real*.

Bézout's Theorem, or rather the problem of intersecting hypersurfaces in $\mathbb{P}^n$, is fully real. This is readily seen for $\mathbb{P}^2$ and the argument generalizes to $\mathbb{P}^n$. Suppose X_0 consists of d real lines, Y_0 of e real lines, and X_0 meets Y_0 transversally in (necessarily) $d \cdot e$ real points. Let X and Y be defined by suitably small generic real deformations of the forms defining X_0 and Y_0. Then X and Y are smooth real plane curves of degrees d and e meeting transversally in $d \cdot e$ real points.

This argument was based upon a degenerate case free of multiplicities; X_0 and Y_0 are reduced and meet transversally. While it is typical to introduce multiplicities (for example, in the proof of Bézout's Theorem in [**36**]) to establish enumerative formulas, multiplicities may lead to complex conjugate pairs of solutions, complicating the search for real solutions.

All Schubert-type enumerative problems involving lines in $\mathbb{P}^n$ are fully real [**45**]. This follows from the existence of (multiplicity-free) deformations of generically transverse intersections of Schubert varieties into sums of Schubert varieties. Refining this method of multiplicity-free deformations [**46**] yields techniques for showing other enumerative problems are fully real. Ronga, Tognoli, and Vust [**39**] have shown the problem of 3264 conics tangent to five general plane conics is fully real. Their analysis utilizes degenerate conditions having multiplicities.

Enumerative problems that we know are not fully real share a common flaw: they do not involve intersecting general subvarieties. For example, Klein [**27**] showed that at most $n(n-2)$ of the $3n(n-2)$ flexes on a real plane curve of degree n can be real. These flexes are the intersection of the curve with its Hessian determinant, *not* with a general curve of degree $3(n-2)$. This problem has been revisited by Wall [**51**]. For the problem of intersecting hypersurfaces in a complex torus

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defined by polynomials with few monomials, Khovanskii [**24**] gave an upper bound on the number of real points of intersection, independent of the total number of points in such an intersection. These are not generic hypersurfaces with given Newton polytope. Even generic hypersurfaces may fail to have all their points of intersection be real: For general $a, b \in \mathbb{R}$, the system $a = x^5y^3, b = x^7y^9$ has at most 4 real solutions, but has 24 solutions in $(\mathbb{C}^\times)^n$, by the Theorems of Kouchnirenko, Bernstein, and Khovanskii ([**3, 31, 23**]).

We are unaware of a good theoretical framework for studying fully real enumerative problems and it is not known how common it is for an enumerative problem to be fully real. Here are some examples of enumerative problems worth considering:

(1) Are there conditions on lattice polytopes $\Delta_1, \ldots, \Delta_n$ in $\mathbb{Z}^n$ which ensure there exist real polynomials $f_1, \ldots, f_n$ where Δ_i is the Newton polytope of f_i and all solutions to the system $f_1 = \cdots = f_n = 0$ in $(\mathbb{C}^\times)^n$ are real? One could also seek better bounds on the number of real solutions in the spirit of [**49**]. A (conjectural) bound is due to Itenberg and Roy [**21**].

(2) Generalize the results of [**45**] and [**46**]: Are other (all?) Schubert-type enumerative problems on flag varieties fully real?

(3) All known examples involve spherical varieties ([**6, 28, 33**]). Which enumerative problems on spherical varieties are fully real?

(4) In [**46**] all problems of enumerating lines incident upon subvarieties of fixed dimension and degree in $\mathbb{P}^n$ are shown to be fully real. What is the situation for rational curves of higher degree? (A rational curve of degree 0 is a point, so degree 0 is Bézout's Theorem.) For example, for which positive integers d do there exist $3d-1$ real points in $\mathbb{P}^2$ such that the Kontsevich number N_d of degree d rational curves passing through these points ([**30, 41**]) are all real? For an introduction to these questions of quantum cohomology, see the paper by Fulton and Pandharipande [**13**] in this volume.

This note is organized as follows: In §2 we discuss some examples of fully real enumerative problems for which multiplicity-free deformations play a central role. This technique is illustrated in §3, where we show that there are nine real Veronese surfaces in $\mathbb{P}^5$ such that the 11010048 planes meeting all nine are real. Next is a discussion of the work of Ronga, Tognoli, and Vust [**39**] on the problem of conics. We conclude with a description of some computational work on related questions and a conjecture of Shapiro and Shapiro regarding the second question above.

2. Effective Rational Equivalence

A common feature of many fully real enumerative problems are multiplicity-free deformations of intersection cycles. Effective rational equivalence is a precise formulation of this for Grassmann and flag varieties.

2.1. Real effective rational equivalence. Varieties will be quasi-projective, reduced, complex and defined over the real numbers, $\mathbb{R}$. Let X be a Grassmann or flag variety, G a linear algebraic group acting transitively on X, and B a Borel subgroup of G. The letters U and V will denote smooth rational varieties. Let the real points $Y(\mathbb{R})$ of a variety Y be equipped with the classical topology.

A subvariety $\Xi \subset U \times X$ (or $\Xi \to U$) with generically reduced equidimensional fibres over U is a *family of (multiplicity-free) cycles on X over U*. We assume all families are G-stable; if Y is a fibre of Ξ over U, then so are all translates of Y. Let $\mathit{Chow}\,X$ be the Chow variety of X parameterizing cycles of the same dimension and degree as Ξ_u [**42**, §I.9]. Associating a point u of U to the fundamental cycle of the fibre Ξ_u determines a morphism $\phi : U \to \mathit{Chow}\,X$. In fact, it suffices for U to be normal ([**29**, §1] or [**11**, §3]). One may also see this as follows: If $C \subset U$ is a smooth curve, then $\Xi|_C \to C$ is flat and the canonical map of the Hilbert scheme to the Chow variety [**37**, §5.4] shows $\phi|_C$ is a morphism. By Hartogs' Theorem on separate analyticity, ϕ is in fact a morphism. For a discussion of Chow varieties in the analytic category (which suffices for our purposes), see [**2**].

Any positive cycle Y on X is rationally equivalent to a positive integral linear combination of Schubert cycles. This rational equivalence occurs within the closure of $B \cdot Y$ in *Chow* X since B-stable cycles of X (B-fixed points in $B \cdot Y$) are integral linear combinations of Schubert cycles [**17**]. If any coefficients in this linear combination exceed 1, this stable cycle has multiplicities.

A family $\Xi \to U$ of multiplicity-free cycles on X has *effective rational equivalence* with *witness* Z if there is a cycle $Z \in \overline{\phi(U)}$ which is a sum of distinct Schubert varieties, hence multiplicity-free. An effective rational equivalence is *real* if $Z \in \overline{\phi(U(\mathbb{R}))}$ and each component of Z is a Schubert variety defined by a real flag.

Suppose $\Xi_1 \to U_1, \ldots, \Xi_b \to U_b$ are G-stable families of multiplicity-free cycles on X. By Kleiman's Transversality Theorem [**25**], there is a nonempty open set U of $\prod_{i=1}^b U_i$ consisting of b-tuples $(u_1, \ldots, u_b)$ such that the fibres $(\Xi_1)_{u_1}, \ldots, (\Xi_b)_{u_b}$ meet generically transversally. Let $\Xi \subset U \times X$ be the resulting family of intersection cycles and call $\Xi \to U$ the *intersection problem* given by $\Xi_1, \ldots, \Xi_b$.

THEOREM 1 ([**45**]). *Any intersection problem given by families of Schubert varieties in the Grassmannian of lines in projective space has real effective rational equivalence.*

We present a synopsis of the proof: Let X be the Grassmannian of lines in $\mathbb{P}^n$ and suppose $\Xi \to U$ is an intersection problem given by families of Schubert varieties. A sequence $\Psi_0 \to V_0, \ldots, \Psi_c \to V_c$ of families of multiplicity-free cycles on X is constructed with each V_i rational, where $\Psi_0 \to V_0$ is the family $\Xi \to U$, V_c is a point, and Ψ_c a union of distinct real Schubert varieties. For each $i = 0, \ldots, c$, let $\mathcal{G}_i \subset$ *Chow* X be $\phi(V_i(\mathbb{R}))$, the set of fibres of the family $\Psi_i \to V_i$ over $V_i(\mathbb{R})$.

Then $\mathcal{G}_i \subset \overline{\mathcal{G}_{i-1}}$: For any $v \in V_i(\mathbb{R})$ a family $\Gamma \to C$ of cycles is constructed with C a smooth rational curve, the cycle $(\Psi_i)_v$ a fibre over $C(\mathbb{R})$, and all other fibres of Γ are fibres of Ψ_{i-1}. This family induces a morphism $\phi : C \to$ *Chow* X, which shows $(\Psi_i)_v \in \overline{\mathcal{G}_{i-1}}$ since $\phi(C(\mathbb{R})) - \{(\Psi_i)_v\} \subset \mathcal{G}_{i-1}$. It follows that $\Psi_c \in \overline{\mathcal{G}_0} = \overline{\phi(U(\mathbb{R}))}$, showing $\Xi \to U$ has real effective rational equivalence. □

While it is difficult to describe an intersection of several Schubert varieties, it is sometimes possible to describe the limiting position of such an intersection as the Schubert varieties degenerate to the point of attaining excess intersection. This is the aim of effective rational equivalence. For example, the 'limit cycle' Ψ_c in the discussion of Theorem 1 is generally not an intersection of Schubert varieties. However, it is a deformation of such cycles.

An *enumerative problem* of degree d is an intersection problem $\Xi \to U$ with finite fibres of cardinality d. It is *fully real* if there is a fibre Ξ_u with $u \in U(\mathbb{R})$ consisting entirely of real points. Here, $u = (u_1, \ldots, u_b)$ with $u_i \in U_i(\mathbb{R})$ and Ξ_u is the transverse intersection of the cycles $(\Xi_1)_{u_1}, \ldots, (\Xi_b)_{u_b}$.

The set $\mathcal{M} \subset Sym^d X$ of degree d zero cycles consisting of d distinct real points of X is an open subset of $(Sym^d X)(\mathbb{R})$. Thus $\Xi \to U$ is fully real if and only if it has real effective rational equivalence. Hence Theorem 1 has the following consequence:

COROLLARY 2 ([**45**]). *Any enumerative problem given by Schubert conditions on lines in projective space is fully real.*

2.2. Products in A^*X. The variety X is the quotient G/P of G by a parabolic subgroup P. A Schubert subvariety $\Omega_w F_\bullet$ of X is given by a complete flag $F_\bullet$ and a coset w of the corresponding parabolic subgroup of the symmetric group [**5**, Ch. IV, §2.5]. Call w the *type* of $\Omega_w F_\bullet$. A Schubert class σ_w is the cycle class of $\Omega_w F_\bullet$.

Let $\Xi_1 \to U_1, \ldots, \Xi_b \to U_b$ be families of cycles on X giving an intersection problem $\Xi \to U$. Then fibres of $\Xi \to U$ have cycle class $\prod_i \beta_i$, where β_i is the cycle class of fibres of $\Xi_i \to U_i$. Suppose $\Xi \to U$ has effective rational equivalence with witness Z. Let c_w count the components of Z of type w. Since Z is rationally equivalent to fibres of $\Xi \to U$, we deduce the formula in A^*X.

$$\prod_{i=1}^{b} \beta_i \quad = \quad \sum_w c_w \cdot \sigma_w .$$

2.3. Pieri-type formulas. Given such a product formula with each $c_w \leq 1$, the action of a real Borel subgroup B of G shows that the family of intersection cycles $\Xi \to U$ has real effective rational equivalence: Let Y be a fibre of $\Xi \to U$ over a real point of U. Then the closure of the orbit $B(\mathbb{R}) \cdot Y$ in $\mathit{Chow}\, X(\mathbb{R})$ contains a $B(\mathbb{R})$-fixed point Z as Borel's fixed point Theorem [**4**, III.10.4] holds for $B(\mathbb{R})$-stable real analytic sets. Moreover, Z is multiplicity-free as $c_w \leq 1$.

In the Grassmannian of k-planes in $\mathbb{P}^n$, a *special Schubert variety* is the locus of k-planes having excess intersection with a fixed linear subspace. A *special Schubert variety* of a flag variety is the inverse image of a special Schubert variety in a Grassmannian projection. Pieri's formula for Grassmannians [**16, 18**] and the Pieri-type formulas for flag varieties [**32, 44**] show that the coefficients c_w in the product of a Schubert class with a special Schubert class are either 0 or 1. It then follows from the previous paragraph that any intersection problem given by a Schubert variety and a special Schubert variety has real effective rational equivalence. We use this to prove the following theorem.

THEOREM 3 ([**46**]). *Any enumerative problem in any flag variety given by five Schubert varieties, three of which are special, is fully real.*

PROOF. First pair each non-special Schubert variety with a special Schubert variety. The associated families $\Xi \to U$ and $\Xi' \to U'$ of intersection cycles have real effective rational equivalence with witnesses Z and Z', respectively.

Since the coefficients c_w in the Pieri-type formulas are either 0 or 1, a zero-dimensional intersection of three real Schubert varieties in general position where one is special is a single real point. Considering components of Z and Z' separately, we see that if Z, Z', and the third special Schubert variety Y are in general position with Y real, then they intersect transversally with all points of intersection real. Suitably small deformations of Z and Z' into real fibres of Ξ and Ξ' preserve the number of real points of intersection, completing the proof. □

3. The Grassmannian of planes in $\mathbb{P}^5$

The Grassmannian of planes in $\mathbb{P}^5$, $\mathbb{G}_{2,5}$, is a 9-dimensional variety. If K is a plane in $\mathbb{P}^5$, then the set $\Omega(K)$ of planes which meet K is a hyperplane section of $\mathbb{G}_{2,5}$ in its Plücker embedding. Thus the number of planes which meet 9 general planes is the degree of $\mathbb{G}_{2,5}$, which is $\frac{1!2!9!}{3!4!5!} = 42$ [**43**]. This variety is the smallest dimensional flag variety for which an analog of Corollary 2 is not known. We illustrate the methods of §2 to prove the following result:

THEOREM 4. *There are 9 real planes in $\mathbb{P}^5$ such that the 42 planes meeting all 9 are real.*

The Veronese surface in $\mathbb{P}^5$ is the image of $\mathbb{P}^2$ under an embedding induced by the complete linear system $|\mathcal{O}(2)|$, and so it has degree 4.

COROLLARY 5. *There are 9 real Veronese surfaces in $\mathbb{P}^5$ such that the 11010048 ($= 4^9 \cdot 42$) planes meeting all 9 are real.*

PROOF. Let x_{ij} , $1 \leq i \leq j \leq 3$, be real coordinates for $\mathbb{P}^5$. For $t \neq 0$

$$(1)\quad \langle \underline{x_{11}x_{33}} - t^4\, x_{13}^2,\ \underline{x_{11}x_{22}} - t^2\, x_{12}^2,\ \underline{x_{11}x_{23}} - t\, x_{12}x_{13},\ \underline{x_{12}x_{33}} - t\, x_{13}x_{23},\ \underline{x_{13}x_{22}} - t\, x_{12}x_{23},\ \underline{x_{22}x_{33}} - t^2\, x_{23}^2\rangle$$

generates the ideal of a Veronese surface, $\mathcal{V}(t)$ (cf. [**50**, p. 142]), which is real for $t \in \mathbb{R}$. This family of Veronese surfaces is induced by the (real) $\mathbb{C}^\times$-action on the space of linear forms on $\mathbb{P}^5$:

$$x_{ij} \quad \mapsto \quad t^{j-i}x_{ij} \qquad \text{for } t \in \mathbb{C}^\times .$$

The ideal of the special fibre $\mathcal{V}(0)$ of this family is generated by the underlined terms, so $\mathcal{V}(0)$ is the union of the four planes given by the ideals:

$$(2)\qquad \langle x_{11}, x_{22}, x_{33}\rangle \qquad \langle x_{ii}, x_{jj}, x_{ij}\rangle, \quad ij = 12, 13, 23.$$

By Theorem 4, there exist 9 real planes $K_1, \ldots, K_9$ such that $\bigcap_{i=1}^9 \Omega(K_i)$ is a transverse intersection consisting of 42 real planes. This property of $K_1, \ldots, K_9$ is preserved by small real deformations. So for each $1 \leq i \leq 9$, there is a neighborhood W_i of K_i in $\mathbb{G}_{2,5}(\mathbb{R})$ such that if $K_i' \in V_i$ for $1 \leq i \leq 9$, then $\bigcap_{i=1}^9 \Omega(K_i')$ is transverse and consists of 42 real planes.

For each $1 \leq i \leq 9$, choose a set of real coordinates for $\mathbb{P}^5$ so that the four planes, $K_{i,j}$, for $j = 1, 2, 3, 4$, defined by the ideals of (2) are in W_i. In these same coordinates, consider the family $\mathcal{V}_i(t)$ of real Veronese surfaces given by the ideals (1) with special member $\mathcal{V}_i(0) = K_{i,1}+K_{i,2}+K_{i,3}+K_{i,4}$. If the sets of coordinates are chosen sufficiently generally, then there exists $\epsilon > 0$ such that whenever $t \in (-\epsilon, \epsilon)$, there are exactly $4^9 \cdot 42$ real planes meeting each of $\mathcal{V}_1(t), \ldots, \mathcal{V}_9(t)$. Indeed, the set of planes meeting each of $\mathcal{V}_1(0), \ldots, \mathcal{V}_9(0)$ is

$$\bigcap_{i=1}^{9} \left(\Omega(K_{i,1}) + \Omega(K_{i,2}) + \Omega(K_{i,3}) + \Omega(K_{i,4})\right),$$

which is a transverse intersection consisting of $4^9 \cdot 42$ real planes: Since $K_{i,j} \in W_i$ for $1 \leq i \leq 9$ and $1 \leq j \leq 4$, this follows if the 4^9 sets of 42 planes $\bigcap_{i=1}^9 \Omega(K_{i,l_i})$ given by all sequences l_i, where $1 \leq l_i \leq 4$ for $1 \leq i \leq 9$, are pairwise disjoint. But this may be arranged when choosing the sets of coordinates. □

LEMMA 6. *The intersection problem of planes meeting 4 given planes in $\mathbb{P}^5$ has real effective rational equivalence.*

PROOF OF THEOREM 4 USING LEMMA 6. Partition the 9 planes into two sets of 4 and a singleton. Apply Lemma 6 to the intersection problems $\Xi \to U$, $\Xi' \to U'$ given by each set of 4, obtaining witnesses Z and Z'. Arguing as for Theorem 3 completes the proof. □

PROOF OF LEMMA 6. We use an economical notation for Schubert varieties. A partial flag $A_0 \subset A_1 \subset A_2 \subset \mathbb{P}^5$ determines a Schubert subvariety of $\mathbb{G}_{2,5}$:

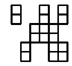
$$\Omega(A_0, A_1, A_2) := \{H \in \mathbb{G}_{2,5} \mid \dim H \bigcap A_i \geq i,\ \text{for } i = 0, 1, 2\}.$$

If A_j is a hyperplane in A_{j+1} or if $A_2 = \mathbb{P}^5$, then it is no additional restriction for $\dim H \bigcap A_j \geq j$. We omit such inessential conditions. Thus, if $\mu \subset M \subset \Lambda$ is a partial flag, then $\Omega(\mu), \Omega(\bullet, M)$, and $\Omega(\mu, \bullet, \Lambda)$ are, respectively, those planes H which meet μ, those H with $\dim H \cap M \geq 1$, and those $H \subset \Lambda$ which also meet μ.

Let $\Xi \subset U \times \mathbb{G}_{2,5}$ be the intersection problem of planes meeting four given planes. Then $U \subset (\mathbb{G}_{2,5})^4$ is the set of 4-tuples of planes (K_1, K_2, K_3, K_4) such that $\bigcap_{i=1}^4 \Omega(K_i)$ is a generically transverse intersection and the fibre of Ξ over (K_1, K_2, K_3, K_4) is $\bigcap_{i=1}^4 \Omega(K_i)$. We show $\Xi \to U$

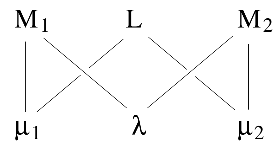
has real effective rational equivalence by exhibiting a family $\Psi \subset V \times \mathbb{G}_{2,5}$, satisfying the four conditions:

(a) V is rational. In fact V is a dense subset of Magyar's configuration variety $\mathcal{F}_D$ [**34**], where D is the diagram



(b) V has a dense open subset V° such that the fibres of $\Psi|_{V^\circ}$ are also fibres in the family $\Xi \to U$.
(c) V has a rational subset V' such that the fibres of $\Psi|_{V'}$ are unions of distinct Schubert varieties, real for real points of V'.
(d) $V'(\mathbb{R}) \subset \overline{V^\circ(\mathbb{R})}$. Hence $\phi(V'(\mathbb{R})) \subset \overline{\phi(U(\mathbb{R}))}$. Together with (c), this shows $\Xi \to U$ has effective rational equivalence.

Let $V \subset (\mathbb{G}_{1,5})^3 \times (\mathbb{G}_{3,5})^3$ be the locus of sextuples $(\mu_1, \mu_2, \lambda; M_1, M_2, L)$ such that $\mu_i \subset M_i$, $i = 1, 2$, $\mu_1, \mu_2 \subset L$, $\lambda \subset M_1 \bigcap M_2$, and $\mu_i \not\subset M_j, i \neq j$. We illustrate the inclusions:



Let $V^\circ \subset V$ be the dense locus where $\langle \mu_i, M_j\rangle = \mathbb{P}^5$ for $i \neq j$. Then $\lambda = M_1 \bigcap M_2$ and $L = \langle \mu_1, \mu_2\rangle$. Let $V' \subset V$ be the locus where $\mu_1 \bigcap \mu_2$ is a point, so that $\langle M_1, M_2\rangle$ is a hyperplane. Then V' is rational and $V'(\mathbb{R}) \subset \overline{V^\circ(\mathbb{R})}$, proving (d).

We define the family Ψ. For $v = (\mu_1, \mu_2, \lambda; M_1, M_2, L) \in V$, let Ψ_v be the cycle

$$\Omega(\mu_1) \bigcap \Omega(\bullet, M_2) \;+\; \Omega(\mu_2) \bigcap \Omega(\bullet, M_1) \;+ \\ \{H \in \Omega(\mu_1, L) \mid H \bigcap \mu_2 \neq \emptyset\} \;+\; \{H \in \Omega(\lambda, M_1) \mid \dim H \bigcap M_2 \geq 1\}. \tag{3}$$

Let $\Psi \subset \mathbb{G}_{2,5} \times V$ be the subvariety with fibre Ψ_v over points $v \in V$.

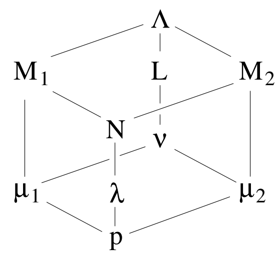
Suppose $v \in V^\circ$. Since $L = \langle \mu_1, \mu_2\rangle$ and $\mu_1 \bigcap \mu_2 = \emptyset$,

$$\Omega(\mu_1) \bigcap \Omega(\mu_2) \;=\; \{H \in \Omega(\mu_1, L) \mid H \bigcap \mu_2 \neq \emptyset\},$$

as any plane meeting both μ_1 and μ_2 must intersect their span L in at least a line. Similarly, $\Omega(\bullet, M_1) \bigcap \Omega(\bullet, M_2)$ is the fourth term of the cycle (3): If l_i is a line in $H \bigcap M_i$ for $i = 1, 2$, then $l_1 \bigcap l_2 \subset \lambda = M_1 \bigcap M_2$. Thus we see that $\Psi_v = \bigcap_{i=1}^{2} (\Omega(\mu_i) + \Omega(\bullet, M_i))$. This intersection is generically transverse as each pair of subspaces (μ_1, μ_2), (M_1, M_2), and (μ_i, M_j) for $i \neq j$ is in general position.

We claim Ψ_v is a fibre of $\Xi \to U$: Let K_i, K_i' for $i = 1, 2$ be planes such that $\mu_i = K_i \bigcap K_i'$ and $M_i = \langle K_i, K_i'\rangle$. Then $\Omega(K_i) \bigcap \Omega(K_i') = \Omega(\mu_i) + \Omega(\bullet, M_i)$: If a plane H meets both K_i and K_i', either it meets their intersection μ_i, or else it intersects their span M_i in at least a line. Moreover, while K_i, K_i' are not in general position, this intersection is generically transverse as a proper intersection of a Schubert variety with a special Schubert variety is necessarily generically transverse [**47**, §2.7]. Thus $\Psi_v = \Xi_{(K_1, K_1', K_2, K_2')}$, proving (b).

To show (c), let $v = (\mu_1, \mu_2, \lambda; M_1, M_2, L) \in V'$. Set $p = \mu_1 \bigcap \mu_2$, a point and $\Lambda = \langle M_1, M_2\rangle$, a hyperplane. Then $\langle \mu_1, \mu_2\rangle$ is a plane ν contained in L and $M_1 \bigcap M_2$ is a plane N containing λ. We illustrate these inclusions:



To complete the proof, we show Ψ_v is the sum of Schubert varieties

$$\Omega(\mu_1,\bullet,\Lambda) \;+\; \Omega(p,M_2) \;+\; \Omega(\mu_2,\bullet,\Lambda) \;+\; \Omega(p,M_1) \;+\; \Omega(\bullet,\nu) \;+\; \Omega(p,L) \;+\; \Omega(\lambda,\bullet,\Lambda) \;+\; \Omega(\bullet,N).$$

First note that

$$\Omega(\mu_1)\bigcap\Omega(\bullet,M_2) \;=\; \Omega(\mu_1,\bullet,\Lambda)+\Omega(p,M_2):$$

If $H\in\Omega(\mu_1)\bigcap\Omega(\bullet,M_2)$, then either $H\bigcap\mu_1\not\subset M_2$, so that $H\subset\langle\mu_1,M_2\rangle=\Lambda$, or else $p\in H$ so that $H\in\Omega(p,M_2)$. Similarly, we have $\Omega(\mu_2)\bigcap\Omega(\bullet,M_1)=\Omega(\mu_2,\bullet,\Lambda)+\Omega(p,M_1)$. These intersections are generically transverse, as they are proper.

Furthermore,

$$\{H\in\Omega(\mu_1,L)\,|\,H\bigcap\mu_2\neq\emptyset\} \;=\; \Omega(\bullet,\nu) \;+\; \Omega(p,L):$$

Either $H\bigcap\mu_1\bigcap\mu_2=\emptyset$, thus $\dim H\bigcap\langle\mu_1,\mu_2\rangle\geq 1$, and so $H\in\Omega(\bullet,\nu)$, or else $p\in H$, so that $H\in\Omega(p,L)$. Finally,

$$\{H\in\Omega(\lambda,M_1)\,|\,\dim H\bigcap M_2\geq 1\} \;=\; \Omega(\lambda,\bullet,\Lambda) \;+\; \Omega(\bullet,N):$$

Either $H\bigcap M_1\not\subset M_2$ thus $H\subset\langle M_1,M_2\rangle=\Lambda$ and so $H\in\Omega(\lambda,\bullet,\Lambda)$, or else $\dim H\bigcap M_1\bigcap M_2\geq 1$, so that $H\in\Omega(\bullet,N)$. □

Note that for $v\in V'$, the fibre Ψ_v is *not* an intersection of four Schubert varieties of type $\Omega(K)$, for K a plane: The Schubert subvariety $\Omega(\bullet,N)$ is the locus of planes which contain a line $l\subset N$ and hence it consists of all planes of the form $\langle q,l\rangle$, where $l\subset N$ is a line and $q\in\mathbb{P}^5\setminus l$ is a point. Suppose $\Psi_v\subset\Omega(K)$ so that $\Omega(\bullet,N)\subset\Omega(K)$. Then for every line $l\subset N$ and point $q\in\mathbb{P}^5\setminus l$, we have $K\bigcap\langle q,l\rangle\neq\emptyset$. This implies that $K\bigcap l\neq\emptyset$ for every line $l\subset N$, and hence that $\dim K\bigcap N\geq 1$. Similarly, $\dim K\bigcap\nu\geq 1$, and so $K\bigcap N\bigcap\nu\neq\emptyset$, thus $p\in K$. This shows $\Omega(p)\subset\Omega(K)$ and so if $\Psi_v\subset\Omega(K_1)\bigcap\Omega(K_2)\bigcap\Omega(K_3)\bigcap\Omega(K_4)$, then this intersection must contain $\Omega(p)$. Hence Ψ_v is a proper subset of the intersection.

If $a_i=\dim A_i$, then $\sigma_{a_1a_2a_3}$ is the rational equivalence class of $\Omega(A_0,A_1,A_2)$. By the observation of §2.2, Lemma 6 implies the formula in $A^*\mathbb{G}_{2,5}$:

$$(\sigma_{245})^4 \;=\; 3\cdot\sigma_{035} \;+\; 2\cdot\sigma_{125} \;+\; 3\cdot\sigma_{134},$$

which may be determined by other means from the classical Schubert calculus.

4. Real Plane Conics

In 1864 Chasles [**9**] showed there are 3264 plane conics tangent to five general conics. Fulton [**12**] asked how many of the 3264 conics tangent to five general (real) conics can be real. He later determined that all can be real, but did not publish that result. More recently, Ronga, Tognoli, and Vust [**39**] rediscovered this using different methods. We conclude this note with an outline of their work. The author is grateful to Bill Fulton and Felice Ronga for explaining these ideas.

Let X be the variety of complete plane conics, a smooth variety of dimension 5. Let the hypersurfaces H_p, H_l, and H_C, be, respectively those conics containing a point p, those tangent to a line l, and those tangent to a conic, C. If $\hat{p}, \hat{l}$, and $\widehat{C}$ are, respectively, their cycle classes in A^1X, then

$$\widehat{C} \;=\; 2\hat{p} \;+\; 2\hat{l},$$

which may be seen by degenerating a conic into two lines (cf. Figure 2). Then the number of conics tangent to five general conics is the degree of

$$\widehat{C}^5 \;=\; 32(\hat{p}^{\,5} \;+\; 5\hat{p}^{\,4}\cdot\hat{l} \;+\; 10\hat{p}^{\,3}\cdot\hat{l}^{\,2} \;+\; 10\hat{p}^{\,2}\cdot\hat{l}^{\,3} \;+\; 5\hat{p}\cdot\hat{l}^4 \;+\; \hat{l}^{\,5}).$$

The monomials $\hat{p}^{\,j}\cdot\hat{l}^{5-j}$ for $j = 0,\ldots,5$, have degrees $1,2,4,4,2,1$, giving Chasles' number of $32(1+10+40+40+10+1) = 3264$ [**26**, §9].

THEOREM 7 (Ronga-Tognoli-Vust). *There are five real conics in general position such that all of the 3264 conics tangent to the five are real.*

SKETCH OF PROOF. The strategy is to realize the five conics as a deformation of five degenerate conics consisting of a (double) point on a (double) line giving a maximal number of real conics. The first step is to show that for each j, there are j lines and $5-j$ points such that the $2^{\min\{j,5-j\}}$ conics tangent to the lines and containing the points are real. In [**39**], this step is done explicitly with a precise determination of which configurations of points and lines are 'maximal'; that is, have all solutions real. Remarkably, there are five lines $l_1,\ldots,l_5$ and five real points $p_1,\ldots,p_5$ with $p_i \in l_i$ such that each of the 32 terms in

$$\bigcap_{i=1}^{5}(H_{p_i} \;+\; H_{l_i})$$

is a transverse intersection with all points of intersection real. This gives 102 real conics. Such a configuration is illustrated in Figure 1, which also shows the 4 conics incident upon p_1 and p_3, and tangent to l_2, l_4, and l_5 (solid lines).

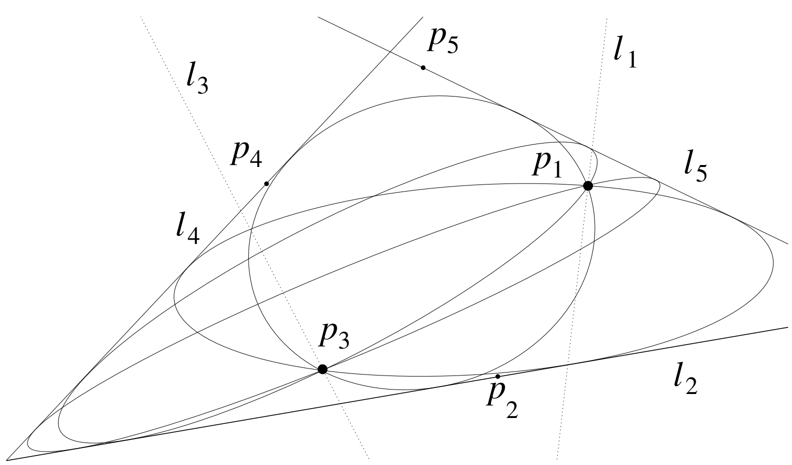


FIGURE 1. A Maximal Configuration

The maximality of such a configuration is stable under small real deformations of its points and lines. Thus we may choose real lines $l'_1,\ldots,l'_5$ where

(1) $p_i \in l'_i$ and l'_i is distinct from l_i, for $i = 1, \ldots, 5$,
(2) Any configuration obtained from a maximal configuration by substituting some primed lines for the corresponding unprimed lines is maximal.
(3) The lines l_i and l'_i partition the real tangent directions at p_i into two intervals. The configurations described in condition (2) give finitely many (273) real conics passing through p_i. We require that all tangent directions to these conics at p_i lie within the interior of *one* of these two intervals.

The relation $\widehat{C} = 2\hat{p} + 2\hat{l}$ may be obtained by considering a conic C near a degenerate conic consisting of two lines l, l' meeting at a point p, and a pencil of conics. For any conic q in that pencil tangent to one of the lines, there is a nearby conic q' in that pencil tangent to C. However, for every conic Q in the pencil containing p, there are *two* nearby conics Q', Q'' in that pencil tangent to C. Moreover, if Q is real, then Q' and Q'' are real if and only if the real tangent line to Q at p does not intersect C. This is illustrated in Figure 2.

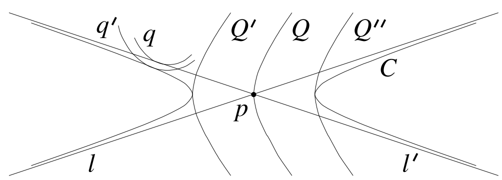


FIGURE 2. Deforming a degenerate conic

By condition (3), we may choose real conics $C_1, \ldots, C_5$ with C_i near the degenerate conic $l_i + l'_i$ and, if Q is a conic in

$$\bigcap_{i=1}^{5} \left(H_{p_i} + H_{l_i} + H_{l'_i}\right) \tag{4}$$

containing p_i, the the real tangent line to Q at p_i does not intersect C_i. If, in addition, the conics C_i are sufficiently close to each degenerate conic, then there will be 3264 real conics tangent to each of $C_1, \ldots, C_5$.

Indeed, suppose H_{C_1} replaces $H_{p_1} + H_{l_1} + H_{l'_1}$ in the intersection (4). Then for any conic q in (4) that is tangent to either l_1 or l'_1, there is a nearby real conic q' tangent to C_1 which satisfies the other conditions on q (since these other conditions determine a pencil of conics). Similarly, if Q is a conic in (4) containing p_1, then there are two nearby real conics Q' and Q'' tangent to C_1 which satisfy the other conditions on Q. If H_{C_2} now replaces $H_{p_2} + H_{l_2} + H_{l'_2}$ in the new intersection $H_{C_1} \cap \bigcap_{i=2}^{5} \left(H_{p_i} + H_{l_i} + H_{l'_i}\right)$, then each conic tangent to l_2 and l'_2 gives a conic tangent to C_2, but each conic through p_2 gives two conics tangent to C_2. Replacing H_{C_3}, H_{C_4}, and H_{C_5} in turn completes the argument. □

Ronga, Tognoli, and Vust gave a careful version of the argument in the previous paragraph. They considered the incidence variety of sextuples of conics, the first tangent to the last five, and studied the singularities of the projection to the last five conics at points lying above the five-tuple of degenerate conics $(l_1 + l'_1, \ldots, l_5 + l'_5)$.

Fulton's argument differs primarily in the last step. He first checks that a maximal configuration is obtained by a small generic deformation of the configuration of lines defining a regular pentagon and points at the midpoints of the sides. This defines 102 real conics. For each of these lines, he chooses a hyperbola where each branch lies very close to the line on one side of the point and crosses the line near the point. Then, for each conic among the 102, he argues there are 32 nearby conics tangent to all five hyperbolas.

The proof we gave used the effective rational equivalence illustrated in Figure 2:

$$H_C \;\sim\; 2H_p \;+\; H_l \;+\; H_{l'},$$

where l, l' form a degenerate conic with $p = l \bigcap l'$. This deformation to a cycle having multiplicities (the coefficient 2 of H_p) is unavoidable: The variety X, and thus $\mathit{Chow}\, X$ has an action of $G = PGL(3,\mathbb{C})$. The locus of hypersurfaces H_C on $\mathit{Chow}\, X$ is a single 5-dimensional G-orbit. This family cannot have effective rational equivalence. If Z is a cycle in the closure of this locus, then Z is in a G-orbit of dimension at most 4. Thus if $Z = H_p + H_{p'} + H_l + H_{l'}$, then the dimension of the G-orbit of (p, p', l, l') in the product of $\mathbb{P}^2$'s and their duals is at most 4. But this is impossible unless either $p = p'$ or $l = l'$. Figure 2 illustrates the case $p = p'$.

5. Computational aspects

Multiplicity-free deformations have applications beyond showing the existence of real solutions. When the deformations are explicitly described, it is possible to obtain explicit solutions to the enumerative problem using continuation methods of numerical analysis [**1**] to follow real points in the degenerate configuration backwards along the deformation. Algorithms to accomplish this have been developed for intersecting hypersurfaces in a *complex* torus [**10, 20**]. Huber has implemented one in the software package PELICAN (`http://math.cornell.edu/~birk`).

Recently, the Pieri-type deformations of [**45, 47**] have been used to construct continuation algorithms for finding explicit solutions to Pieri-type enumerative problems on Grassmann varieties [**19**]. The polynomial systems which arise are over-determined, a novel feature of these continuation algorithms. Techniques based upon flat deformations of the Grassmann variety, either into planes (induced by a Gröbner basis of the Plücker ideal with square-free initial ideal [**50**]) or into a toric variety (induced by a SAGBI (Subalgebra Analog of Gröbner Basis for Ideals) basis [**22, 38**] for the bracket algebra [**48**]) are also employed for problems involving Schubert hypersurfaces [**19**]. These techniques have an advantage over a traditional continuation (cf. [**8, 7**]) in that no divergent paths are followed. These last two methods may apply when a variety has a square-free initial ideal, or when its coordinate ring has a SAGBI basis inducing a flat deformation into a toric variety. This is the case, for example, for flag varieties and many Schubert varieties [**14**].

We conclude with a discussion of an intriguing conjecture of Shapiro and Shapiro. A point on the rational normal curve defines an *osculating flag*; the k-dimensional subspace of that flag is the span of the point and the first $k-1$ derivatives of the curve at that point.

CONJECTURE 8 (B. Shapiro and M. Shapiro). *An enumerative problem involving Schubert conditions on a Grassmann or flag variety has all real solutions if the conditions are given by osculating flags at (distinct) real points.*

Such enumerative problems arise in the control of linear systems by output feedback [**7**]. Based upon a degenerate case, J. Rosenthal suggested that there exist real osculating flags giving all real solutions. This inspired us to use computer algebra to search for some evidence of this conjecture. The results of this search are compelling. For each enumerative problem involving Schubert hypersurfaces on a flag variety of dimension at most 10, we have verified the conjecture in at least 2 and as many as 40 instances. We describe one such calculation in detail.

Let K be a 7×5 matrix whose first two rows are indeterminants and last 5 constitute an identity matrix. This matrix K gives local coordinates on the Grassmannian of 5-dimensional subspaces of $\mathbb{C}^7$: the column span of K is a 5-dimensional subspace of $\mathbb{C}^7$. Let $F(s)$ be the transpose of the matrix:

$$\begin{bmatrix} 1000000 & 100000s & 10000s^2 & 1000s^3 & 100s^4 & 10s^5 & s^6 \\ 0 & 100000 & 20000s & 3000s^2 & 400s^3 & 50s^4 & 6s^5 \end{bmatrix}$$

The column span of $F(s)$ is a 2-dimensional subspace which osculates the rational normal curve defined by the first column of $F(s)$. Set $f(s) := det[F(s) : K]$, the equation for K to meet $F(s)$, which defines a Schubert hypersurface. Thus

$$\langle f(1), f(2), \ldots, f(10) \rangle$$

is the ideal of those 5-dimensional subspaces which meet each of $F(1), \ldots, F(10)$. We used the computer algebra system SINGULAR [**15**] to compute an elimination ideal of this system, obtaining a degree 42 polynomial g as the eliminant. Then the `realroot` routine of MAPLE showed that g had 42 real roots.

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