ENUMERATIVE GEOMETRY FOR THE REAL GRASSMANNIAN OF LINES IN PROJECTIVE SPACE

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Abstract. Given Schubert conditions on lines in projective space which generically determine a finite number of lines, we show there exist general real conditions determining the expected number of real lines. This extends the classical Schubert calculus of enumerative geometry for the Grassmann variety of lines in projective space from the complex realm to the real. Our main tool is an explicit geometric description of rational equivalences, which also constitutes a novel determination of the Chow rings of these Grassmann varieties of lines. The combinatorics of these rational equivalences suggests a non-commutative, associative product on the free Abelian group on Young tableaux. We conclude by considering some of these enumerative problems over finite fields.

1. Introduction

Describing the common zeroes of a set of polynomials is more problematic over non-algebraically closed fields. For systems of polynomials with few monomials on a complex torus ("fewnomials"), Khovanskii [8] showed that the number of real zeroes are at most a small fraction of the the number of complex zeroes. Fulton ([5], §7.2) asked how many solutions to a problem of enumerative geometry can be real; for example, how many of the 3264 conics tangent to five general real conics can be real. He later showed that all, in fact, can be real. This was rediscovered by Ronga, Tognoli, and Vust [13]. Robert Speiser suggested the classical Schubert calculus of enumerative geometry would be a good testing ground for this question. For problems of enumerating lines in $\mathbf{P}^n$ incident to real linear subspaces in general position, we show that all solutions can be real.

Let $\mathbf{G}_1\mathbf{P}^n$ be the Grassmannian of lines in $\mathbf{P}^n$. A flag and a partition $\lambda = (\alpha, \beta)$ determine a Schubert subvariety of type λ, which

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has codimension $|\lambda| = \alpha + \beta$. The automorphism group PGL_{n+1} of $\mathbf{P}^n$ acts transitively on $\mathbf{G}_1\mathbf{P}^n$ and on the set of Schubert varieties of a fixed type. Over fields k of characteristic zero, Kleiman's Transversality Theorem [9] shows that a general collection of Schubert subvarieties of $\mathbf{G}_1\mathbf{P}^n$ will intersect generically transversally. In §5, we extend this to fields of positive characteristic. A consequence is that for Schubert-type enumerative problems in $\mathbf{G}_1\mathbf{P}^n$, the basic principle of the Schubert calculus remains valid in positive characteristic: each component of a general intersection of Schubert varieties appears with multiplicity one. For partitions $\lambda^1, \dots, \lambda^m$, let $\mathcal{G}(\lambda^1, \dots, \lambda^m)$ be the (non-empty) set of points of the Chow variety of $\mathbf{G}_1\mathbf{P}^n$ representing cycles arising as generically transverse intersections of Schubert varieties of types $\lambda^1, \dots, \lambda^m$. Any generically transverse intersection of Schubert varieties is rationally equivalent to a sum of Schubert varieties; the Schubert calculus gives algorithms for determining how many of each type. In §4, we prove

Theorem A. *Let $\lambda^1, \dots, \lambda^m$ be partitions. Then there is a cycle Φ (depending upon $\lambda^1, \dots, \lambda^m$) whose components are explicitly described Schubert varieties, such that Φ is in the Zariski closure of $\mathcal{G}(\lambda^1, \dots, \lambda^m)$. Moreover, for each cycle X in $\mathcal{G}(\lambda^1, \dots, \lambda^m)$, there is an explicit chain of rational curves between X and Φ with each curve lying in the Zariski closure of $\mathcal{G}(\lambda^1, \dots, \lambda^m)$.*

The proof of Theorem A constitutes an explicitly geometric determination of the Schubert calculus of enumerative geometry for lines in $\mathbf{P}^n$. In fact, Theorem A shows these 'Schubert-type' enumerative problems may be solved *without* reference to the Chow ring, a traditional tool in enumerative geometry. As well, it determines products in the Chow ring: Let σ_λ be the rational equivalence class of a Schubert variety of type λ. Equating the rational equivalence class of cycles in $\mathcal{G}(\lambda^1, \dots, \lambda^m)$ to that of Φ yields a formula for products in $A^*\mathbf{G}_1\mathbf{P}^n$, the Chow ring of $\mathbf{G}_1\mathbf{P}^n$.

Corollary B. *Let c^λ be the number of components of Φ of type λ. Then*

$$\prod_{i=1}^{m} \sigma_{\lambda^i} = \sum_{\lambda} c^\lambda \sigma_\lambda.$$

Thus the structure of these Chow rings is determined in a strong sense: All products among classes from the Schubert basis are expressed as linear combinations of basis elements and these expressions are obtained by exhibiting rational equivalences between a generically transverse intersection of Schubert varieties and the cycle Φ. Moreover, these expressions require only a 'set-theoretic' understanding, as the cycle Φ is free of multiplicities.

Let $\mathbf{Fl}$ be the manifold of real flags which parameterizes real Schubert varieties of fixed type. When $k = \mathbf{R}$, Theorem A has the following consequence.

Theorem C. *Let $\lambda^1, \ldots, \lambda^m$ be partitions with $|\lambda^1| + \cdots + |\lambda^m| = 2n-2$, the dimension of $\mathbf{G}_1\mathbf{P}^n$. Then there exists a non-empty classically open subset of $(\mathbf{Fl})^m$ consisting of m-tuples of flags whose corresponding Schubert varieties (of respective types $\lambda^1, \ldots, \lambda^m$) meet transversally, with all points of intersection real.*

To the best of the author's knowledge, this is the first result showing that a large class of non-trivial enumerative problems can have all of their solutions real. Theorem C may have applications outside of geometry. For example, some problems involving real matrices, such as the pole assignment problem in systems control theory [1], may be expressed as intersection problems on a real Grassmannian.

The construction of the cycle Φ and rational curves of Theorem A uses the combinatorics of Young tableaux and suggests a non-commutative, associative algebra with additive basis the set of Young tableaux, described in §7. This algebra has surjections to the Chow rings of Grassmann varieties and the algebra of symmetric functions. However, it differs fundamentally from the plactic algebra of Lascoux and Schützenberger [11], which is also non-commutative, associative, constructed from Young tableaux, and related to symmetric functions. In §8, we ask which enumerative problems may be solved over which (finite) fields and give the answer for two classes of Schubert-type enumerative problems. We also show how some of our constructions may be carried out over finite fields.

The rational equivalences of Theorem A arise from a sequence of deformations which transform generically transverse intersections of Schubert varieties into Φ, a sum of distinct Schubert varieties. We regard this as the classical method of degeneration, with a shift in focus. Traditionally, a product of cycles is computed by moving the cycles into special position and then studying the resulting intersection cycle. This may fail when applied to more than a few cycles; an intersection typically becomes improper before the positions are special enough for the intersection to be easily recognized. Rather than considering deformations of cycles to be intersected, we use an idea of Chaivacci and Escamilla-Castillo [2] and study deformations of intersection cycles. In particular, the cycle Φ of Theorem A is usually *not* an intersection of Schubert varieties. Theorem C is deduced from Theorem A and 'real' conservation of number; the number of real points in a real zero-cycle is constant under small real deformations.

2. Preliminaries

Let k be an infinite field. Varieties will be quasi-projective, reduced (not necessarily irreducible), and defined over k. All subvarieties (except some curves) are assumed to be closed. When $k = \mathbf{R}$, let $X(\mathbf{R})$ be the $\mathbf{R}$-valued points of X, equipped with the classical topology. Note that $X(\mathbf{R})$ need not be topologically connected, even when X is irreducible and projective.

Let X be a smooth variety, U and W subvarieties of X, and set $Z = U \cap W$. We say U and W meet *properly* if either Z is empty or the codimension of Z in X is the sum of the codimensions of U and W. We say U and W meet *generically transversally* if each irreducible component of Z has an open subset along which U and W are nonsingular and meet transversally. In this case, Z is *generically reduced* (reduced at the generic point of each component), the fundamental cycle $[Z]$ of Z is multiplicity free, and in the Chow ring A^*X of X

$$[U] \cdot [W] \;=\; [U \cap W] \;=\; [Z] \;=\; \sum_{i=1}^{r} [Z_i],$$

where $Z_1, \ldots, Z_r$ are the irreducible components of Z.

2.1. Chow varieties. Suppose X is projective. Then positive cycles of a fixed dimension and degree on X are parameterized by $\mathit{Chow}\, X$, a Chow variety of X. We write $\mathit{Chow}\, X$ for any Chow variety of X; context will indicate the dimension and degree intended. Let U be a normal variety and Ξ a subvariety of $X \times U$ with generically reduced equidimensional fibres over U. The association of a point u of U to the fundamental cycle of the fibre Ξ_u determines a function ψ from U to $\mathit{Chow}\, X$. Since U is normal, this function is algebraic ([10], [4]). Moreover, if X, U, and Ξ are defined over k, then so are $\mathit{Chow}\, X$ and the map $\psi : U \to \mathit{Chow}\, X$ ([15], §I.9). Cycles represented by points on a rational curve in $\mathit{Chow}\, X$ are rationally equivalent; the converse to this statement is not true. If Y and Z are rationally equivalent, then there is a third cycle W such that $Y + W$ and $Z + W$ are connected by a chain of rational curves, as points in some Chow variety of X.

2.2. Grassmannians and Schubert varieties. For $S \subset \mathbf{P}^n$, let $\langle S \rangle$ denote the linear span of S. For a vector space V, let $\mathbf{P}V$ be the projective space of all one dimensional subspaces of V. Suppose $K = \mathbf{P}U$ and $M = \mathbf{P}W$. Define $\mathrm{Hom}(K, M)$ to be $\mathrm{Hom}(U, W)$, the vector space of linear maps from U to W. If $K \subset M$, set $M/K = \mathbf{P}(W/U)$.

A complete flag $F_\bullet$ is a collection of linear subspaces $F_n \subset \cdots \subset F_1 \subset F_0 = \mathbf{P}^n$, where $\dim F_i = n - i$. We adopt the convention that $F_p = \emptyset$ for $p > n$.

Let $\mathbf{G}_1\mathbf{P}^n$ be the Grassmannian of lines in $\mathbf{P}^n$, a $(2n-2)$-dimensional variety. For a partition $\lambda = (\alpha, \beta)$, let $\mathbf{Fl}_\lambda$ denote the variety of partial flags of *type* λ; those $K \subset M$ with K an $(n-\alpha-1)$-plane and M an $(n-\beta)$-plane. If $\mathbf{Fl}_\lambda$ is non-empty, then necessarily $0 \leq \beta < \alpha + 1 \leq n$, so $\alpha + \beta \leq 2n-2$. A partial flag $K \subset M$ determines a *Schubert variety* $\Omega(K, M)$ comprising lines contained in M which also meet K. The *type* of $\Omega(K, M)$ is the type, $\lambda = (\alpha, \beta)$, of its defining partial flag $K \subset M$, and its codimension is $|\lambda| = \alpha + \beta$. If $\alpha = \beta$, then $\Omega(K, M) = \mathbf{G}_1 M$, the Grassmannian of lines in M. If $M = \mathbf{P}^n$, so $\beta = 0$, then we write Ω_K for this Schubert variety. The tangent space to $\ell \in \mathbf{G}_1\mathbf{P}^n$ is naturally identified with the linear space $\mathrm{Hom}(\ell, \mathbf{P}^n/\ell)$. In fact, ℓ has an affine neighbourhood isomorphic to $\mathrm{Hom}(\ell, \mathbf{P}^n/\ell)$. It is not hard to verify the following lemma, whose proof we omit.

2.3. **Lemma.** *Let K, M be subspaces of $\mathbf{P}^n$*

(1) *If $K \subset M$, then the smooth locus of $\Omega(K, M)$ consists of those lines ℓ with $\ell \not\subset K$. For such ℓ,*

$$T_\ell\Omega(K, M) \;=\; \{\phi \in Hom(\ell, \mathbf{P}^n/\ell) \mid \phi(\ell) \subset M/\ell \text{ and } \phi(\ell \cap K) \in \langle K, \ell\rangle/\ell\}.$$

(2) *We have $\Omega_K \bigcap \mathbf{G}_1 M \;=\; \Omega(K \cap M,\, M)$. This intersection is transverse at the smooth points of $\Omega(K \cap M,\, M)$ if and only if K and M meet properly in $\mathbf{P}^n$.*

(3) *Let $K_i \subset M_i$, for $i = 1, 2$. If the intersection $\Omega(K_1, M_1) \bigcap \Omega(K_2, M_2)$ is proper, then M_i meets K_j properly for $i \neq j$ and $M_1 \cap M_2$ is proper.*

An intersection of two Schubert varieties may be generically transverse and reducible. In fact, this observation is at the heart of our methods.

2.4. **Lemma.** *Let $H \subset \mathbf{P}^n$ be a hyperplane, $P \not\subset H$ a linear subspace, and $F \subset P \cap H$ a proper linear subspace. Let $N \not\subset H$ be a linear subspace meeting F—and hence P—properly, and set $L = N \cap H$. Then $\Omega(F, P)$ and Ω_L meet generically transversally in $\mathbf{G}_1\mathbf{P}^n$,*

$$\Omega(F, P) \bigcap \Omega_L \;=\; \Omega(N \cap F, P) + \Omega(F, P \cap H) \bigcap \Omega_N,$$

and the second component is itself a generically transverse intersection.

Proof: The right hand side is a subset of the left; we show the other inclusion. Let $\ell \in \Omega(F, P) \bigcap \Omega_L$. If ℓ meets $L \cap F = N \cap F$, then

$\ell \in \Omega(N \cap F, P)$. Otherwise, ℓ is spanned by its intersections with F and L, hence $\ell \subset P \cap \langle F, L\rangle \subset P \cap H$ and so $\ell \in \Omega(F, P \cap H) \bigcap \Omega_N$.

We use (1) of Lemma 2.3 to verify these intersections are generically transverse. Let $\ell \in \Omega(F, P \cap H) \bigcap \Omega_N$ and suppose that $p = \ell \cap F$ is distinct from $q = \ell \cap N$, so that each Schubert variety is smooth at $\ell = \langle p, q\rangle$. Then $\langle F, q\rangle = \langle F, \ell\rangle \subset P \cap H$ and $\langle N, p\rangle = \langle N, \ell\rangle$, so that

$$\begin{aligned} T_\ell\Omega(F, P\cap H) &= \{\phi \in \mathrm{Hom}(\ell, \mathbf{P}^n/\ell) \mid \phi(q) \in P \cap H/\ell \text{ and } \phi(p) \in \langle F, q\rangle/\ell\} \\ T_\ell\Omega_N &= \{\phi \in \mathrm{Hom}(\ell, \mathbf{P}^n/\ell) \mid \phi(q) \in \langle N, p\rangle/\ell\}. \end{aligned}$$

The assumptions on F, P, N, and H ensure these tangent spaces meet properly, proving $\Omega(F, P\cap H)$ and Ω_N meet transversally at ℓ. Similar arguments may be used to verify the other transversality assertions. ∎

2.5. **Young tableaux I.** The *Young diagram* of a partition $\lambda = (\alpha, \beta)$ is a two-rowed array of boxes with α boxes in the first row and β in the second. Note that $\alpha \geq \left\lfloor \frac{|\lambda|+1}{2} \right\rfloor \geq \left\lfloor \frac{|\lambda|}{2} \right\rfloor \geq \beta$. Here, $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r. We make no distinction between a partition and its Young diagram.

A *Young tableau* T of *shape* λ is a filling of the boxes of λ with the integers $1, 2, \ldots, |\lambda|$. These integers increase left to right across each row and down each column. Thus the ith entry in the second row of T must be at least $2i$. Call $|\lambda|$ the *degree* of T, denoted $|T|$. If $\alpha = \beta$, then T is *rectangular.* Here are three Young tableaux; the first is rectangular.

1	2	5
3	4	6

1	2	5	7
3	4	6	

1	3	4	6	7	8
2	5	9			

2.6. **Arrangements.** A key to our proof of Theorem A is the form of the cycle Φ as well as the cycles which are intermediate in the rational equivalences. These cycles depend upon a particular lattice of subspaces of $\mathbf{P}^n$ defined by an arrangement of hyperplanes $H_2, \ldots, H_{2n-2}$ with specified linear dependencies. Given hyperplanes $H_2, \ldots, H_{2n-2}$ and a subset $A \subset \{2, 3, \ldots, 2n-2\}$, define H_A by $H_\emptyset = \mathbf{P}^n$ and, if $A \neq \emptyset$, $H_A = \bigcap_{j\in A} H_j$.

2.7. **Definition.** An *arrangement $\mathcal{F}$ in $\mathbf{P}^n$* is a collection of $2n-3$ distinct hyperplanes $H_2, \ldots, H_{2n-2}$ in $\mathbf{P}^n$ such that for $m = 2, \ldots, n$, the following two conditions hold:

(A.1) $F_m := H_{\{2,3,\ldots,2m-2\}}$ has codimension m, and if $m \neq n$, then $F_m \subset H_{2m-1}$ but $F_m \not\subset H_{2m}$. Thus $F_{m+1} = F_m \cap H_{2m}$, and if $A \subset \{2, \ldots, 2m-1\}$, then $H_A \not\subset H_{2m}$.

(A.2) If $A \subset \{2, \ldots, 2m-2\}$, and $F_m \neq H_A$, then $H_A \not\subset H_{2m-1}$.

A consequence of (A.1) is that the subspaces $F_2, F_3, \ldots, F_n$ form part of a complete flag in $\mathbf{P}^n$, and that if $l \geq 2$, then $F_l \subset H_j$ if and only if $l \geq \lfloor \frac{j}{2} \rfloor + 1$. A main point of these conditions is that if $A \subset \{2, \ldots, l\}$, then H_A meets H_{l+1} properly unless l even and $H_A = F_{\frac{l}{2}+1}$, or dropping the condition that l is even, unless $H_A = F_{\lfloor \frac{l+1}{2} \rfloor + 1}$ (which implies l is even, by (A.1)). Arrangements can be constructed over an arbitrary infinite field. The potential obstruction for finite fields is nonexistence of hyperplanes H_{2m-1} satisfying (A.2). In §8, we describe an inductive construction of arrangements and estimate over which finite fields it is possible to have an arrangement.

2.8. **An arrangement in $\mathbf{P}^5$.** We illustrate this definition with an arrangement in $\mathbf{P}^5$. Let $x_0, x_1, \ldots, x_5$ be homogeneous coordinates for $\mathbf{P}^5$. Define hyperplanes $H_2, \ldots, H_8$ by the linear forms:

$$\begin{array}{lcl} H_2 & : & x_1 \\ H_4 & : & x_2 \\ H_6 & : & x_3 \\ H_8 & : & x_4 \end{array} \qquad \begin{array}{lcl} H_3 & : & x_0 \\ H_5 & : & x_0 + x_1 + x_2 \\ H_7 & : & x_0 + 2x_1 + 3x_2 + x_3 \end{array} \tag{1}$$

The flag $F_\bullet$ associated to this arrangement is the standard flag, where F_i is defined by the equations $0 = x_0 = \cdots = x_{i-1}$. The hyperplanes $H_2, \ldots, H_6$ restricted to the $\mathbf{P}^4$ defined by $x_5 = 0$ give an arrangement in $\mathbf{P}^4$. Figure 1 shows (part of) the Hasse diagram of the lattice of subspaces in $\mathbf{P}^4$ generated by this arrangement (also part of the Hasse diagram of the original arrangement in $\mathbf{P}^5$).

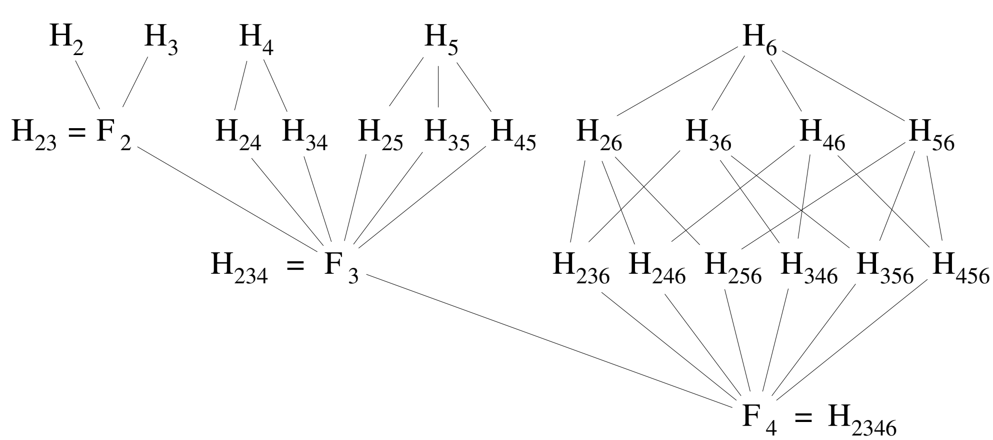


FIGURE 1. The Arrangement in $\mathbf{P}^4$

2.9. **Definition-Lemma.** *Let $\mathcal{F}$ be an arrangement in $\mathbf{P}^n$. For each tableau T with $|T| \leq 2n-2$, define H_T to be H_A, where A is the set of entries in the second row of T. If T has shape (α, β) with $\beta \geq 1$, then the following hold.*

(1) $\dim H_T = n - \beta$.
(2) $H_T \neq F_\beta$.
(3) *If S is another tableau with shape (α, β), then $H_S = H_T$ only if $S = T$.*
(4) *If T is rectangular with degree 2β, then $H_T \cap H_{2\beta+1} = F_{\beta+1}$.*

Proof: Let the second row of T be the set $A \cup \{j\}$, where j is the largest entry in the second row of T. We prove the first part by induction on β. Let S be the tableau obtained from T by removing the entries $j, j+1, \ldots, \alpha+\beta$. Then $H_S = H_A$ has dimension $n - \beta + 1$ and so H_T will have dimension $n - \beta$ if $H_S \not\subset H_j$, since $H_T = H_S \cap H_j$. As $A \subset \{2, \ldots, j-1\}$, we see by (A.1) and (A.2) that $H_A \subset H_j$ only if $j = 2m - 1$ and $F_m = H_A$. In this case, $m = \beta - 1$, thus $j = 2\beta - 3$. But this contradicts $j \geq 2\beta$, as j is the βth entry in the second row of T.

To prove the second statement, note that $F_\beta \not\subset H_j$, since $F_{\lfloor \frac{j}{2} \rfloor} \not\subset H_j$ and $F_{\lfloor \frac{j}{2} \rfloor} \subset F_\beta$, as $j \geq 2\beta$. Since $H_T \subset H_j$, it follows that $F_\beta \neq H_T$.

For (3), suppose that β is minimal with respect to the existence of tableaux $S \neq T$ of shape (α, β) where $H_S = H_T$. Let j be the largest entry in the second row $A \cup \{j\}$ of T and let m be the largest entry in the second row $B \cup \{m\}$ of S. If $m \neq j$, suppose $m < j$. Then $H_S \subset H_j$, as $H_T \subset H_j$. Since $B \cup \{m\} \subset \{2, \ldots, j-1\}$, this implies $H_S = F_\beta$, contradicting (2). Suppose now that $j = m$. By the minimality of β, we have $H_A \neq H_B$ and so $H_{A \cup B} = H_A \cap H_B = H_T \subset H_j$. Then (A.1) implies j is odd and (A.2) implies $H_T = F_\beta$, as $A \cup B \subset \{2, \ldots, j-1\}$, again contradicting (2).

For the last statement, suppose T is rectangular with $|T| = 2\beta$. Since the second row of T is a subset of $\{2, \ldots, 2\beta\}$ and $F_{\beta+1} = H_{\{2,\ldots,2\beta\}}$, we have $F_{\beta+1} \subset H_T$, and so $F_{\beta+1}$ is a hyperplane in H_T. Then (4) is immediate, as $F_{\beta+1} \subset H_{2\beta+1}$ but $H_T \not\subset H_{2\beta+1}$, by (A.2). ∎

2.10. **Definition-Corollary.** *Let $\mathcal{F}$ be an arrangement and T a tableau of shape (α, β) with $\alpha + \beta \leq 2n-2$. Define $\Psi(T) := \Omega(F_{\alpha+1}, H_T)$. Then*

(1) *If $\alpha < n$, then $F_{\alpha+1} \subset H_T$ is a partial flag of type (α, β).*
(2) *If $\alpha \geq n$ then $\Psi(T) = \emptyset$; otherwise, $\Psi(T)$ is a Schubert variety of type (α, β).*
(3) *If S is a tableau and $\Psi(S) = \Psi(T) \neq \emptyset$, then $S = T$.*

Proof: For the first statement, note that the second row of T is a subset of $\{2, \ldots, \alpha+\beta\}$, so that $F_{\lfloor\frac{\alpha+\beta}{2}\rfloor+1} \subset H_T$, by (A.1). This shows $F_{\alpha+1} \subset H_T$, as $\alpha \geq \lfloor\frac{\alpha+\beta}{2}\rfloor$. Since $\dim H_T = n - \beta$ by Lemma 2.9(1), $F_{\alpha+1} \subset H_T$ is a partial flag of type (α, β).

The second statement is a consequence of the first. For the final statement, first note that if $\emptyset \neq K \subset M$, then M is the union of all lines in $\Omega(K, M)$. Thus $\Psi(S) = \Psi(T) \neq \emptyset$ implies that $H_T = H_S$. But then $S = T$, by Lemma 2.9(3). ∎

3. The Cycles $\Phi(\mathcal{T})$ and $\Phi(\mathcal{T}_{s,\alpha}; L)$

We describe the cycle $\Phi = \Phi(\mathcal{T})$ of Theorem A as well as the cycles $\Phi(\mathcal{T}_{s,\alpha}; L)$ which arise intermediately in the rational equivalences between X and Φ in Theorem A. These are defined with respect to a fixed arrangement $\mathcal{F}$, though our notation suppresses the dependence on $\mathcal{F}$. These cycles are also defined with respect to a fixed set $\mathcal{T}$ of Young tableaux, all of which have the same degree l. In other words, $|T| = l$ for every $T \in \mathcal{T}$.

Define $\Phi(\mathcal{T})$, a multiplicity-free cycle on $\mathbf{G}_1\mathbf{P}^n$ with pure dimension, by

$$\Phi(\mathcal{T}) \ := \ \sum_{T\in\mathcal{T}} \Psi(T).$$

By Corollary 2.10, each Schubert variety summand of $\Phi(\mathcal{T})$ has dimension $2n-2-l$ and they are all distinct, showing $\Phi(\mathcal{T})$ is multiplicity-free and of pure dimension.

3.1. **Young tableaux II.** Suppose T is a tableau with $|T| = l$ and s a positive integer. Let $T(s)$ be the tableau obtained from T by adjoining the consecutive integers $l+1, \ldots, l+s$ to the first row of T. Let T^{+s} be the tableau obtained by adjoining $l+1, \ldots, l+s$ to the second row of T, if that is possible. For example, the following picture shows T, $T(3)$ and T^{+3}. Note that T^{+4} is not defined.

1	2	3	5
4			

1	2	3	5	6	7	8
4						

1	2	3	5
4	6	7	8

With this notation, $H_{T(s)} = H_T$ and $H_{T^{+s}} = H_T \cap H_{l+1} \cap \cdots \cap H_{l+s}$. These relations are the motivation for arrangements and our indexing of subspaces by tableaux.

If $a, b \geq 0$, let $T^{+a}(b)$ be $(T^{+a})(b)$. In $T^{+a}(b)$, the consecutive integers $l+1, \ldots l+a+b$ occur in distinct columns, in order left to right. If $\mathcal{T}$

is a set of Young tableaux for which $|T| = l$ for every $T \in \mathcal{T}$, then set

$$\mathcal{T}(s) := \bigcup_{T\in\mathcal{T}} T(s).$$

Similarly define $\mathcal{T}^{+a}$ and $\mathcal{T}^{+a}(b)$. For integers $0 \leq s \leq \alpha$, recursively define $\mathcal{T}_{s,\alpha}$ by

$$\begin{aligned}\mathcal{T}_{0,\alpha} &:= \mathcal{T}(\alpha),\\ \mathcal{T}_{s,\alpha} &:= \mathcal{T}_{s-1,\alpha}\bigcup\mathcal{T}^{+s}(\alpha-s)\\ &= \mathcal{T}(\alpha)\bigcup\mathcal{T}^{+1}(\alpha-1)\bigcup\cdots\bigcup\mathcal{T}^{+s}(\alpha-s).\end{aligned}$$

3.2. **Intermediate cycles $\Phi(\mathcal{T}_{s,\alpha}; L)$.** The intermediate cycle $\Phi(\mathcal{T}_{s,\alpha}; L)$ is defined with respect to an integer s with $0 \leq s \leq \alpha$, a set $\mathcal{T}$ of tableaux with $|T| = l$ for every $T \in \mathcal{T}$, and a subspace L of codimension $\alpha - s + 1$ in $\mathbf{P}^n$ which has a particular position with respect to the fixed arrangement $\mathcal{F}$. We say that a subspace L of codimension $\alpha - s + 1$ in $\mathbf{P}^n$ meets $\mathcal{F}$ *(l,s)-properly* if the following conditions are satisfied

(1) L meets the subspaces $F_{\lfloor\frac{l+1}{2}\rfloor+1}, \ldots, F_{l+1}$ properly;
(2) $L \cap F_{l+1} = F_{l+\alpha-s+2}$; and
(3) For each tableau T with $|T| = l$, L meets the subspaces $H_{T^{+(s-1)}}$ and $H_{T^{+s}}$ properly.

If $F_{l+\alpha-s+2} \neq \emptyset$, then the last condition is redundant: For any b, either $H_{T^{+b}}$ is undefined, or else $F_{l+1} \subset H_{T^{+b}}$, implying that L meets $H_{T^{+b}}$ properly, as L meets F_{l+1} properly with non-empty intersection. Note also that L is a linear subspace of $\mathbf{P}^n/F_{l+\alpha-s+2}$ with codimension $\alpha - s + 1$.

Suppose l, s, α, and $\mathcal{T}$ are as in the previous paragraph, and that L is a subspace of $\mathbf{P}^n$ which meets $\mathcal{F}$ (l,s)-properly. Let $T \in \mathcal{T}_{s-1,\alpha}$. Then the first row of T has length $b \geq \lfloor\frac{l+s}{2}\rfloor + 1 + \alpha - s$. Define the Schubert variety

$$\Psi(T;L) = \begin{cases}\Psi(T)\ (=\Omega(F_{b+1}, H_T)) & \text{if } b \geq l+\alpha-s+2\\ \Omega(L\cap F_{b-\alpha+s}, H_T) & \text{otherwise}\end{cases}$$

and set

$$\Phi(\mathcal{T}_{s-1,\alpha}; L) := \sum_{T\in\mathcal{T}_{s-1,\alpha}} \Psi(T;L).$$

The cycle $\Phi(\mathcal{T}_{s-1,\alpha}; L)$ is well-defined and multiplicity-free:

3.3. **Lemma.** *For $T \in \mathcal{T}_{s-1,\alpha}$, $\Psi(T;L)$ is a Schubert variety of type equal to the shape of T. Furthermore, $\Phi(\mathcal{T}_{s-1,\alpha};L)$ is a multiplicity-free cycle with each component having dimension $2n-2-l-\alpha$.*

Proof: Let $T \in \mathcal{T}_{s-1,\alpha}$ have shape (b,a). Then $|T| = a+b = l+\alpha$ and no entry in the second row of T exceeds $l+s-1$. If the length of the first row of T is at least $l+\alpha-s+2$, then $\Psi(T;L) = \Psi(T)$ is a Schubert variety of type (b,a). Otherwise $\lfloor\frac{l+s}{2}\rfloor+1 \leq b-\alpha+s \leq l+1$, so L meets $F_{b-\alpha+s}$ properly in a space of codimension $b+1$. Since no entry in the second row of T exceeds $l+s-1$, H_T contains $F_{\lfloor\frac{l+s}{2}\rfloor+1}$, hence also $F_{b-\alpha+s}\cap L$, so $\Psi(T;L)$ is a Schubert variety of type (b,a).

To see that $\Phi(\mathcal{T}_{s-1,\alpha};L)$ is multiplicity-free, suppose $\Psi(T;L) = \Psi(S;L)$ for some $S,T \in \mathcal{T}_{s-1,\alpha}$. Then S and T have the same shape and $H_S = H_T$, so $S = T$ by Lemma 2.9(3). $\blacksquare$

3.4. **Young tableaux III.** Let T be a tableau of shape $(a+s,a)$, and let α be a positive integer. Define a set $T*\alpha$ of tableaux by

$$T*\alpha \ := \ \{T(\alpha), T^{+1}(\alpha-1), \ldots, T^{+\alpha}\}.$$

(It is impossible to form the tableau $T^{+(s+1)}$, so if $\alpha > s$, then the last tableau listed will be $T^{+s}(\alpha-s)$.) If $|T| = l$, then $T*\alpha$ consists of all tableaux obtained from T by adjoining the consecutive integers $l+1,\ldots,l+\alpha$ to T in distinct columns and in order from left to right. For example,

$$\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4\\\cline{1-2}\end{array} * 3 = \left\{ \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&5&6&7&8&9\\\hline 3&4\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|c|c|c|}\hline 1&2&5&6&8&9\\\hline 3&4&7\\\cline{1-3}\end{array}\ ,\ \begin{array}{|c|c|c|c|c|}\hline 1&2&5&6&9\\\hline 3&4&7&8\\\cline{1-4}\end{array} \right\}$$

Given a set $\mathcal{T}$ of tableaux, define $\mathcal{T}*\alpha := \bigcup_{T\in\mathcal{T}} T*\alpha$. Then, by the definition of $*$,

$$\mathcal{T}*\alpha \ = \ \mathcal{T}(\alpha)\bigcup\mathcal{T}^{+1}(\alpha-1)\bigcup\cdots\bigcup\mathcal{T}^{+\alpha} \ = \ \mathcal{T}_{\alpha,\alpha}.$$

Let $\alpha_1,\ldots,\alpha_m$ be positive integers and $\emptyset$ the empty tableau, the only tableau of shape $(0,0)$. Define the set of tableaux

$$\alpha_1*\cdots*\alpha_m \ := \ (\cdots((\emptyset*\alpha_1)*\alpha_2)*\cdots)*\alpha_m.$$

Let $s_i = \sum_{j=1}^{i}\alpha_j$. Then $\alpha_1*\cdots*\alpha_m$ is the set of tableaux where, for each $1\leq i\leq m$, the consecutive integers $s_{i-1}+1,\ldots,s_{i-1}+\alpha_i = s_i$ occur in distinct columns, and in order from left to right. In particular $1*\cdots*1$ (l 1's) is the set of all tableaux of degree l, and $2*4*3$ is the set of 11 tableaux shown in Figure 2.

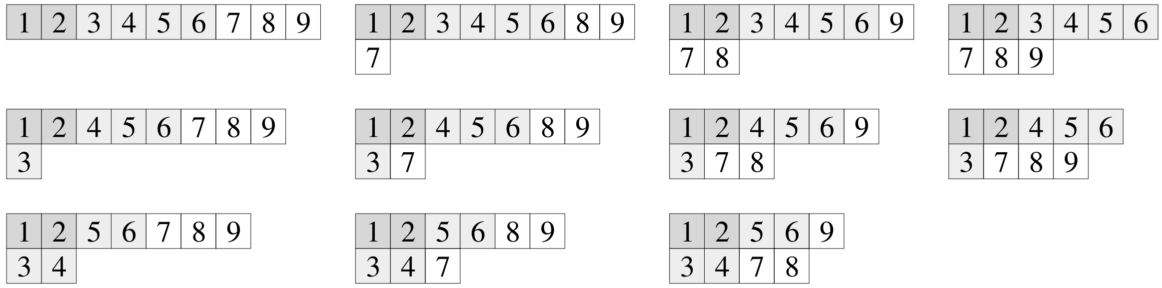


FIGURE 2. $2 * 4 * 3$

3.5. **Families of intermediate cycles.** The cycles $\Phi(\mathcal{T}_{s-1,\alpha}, L)$ are used to construct families of cycles which arise in the proof of Theorem A. These families, $\Xi_{i,s} \to U_{i,s}$, have base $U_{i,s}$ which is an open subset of a product of Grassmann varieties. The different families are compared by considering the subset $\mathcal{G}_{i,s}$ of the Chow variety whose points represent fundamental cycles of fibres of the family $\Xi_{i,s} \to U_{i,s}$. In fact, we only use the Chow variety as a convenient place to compare cycles from different families.

Let $\alpha_1, \ldots, \alpha_m$ be positive integers and $\mathcal{F}$ an arrangement. Fix $1 \leq i \leq m$ and set $\mathcal{T} := \alpha_1 * \cdots * \alpha_{i-1}$. Then every tableau $T \in \mathcal{T}$ has degree $l := \alpha_1 + \cdots + \alpha_{i-1}$. For $0 \leq a \leq n$, let $\mathbf{G}^a\mathbf{P}^n = \mathbf{G}_{n-a}\mathbf{P}^n$, the Grassmannian of codimension a planes in $\mathbf{P}^n$.

Let $\Xi'_{i,0}$ be the subscheme of $\mathbf{G}_1\mathbf{P}^n \times \prod_{j=i}^m \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$ whose fibre at a point $(K_i, \ldots, K_m)$ of $\prod_{j=i}^m \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$ is

$$\Phi(\mathcal{T}) \bigcap \Omega_{K_i} \bigcap \cdots \bigcap \Omega_{K_m}. \tag{2}$$

Let $U_{i,0} \subset \prod_{j=i}^m \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$ consist of those $(m-i+1)$-tuples for which this intersection is generically transverse, and let $\Xi_{i,0}$ be the restriction of $\Xi'_{i,0}$ to $U_{i,0}$, so $\Xi_{i,0} \to U_{i,0}$ is a family with generically reduced equidimensional fibres. This family induces a map $\psi : U_{i,0} \to \mathit{Chow}\,\mathbf{G}_1\mathbf{P}^n$. Let $\mathcal{G}_{i,0} \subset \mathit{Chow}\,\mathbf{G}_1\mathbf{P}^n$ be its image, the set of cycles which are generically transverse intersections of the form (2).

Similarly, for $1 \leq s \leq \alpha_i$, let $V \subset \mathbf{G}^{\alpha_i - s+1}(\mathbf{P}^n/F_{l+\alpha_i-s+2})$ be the (open) subset of those L of codimension $\alpha_i - s + 1$ in $\mathbf{P}^n$ which meet $\mathcal{F}$ (l, s)-properly. Let $\Xi'_{i,s}$ be the subscheme of $\mathbf{G}_1\mathbf{P}^n \times V \times \prod_{j=i+1}^m \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$ whose fibre at $(L, K_{i+1}, \ldots, K_m)$ is

$$\left[\Phi(\mathcal{T}_{s-1,\alpha_i}; L) + \Phi(\mathcal{T}^{+s}) \bigcap \Omega_L\right] \bigcap \Omega_{K_{i+1}} \bigcap \cdots \bigcap \Omega_{K_m}.$$

Define $U_{i,s} \subset V \times \prod_{j=i+1}^m \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$, $\Xi_{i,s}$, and $\mathcal{G}_{i,s}$ analogously to $U_{i,0}$, $\Xi_{i,0}$, and $\mathcal{G}_{i,0}$. Set $\mathcal{G}_{m+1,0}$ to be the singleton $\{\Omega(\alpha_1 * \cdots * \alpha_m)\}$. When

$k = \mathbf{R}$, set $\mathcal{G}_{i,s;\mathbf{R}} = \psi(U_{i,s}(\mathbf{R}))$, the set of fundamental cycles of fibres of $\Xi_{i,s}$ over real points of $U_{i,s}$.

4. Main Results

Kleiman's Transversality Theorem [9] establishes the transversality of a general translate in characteristic zero. In positive characteristic, there exists a smooth subvariety of $\mathbf{G}_1\mathbf{P}^3$ which does not meet a particular smooth Schubert variety, nor any PGL_4-translate of that Schubert variety, generically transversally ([9], §9). However, in §5, we show (Theorem E) that general translates of Schubert varieties of $\mathbf{G}_1\mathbf{P}^n$ do meet generically transversally and use this to show the sets $U_{i,s}$ are non-empty, proving the following lemma.

4.1. **Lemma.** *Let $\alpha_1, \ldots, \alpha_m$ be positive integers and let $\mathcal{F}$ be any arrangement. Then for all $1 \leq i \leq m$ and $0 \leq s \leq \alpha_i$, $U_{i,s}$ is a dense open subset of the corresponding product of Grassmannians.*

Thus $\mathcal{G}_{i,s}$ is a unirational subset of $Chow\,\mathbf{G}_1\mathbf{P}^n$. When $k = \mathbf{R}$, $\mathcal{G}_{i,s;\mathbf{R}}$ is the image of the (non compact) real algebraic manifold $U_{i,s}(\mathbf{R})$ under the morphism ψ. In §6, we prove:

Theorem D. *Let $\alpha_1, \ldots, \alpha_m$ be positive integers, $\mathcal{F}$ an arrangement, and $1 \leq i \leq m$.*

1. *Let X be a closed point of $\mathcal{G}_{i+1,0}$. Then there is an open subset U of $\mathbf{P}^1 \setminus \{0\}$ and a family of cycles $\mathcal{X}$ over $U \cup \{0\}$ such that X is the fibre of $\mathcal{X}$ over 0 and fibres of $\mathcal{X}|_U$ are in $\mathcal{G}_{i,\alpha_i}$. Thus $\psi(U)$ is a rational curve in $\mathcal{G}_{i,\alpha_i}$ such that $X \in \psi(U) \cup \{0\}$.*
2. *Fix s with $0 \leq s < \alpha_i$. Let X be a closed point of $\mathcal{G}_{i,s+1}$. Then there is an open subset U of $\mathbf{P}^1 \setminus \{0\}$ and a family of cycles $\mathcal{X}$ over $U \cup \{0\}$ such that X is the fibre of $\mathcal{X}$ over 0 and fibres of $\mathcal{X}|_U$ are in $\mathcal{G}_{i,s}$. Thus $\psi(U)$ is a rational curve in $\mathcal{G}_{i,s}$ such that $X \in \psi(U) \cup \{0\}$.*

4.2. **Proof of Theorem A using Theorem D.** Suppose $\lambda^i = (\alpha_i, 0)$ for $1 \leq i \leq m$. Let $\mathcal{F}$ be an arrangement and let $\Phi(\alpha_1 * \cdots * \alpha_m)$ be the cycle Φ of Theorem A. Then $\mathcal{G}_{1,0} = \mathcal{G}(\lambda^1, \ldots, \lambda^m)$ and $\mathcal{G}_{m+1,0} = \{\Phi(\alpha_1 * \cdots * \alpha_m)\}$. Since $\mathcal{G}_{i,s}$ is unirational, any two points of $\mathcal{G}_{i,s}$ are connected by a chain of rational curves, each lying within the closure of $\mathcal{G}_{i,s}$. Downward induction on the lexicographic order on pairs (i, s), together with and Theorem D proves the existence of a chain of rational curves between $\Phi(\alpha_1 * \cdots * \alpha_m)$ and an arbitrary cycle $X \in \mathcal{G}(\lambda^1, \ldots, \lambda^m)$. Thus Theorem D implies Theorem A when each partition λ^i is a single row.

Now suppose $\lambda^i = (\alpha_i + \beta_i, \beta_i)$ for $1 \leq i \leq m$ and set $\beta = \beta_1 + \cdots + \beta_m$. For a linear subspace $M_0 \subset \mathbf{P}^n$ of codimension β, let $\mathcal{F}$ be an arrangement in M_0, and put $\Phi = \Phi(\alpha_1 * \cdots * \alpha_m)$. Define $U_{i,s}$ and $\mathcal{G}_{i,s}$ as in §3.5, with M_0 replacing $\mathbf{P}^n$. Then $\mathcal{G}_{1,0} \subset \mathcal{G}(\lambda^1, \ldots, \lambda^m)$. In fact, if PGL_{n+1} is the automorphism group of $\mathbf{P}^n$, then $\mathcal{G}(\lambda^1, \ldots, \lambda^m)$ is the union of all translates of $\mathcal{G}_{1,0}$ under the action of PGL_{n+1}. Let $X \in \mathcal{G}(\lambda^1, \ldots, \lambda^m)$, so X is a generically transverse intersection

$$\Omega(K_1, M_1) \bigcap \cdots \bigcap \Omega(K_m, M_m),$$

where $K_i \subset M_i$ has type λ^i for $1 \leq i \leq m$. Set $M = M_1 \cap \cdots \cap M_m$. Iteration of Lemma 2.3(3) shows that if M has codimension β in $\mathbf{P}^n$ and $L_i = M \cap K_i$ has codimension $\alpha_i + 1$ in M. Thus

$$X \;=\; \Omega_{L_1} \bigcap \cdots \bigcap \Omega_{L_m}$$

is a generically transverse intersection in $\mathbf{G}_1 M$.

Let γ be any automorphism of $\mathbf{P}^n$ with $\gamma M = M_0$, so that $\gamma(X) \in \mathcal{G}_{1,0}$, and suppose that Γ is a one parameter subgroup containing γ. The orbit $\Gamma \cdot X$ is a rational curve (or a point) in $\mathcal{G}(\lambda^1, \ldots, \lambda^m)$, and contains $\gamma(X)$. Since $\gamma(X) \in \mathcal{G}_{1,0}$, previous arguments show there exists a chain of rational curves between $\gamma(X)$ and $\Phi(\alpha_1 * \cdots * \alpha_m)$, with each curve contained in the closure of $\mathcal{G}_{1,0}$. ∎

4.3. **The Schubert calculus.** Interpreting Theorem A in terms of products in the Chow ring of $\mathbf{G}_1\mathbf{P}^n$, we have:

Corollary B′. *Let $\lambda^1, \ldots, \lambda^m$ be partitions with $\lambda^i = (\alpha_i + \beta_i, \beta_i)$ for $1 \leq i \leq m$, and set $\beta = \beta_1 + \cdots + \beta_m$. Then in $A^*\mathbf{G}_1\mathbf{P}^n$,*

$$\prod_{i=1}^{m} \sigma_{\lambda^i} \;=\; \sum_{\substack{\lambda = (\beta + a, \beta + b) \\ a+b=\alpha_1+\cdots+\alpha_m}} c^{(a,b)}_{\alpha_1,\ldots,\alpha_m} \sigma_\lambda. \tag{3}$$

where $c^\lambda_{\alpha_1,\ldots,\alpha_m}$ is the number of tableaux of shape λ such that for $1 \leq i \leq m$, the α_i consecutive integers $1 + \sum_{j=1}^{i-1} \alpha_j, \ldots, \sum_{j=1}^{i} \alpha_j$ occur in distinct columns and in order from left to right.

Corollary B′ follows from Theorem A: The integers $c^\lambda_{\alpha_1,\ldots,\alpha_m}$ count the number of tableaux in the set $\alpha_1 * \cdots * \alpha_m$ of shape λ and hence the number of components of $\Phi(\alpha_1 * \cdots * \alpha_m)$ of type λ. Thus Corollary B' asserts the equality of the rational equivalence classes of cycles shown to be rationally equivalent in Theorem A: The left hand side is the rational equivalence class of cycles in $\mathcal{G}(\lambda^1, \ldots, \lambda^m)$ and the right hand side is the rational equivalence class of $\Phi(\alpha_1 * \cdots * \alpha_m)$. The number $c^\lambda_{\alpha_1,\ldots,\alpha_m}$ is in fact the Kostka number $K_{\lambda\mu}$, where $\mu = (\alpha_1, \ldots, \alpha_m)$ (see [14]).

If $\beta_1 = \cdots = \beta_m = 0$ and $\alpha_1 + \cdots + \alpha_m = 2n-2$, then the only non-zero term on the right hand side of (3) is $c^{(n-1,n-1)}_{\alpha_1,\ldots,\alpha_m}\sigma_{(n-1,n-1)}$, or $c^{(n-1,n-1)}_{\alpha_1,\ldots,\alpha_m}$ times the class of a point (line). Since $\sigma_{(\alpha_i,0)}$ is the rational equivalence class of Ω_L, whenever L has dimension $n-\alpha_i-1$, we deduce:

4.4. **Corollary.** *The number of lines meeting general $(n-\alpha_i-1)$-planes for $1 \leq i \leq m$ is equal to the number of tableaux of shape $(n-1, n-1)$ such that for $1 \leq i \leq m$, the consecutive integers $1 + \sum_{j=1}^{i-1}\alpha_j, \ldots, \sum_{j=1}^{i}\alpha_j$ occur in distinct columns and in order from left to right.*

4.5. **Enumerative geometry for the real Grassmannian.** Let $\lambda^1, \ldots, \lambda^m$ be partitions and define $\mathcal{G}_{\mathbf{R}}$ to be those cycles in $\mathcal{G}(\lambda^1, \ldots, \lambda^m)$ which arise as generically transverse intersections of Schubert varieties defined by real flags.

Theorem C′. *Let $\lambda^1, \ldots, \lambda^m$ be partitions with $\lambda^i = (\alpha_i + \beta_i, \beta_i)$, and set $\beta = \sum_{i=1}^m \beta_i$. Let $M \subset \mathbf{P}^n$ be a real $(n-\beta)$-plane, $\mathcal{F}$ a real arrangement in M, and define $\Phi(\alpha_1 * \cdots * \alpha_m)$ in terms of $\mathcal{F}$.*

(1) *$\Phi(\alpha_1 * \cdots * \alpha_m)$ is in the closure of $\mathcal{G}_{\mathbf{R}}$.*
(2) *If $|\lambda^1| + \cdots + |\lambda^m| = 2n-2$, then there is a non-empty classically open subset of $\prod_{i=1}^m \mathbf{Fl}_{\lambda^i}$ whose corresponding Schubert varieties meet transversally, with all points of intersection real.*

Proof: Define the sets $U_{i,s}$ and $\mathcal{G}_{i,s}$ as in §3.5 for the arrangement $\mathcal{F}$ in M and the integers $\alpha_1, \ldots, \alpha_m$. Arguing as in the proof of Theorem A shows $\mathcal{G}_{1,0;\mathbf{R}} \subset \mathcal{G}_{\mathbf{R}}$. Restricting to the real points of the varieties in Theorem D shows $\mathcal{G}_{i,s;\mathbf{R}} \subset \overline{\mathcal{G}_{1,0;\mathbf{R}}}$. Assertion (1) is the case $(i,s) = (m+1, 0)$.

For (2), let $d = c^{(n-1,n-1)}_{\alpha_1,\ldots,\alpha_m}$. Then $\Phi(\alpha_1 * \cdots * \alpha_m)$ consists of d distinct real lines. Hence $\mathcal{G}_{i,s} \subset S^d\mathbf{G}_1\mathbf{P}^n$, the Chow variety of effective degree d zero cycles on $\mathbf{G}_1\mathbf{P}^n$. The real points $S^d\mathbf{G}_1\mathbf{P}^n(\mathbf{R})$ of $S^d\mathbf{G}_1\mathbf{P}^n$ are effective degree d zero cycles stable under complex conjugation. The dense subset of $S^d\mathbf{G}_1\mathbf{P}^n(\mathbf{R})$ of multiplicity free cycles has a topological component $\mathcal{M}$ parameterizing sets of d distinct real lines and $\Phi(\alpha_1 * \cdots * \alpha_m) \in \mathcal{M}$. By (1), $\Phi(\alpha_1 * \cdots * \alpha_m) \in \overline{\mathcal{G}_{\mathbf{R}}}$, which shows $\mathcal{G}_{\mathbf{R}} \bigcap \mathcal{M} \neq \emptyset$, a restatement of (2). ∎

5. Generically Transverse Intersections

We give a proof, valid in any characteristic, that general translates of Schubert varieties of $\mathbf{G}_1\mathbf{P}^n$ meet generically transversally. This extension of Kleiman's characteristic zero transversality Theorem [9] in this situation relies upon his result that general translates have proper intersection.

Theorem E. *Let $\lambda^1, \ldots, \lambda^m$ be partitions. Then the set $U \subset \prod_{i=1}^m \mathbf{Fl}_{\lambda^i}$ of partial flags $(K_1 \subset M_1, \ldots, K_m \subset M_m)$ for which the intersection*

$$\Omega(K_1, M_1) \bigcap \cdots \bigcap \Omega(K_m, M_m)$$

is generically transverse is a dense open subset of $\prod_{i=1}^m \mathbf{Fl}_{\lambda^i}$.

Proof: We begin with an observation that simplifies the geometry. For $1 \leq i \leq m$, let $K_i \subset M_i$ be a partial flag of type $\lambda^i = (\alpha_i + \beta_i, \beta_i)$ and suppose the corresponding Schubert varieties meet properly. By Lemma 2.3(3),

$$\Omega(K_1, M_1) \bigcap \cdots \bigcap \Omega(K_m, M_m) \;=\; \mathbf{G}_1 M \bigcap \Omega_{N_1} \bigcap \cdots \bigcap \Omega_{N_m},$$

where $M = M_1 \cap \cdots \cap M_m$ has codimension $\beta = \beta_1 + \cdots + \beta_m$ and $K_i = N_i \cap M_i$, with N_i meeting M_i properly for $1 \leq i \leq m$.

Fix a codimension β subspace M of $\mathbf{P}^n$. As U is stable under the diagonal action of PGL_{n+1}, it is the union of the translates of $U \cap X$, where X consists of those m-tuples of flags with $M \subset M_i$, for $1 \leq i \leq m$. Moreover, U is open if and only if $V := U \cap X$ is open in X. Let $Y \subset X$ be the set of m-tuples of flags where $M = M_1 \cap \cdots \cap M_m$ and K_i meets M properly. The product of the maps

$$(K_i, M_i) \mapsto K_i \cap M = L_i, \qquad i = 1, \ldots, m$$

exhibits Y as a fibre bundle with base $\prod_{i=1}^m \mathbf{G}^{\alpha_i+1} M$, and V is the inverse image of the set of $(L_1, \ldots, L_m)$ for which $\Omega_{L_1} \bigcap \cdots \bigcap \Omega_{L_m}$ is generically transverse. It suffices to prove the theorem for $\Omega_{L_1} \bigcap \cdots \bigcap \Omega_{L_m}$ in $\mathbf{G}_1 M$, that is, for the case when $\beta_i = 0$ for $1 \leq i \leq m$.

Let Ξ be the subscheme of

$$(\mathbf{P}^n)^m \times \mathbf{G}_1 \mathbf{P}^n \times \prod_{i=1}^m \mathbf{G}^{\alpha_i+1} \mathbf{P}^n$$

consisting of $(2m+1)$-tuples $(p_1, \ldots, p_m, \ell, L_1, \ldots, L_m)$ such that $p_i \in \ell \cap L_i$ for $1 \leq i \leq m$. The projection of Ξ to $(\mathbf{P}^n)^m \times \mathbf{G}_1 \mathbf{P}^n$ exhibits Ξ as a fibre bundle with fibre $\prod_{i=1}^m \mathbf{G}^{\alpha_i+1}(\mathbf{P}^n/p_i)$ and base $\{(p_1, \ldots, p_m, \ell) \mid \text{each } p_i \in \ell\}$. This base has dimension $m+2n-2$, so Ξ is irreducible of dimension

$$m+2n-2+\sum_{i=1}^m (n-\alpha_i-1)(\alpha_i+1) \;=\; 2n-2-\sum_{i=1}^m \alpha_i + \sum_{i=1}^m (n-\alpha_i)(\alpha_i+1).$$

The projection of Ξ to $\prod_{i=1}^m \mathbf{G}^{\alpha_i+1} \mathbf{P}^n$ has image consisting of those $(L_1, \ldots, L_m)$ whose corresponding Schubert varieties have non-empty intersection. This image is a proper subvariety if $2n-2 < \sum_{i=1}^m \alpha_i$. In this case, let U be the complement of this image.

Suppose $2n-2 \geq \sum_{i=1}^m \alpha_i$. Let $W \subset \Xi$ consist of those points where $\Omega_{L_1}, \ldots, \Omega_{L_m}$ meet transversally at ℓ. By Lemma 2.3, W consists of those points such that

(1) $\ell \not\subset L_i$ for $1 \leq i \leq m$; thus $p_i = \ell \cap L_i$ and ℓ is a smooth point of Ω_{L_i}.

(2) The tangent spaces $T_\ell \Omega_{L_i}$ for $1 \leq i \leq m$ meet transversally.

Thus W is an open subset of Ξ. We show $W \neq \emptyset$. Fix $\ell \in \mathbf{G}_1\mathbf{P}^n$ and fix distinct points $p_1, \ldots, p_m$ of ℓ. Suppose $\ell = \mathbf{P}H$ and for each $1 \leq i \leq m$, fix a point $q_i \in H$ such that $p_i = \mathbf{P}\langle q_i \rangle$. Suppose also that $\mathbf{P}^n/\ell = \mathbf{P}M$ and identify $\mathbf{P}^n$ with $\mathbf{P}(H \oplus M)$. Recall that $T_\ell \mathbf{G}_1\mathbf{P}^n = \mathrm{Hom}(\ell, \mathbf{P}^n/\ell) = \mathrm{Hom}(H, M)$. Define the linear map

$$f \ : \ \mathrm{Hom}(H, M) \ \longrightarrow \ M^m$$

by $f(\phi) = (\phi(q_1), \ldots, \phi(q_m))$. If $m \geq 2$ (the case m=1 is trivial), then $f^{-1}(0) = \{0\}$, as $H = \langle q_1, q_2 \rangle$.

For any linear subspace $M_i \subset M$ of dimension $n-1-\alpha_i$, define $L_i := \mathbf{P}\langle q_i, M_i \rangle$. Then $L_i \in \mathbf{G}^{\alpha_i+1}\mathbf{P}^n$ and $p_i = \ell \cap L_i$. Moreover, $T_\ell \Omega_{L_i} = \{\phi \in \mathrm{Hom}(H, M) \mid \phi(q_i) \in M_i\}$. Fix linear subspaces $M_1, \ldots, M_m$ of M where $\dim M_i = n-1-\alpha_i$. Since $(GL(M))^m$ acts transitively on $M^m \setminus \{0\}$, Theorem 2(i) of [9] shows there exists a dense open subset V of $(GL(M))^m$ consisting of $\boldsymbol{g}$ such that

$$f^{-1}(\boldsymbol{g}(M_1 \times \cdots \times M_m))$$

either is $\{0\}$ or has codimension equal to $\sum_{i=1}^m \alpha_i$, the codimension of $M_1 \times \cdots \times M_m$ in M. Let $\boldsymbol{g} = (g_1, \ldots, g_m) \in V$ and set $M_i' := g_i M_i$ and $L_i' := \langle q_i, M_i' \rangle$. Then $L_i' \in \mathbf{G}^{\alpha_i+1}\mathbf{P}^n$, and

$$\begin{aligned} f^{-1}(M_1' \times \cdots \times M_m') &= \{\phi \in \mathrm{Hom}(H, M) \mid \phi(q_i) \in M_i', \text{ for } i = 1, \ldots, m\} \\ &= \bigcap_{i=1}^m T_\ell \Omega_{L_i'}. \end{aligned}$$

Since α_i is the codimension of $T_\ell \Omega_{L_i'}$ in $T_\ell \mathbf{G}_1\mathbf{P}^n$ and $\sum_{i=1}^m \alpha_i \leq 2n-2$, the varieties $\Omega_{L_1'}, \ldots, \Omega_{L_m'}$ meet transversally at ℓ. Thus $W \neq \emptyset$.

Let $Z = \Xi \setminus W$ and let $\pi : \Xi \to \prod_{i=1}^m \mathbf{G}^{\alpha_i+1}\mathbf{P}^n$ be the projection. Then $\Omega_{L_1} \ldots, \Omega_{L_m}$ meet generically transversally if $\dim(\pi^{-1}(L_1, \ldots, L_m) \bigcap Z)$ is less than the expected dimension of a fibre, which is $2n-2-\sum_{i=1}^m \alpha_i$. Take U to be the set of m-tuples $(L_1, \ldots, L_m)$ where

$$\dim(\pi^{-1}(L_1, \ldots, L_m) \bigcap Z) < 2n-2-\sum_{i=1}^m \alpha_i.$$

Thus U is open and non-empty, for otherwise $\dim Z = \dim \Xi$, which would imply $Z = \Xi$ and contradict $W \neq \emptyset$. ∎

We also need a further lemma:

5.1. **Lemma.** *Let $d, \alpha_1, \ldots, \alpha_m$ be positive integers and let Z be a subscheme of $\mathbf{G}_1\mathbf{P}^n$ with $\dim(Z) < d$. Then the set $W \subset \prod_{i=1}^m \mathbf{G}^{\alpha_i+1}\mathbf{P}^n$ consisting of those $(K_1, \ldots, K_m)$ for which*

$$\dim(Z \bigcap \Omega_{K_1} \bigcap \cdots \bigcap \Omega_{K_m}) < d - \sum_{i=1}^m \alpha_i$$

is open and dense.

Proof: Let Ξ be the subscheme of $Z \times \prod_{i=1}^m \mathbf{G}^{\alpha_i+1}\mathbf{P}^n$ whose fibre at $(K_1, \ldots, K_m)$ is $Z \bigcap \Omega_{K_1} \bigcap \cdots \bigcap \Omega_{K_m}$, and let $X_i(\ell) \subset \mathbf{G}^{\alpha_i+1}\mathbf{P}^n$ denote the set of K_i which meet ℓ. Projection to Z exhibits Ξ as a fibre bundle with fibre $X_1(\ell) \times \cdots \times X_m(\ell)$ at a point $\ell \in Z$. The codimension of $X_i(\ell)$ in $\mathbf{G}^{\alpha_i+1}\mathbf{P}^n$ is α_i, so Ξ has dimension

$$\dim Z - \sum_{i=1}^m \alpha_i + \sum_{i=1}^m (n - \alpha_i)(\alpha_i + 1).$$

$W \subset \prod_{i=1}^m \mathbf{G}^{\alpha_i+1}\mathbf{P}^n$ is the locus where the fibre dimension is less than $d - \sum_{i=1}^m \alpha_i$. By upper semicontinuity of fibre dimension, W is open. If W were empty, then all fibres of the projection to $\prod_{i=1}^m \mathbf{G}^{\alpha_i+1}\mathbf{P}^n$ would have dimension at least $d - \sum_{i=1}^m \alpha_i$. This would imply

$$\dim \Xi \geq d - \sum_{i=1}^m \alpha_i + \sum_{i=1}^m (n - \alpha_i)(\alpha_i + 1),$$

a contradiction, as $d > \dim Z$. Thus W must be non-empty. ∎

5.2. **Proof of Lemma 4.1.** We show that for each (i, s) with $1 \leq i \leq m$ and $0 \leq s \leq \alpha_i$, the sets $U_{i,s}$ are open dense subsets of the corresponding products of Grassmannians. We first show this for $U_{i,0}$. Let $\mathcal{T} = \alpha_1 * \cdots * \alpha_{i-1}$. Recall that $U_{i,0}$ consists of those $(K_i, \ldots, K_m)$ such that the intersection

$$\Phi(\mathcal{T}) \bigcap \Omega_{K_i} \bigcap \cdots \bigcap \Omega_{K_m} \tag{4}$$

is generically transverse. The cycle (4) has dimension $d = 2n - 2\sum_{j=1}^m \alpha_j$.

Let Z be the singular locus of $\Phi(\mathcal{T})$. Since $\Phi(\mathcal{T})$ has pure dimension $2n - 2 - \sum_{j=1}^{i-1} \alpha_j$, the intersection (4) is generically transverse if for every component $\Psi(T)$ of $\Phi(\mathcal{T})$, the intersection $\Psi(T) \bigcap \Omega_{K_i} \bigcap \cdots \bigcap \Omega_{K_m}$ is generically transverse, and if $\dim(Z \bigcap \Omega_{K_i} \bigcap \cdots \bigcap \Omega_{K_m}) < d$.

By Corollary 2.10(3), $\Psi(T) = \Psi(S) \neq \emptyset$ implies $T = S$. Thus Z is a union of intersections of components and the singular loci of components, and hence $\dim(Z) < \dim(\Phi(\mathcal{T}))$. By Lemma 5.1, there is an open subset W of $\prod_{j=i}^{m} \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$ consisting of those $(K_i, \ldots, K_m)$ for which $\dim(Z \bigcap \Omega_{K_i} \bigcap \cdots \bigcap \Omega_{K_m}) < d$.

For $T \in \mathcal{T}$, let $U_T \subset \prod_{j=i}^{m} \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$ be the set of $(K_i, \ldots, K_m)$ for which the intersection

$$\Psi(T) \bigcap \Omega_{K_i} \bigcap \cdots \bigcap \Omega_{K_m}$$

is generically transverse. Because $U_{i,0} = \left(\bigcap_{T \in \mathcal{T}} U_T\right) \cap W$, it suffices to show that U_T is a dense open subset of $\prod_{j=i}^{m} \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$ for each $T \in \mathcal{T}$. Let $T \in \mathcal{T}$ and suppose T has shape $\lambda = (\alpha, \beta)$. Define

$$V \subset \mathbf{Fl}_\lambda \times \prod_{j=i}^{m} \mathbf{G}^{\alpha_j+1}\mathbf{P}^n$$

to be the set of flags for which the intersection

$$\Omega(F, H) \bigcap \Omega_{K_i} \bigcap \cdots \bigcap \Omega_{K_m}$$

is generically transverse. By Theorem E, V is dense and open. Note that

$$\{F_{\alpha+1} \subset H_T\} \times U_T \;=\; V \bigcap \left(\{F_{\alpha+1} \subset H_T\} \times \prod_{j=i}^{m} \mathbf{G}^{\alpha_j+1}\mathbf{P}^n \right),$$

so U_T is open. Since $\mathbf{Fl}_\lambda = PGL_{n+1} \cdot \{F_{\alpha+1} \subset H_T\}$, and V is stable under the diagonal action of PGL_{n+1}, we see that $V = PGL_{n+1} \cdot (\{F_{\alpha+1} \subset H_T\} \times U_T)$. Thus U_T is non-empty.

The case of $U_{i,s}$ for $s > 0$ follows by similar arguments. ∎

6. Construction of Explicit Rational Equivalences

We use the following lemma to parameterize the explicit rational equivalences in the proof of Theorem D.

6.1. **Lemma.** *Let $F_\bullet$ be a complete flag in $\mathbf{P}^n$. Suppose L_∞ is a hyperplane not containing F_n. Then there exists a pencil of hyperplanes L_t, for $t \in \mathbf{P}^1 = \mathbf{A}^1 \bigcup \{\infty\}$, such that if $t \neq 0$, then L_t meets the subspaces of $F_\bullet$ properly, and, for each $i \leq n-1$, the family of codimension $i+1$ planes induced by $L_t \bigcap F_i$, for $t \neq 0$, has fibre F_{i+1} at 0.*

Proof: Let $x_0, \ldots, x_n$ be coordinates for $\mathbf{P}^n$ such that L_∞ is given by $x_n = 0$ and F_i by $x_0 = \cdots = x_{i-1} = 0$. Let $e_0, \ldots, e_n$ be a basis for $\mathbf{P}^n$ dual to these coordinates. For $t \in \mathbf{A}^1$, define

$$L_t = \langle t e_j + e_{j+1} \,|\, 0 \leq j \leq n-1 \rangle.$$

For $t \neq 0$, $L_t \bigcap F_i = \langle te_j + e_{j+1} \mid i \leq j \leq n-1\rangle$ and so has codimension $i+1$. The fibre of this family at 0 is $\langle e_{j+1} \mid i \leq j \leq n-1\rangle = F_{i+1}$. ∎

In the situation of Lemma 6.1, write $\lim_{t\to 0} L_t \cap F_i = F_{i+1}$.

6.2. **Proof of Theorem D, Part 1.** Fix an arrangement $\mathcal{F}$ and an integer i with $1 \leq i \leq m$. Set $\mathcal{T} := \alpha_1 * \cdots * \alpha_{i-1}$ and $l := \alpha_1 + \cdots + \alpha_{i-1}$, the degree of all tableaux in $\mathcal{T}$. Suppose X_0 is a cycle in $\mathcal{G}_{i+1,0}$. That is

$$X_0 \;=\; \Phi(\mathcal{T} * \alpha_i) \bigcap \Omega_{K_{i+1}} \bigcap \cdots \bigcap \Omega_{K_m},$$

where the intersection is generically transverse. Let L_∞ be any hyperplane which meets $\mathcal{F}$ (l, α_i)-properly. We use Lemma 6.1 to produce a family of cycles with base $\mathbf{P}^1$ whose special member is X_0, and whose general member is a cycle in $\mathcal{G}_{i,\alpha_i}$.

Apply Lemma 6.1 to the flag induced by $F_{\bullet}$ in $\mathbf{P}^n/F_{l+2}$ and the hyperplane L_∞/F_{l+2} to obtain a pencil L_t of hyperplanes such that if $t \neq 0$ and $i \leq l+1$, then L_t meets F_i properly and $\lim_{t\to 0} L_t \cap F_i = F_{i+1}$. Let $\mathcal{X}$ be the subscheme of $\mathbf{P}^1 \times \mathbf{G}_1\mathbf{P}^n$ whose fibre at non-zero $t \in \mathbf{P}^1$ is

$$X_t \;=\; \left[\Phi(\mathcal{T}_{\alpha_i - 1, \alpha_i}; L_t) + \Phi(\mathcal{T}^{+\alpha_i})\right] \bigcap \Omega_{K_{i+1}} \bigcap \cdots \bigcap \Omega_{K_m}.$$

It is shown below that X_0 is the fibre of $\mathcal{X}$ at 0; we first deduce Part 1 from this fact. Let $U'' \subset \mathbf{P}^1$ be the open subset of those t for which X_t is generically reduced and $\dim X_t = \dim X_0$. As X_0 is generically reduced, U'' is non-empty. Since L_∞ meets $\mathcal{F}$ (l, α_i)-properly, the set $U' \subset \mathbf{P}^1$ consisting of those t where L_t meets $\mathcal{F}$ (l, α_i)-properly is open and dense. Since $\Omega_{L_t} = \mathbf{G}_1\mathbf{P}^n$, we have

$$\Phi(\mathcal{T}^{+\alpha_i}) \;=\; \Phi(\mathcal{T}^{+\alpha_i}) \bigcap \Omega_{L_t}.$$

Thus, for $t \in U := U' \cap U''$, the fibre X_t is a cycle in $\mathcal{G}_{i-1,\alpha_i}$. The restriction of $\mathcal{X}$ to $U \cup \{0\}$ gives a family over a smooth curve with generically reduced equidimensional fibres. Thus the association of a point u of $U \cup \{0\}$ to the fibre X_u gives a morphism $\psi : U \cup \{0\} \to$ *Chow* $\mathbf{G}_1\mathbf{P}^n$ with $\psi(U) \subset \mathcal{G}_{i-1,\alpha_i}$ and $\psi(0) = X_0$, proving Part 1.

We show X_0 is the fibre of $\mathcal{X}$ at 0 by examining components of $\mathcal{X}$ separately. For $T \in \mathcal{T}_{\alpha_i - 1, \alpha_i}$, let $\mathcal{X}_T$ be the subscheme of $\mathbf{P}^1 \times \mathbf{G}_1\mathbf{P}^n$ whose fibre at a non-zero $t \in \mathbf{P}^n$ is

$$(X_T)_t = \Psi(T; L_t) \bigcap \Omega_{K_{i+1}} \bigcap \cdots \bigcap \Omega_{K_m}.$$

Since

$$\mathcal{X} \;=\; \left(\sum_{T\in\mathcal{T}_{\alpha_i-1,\alpha_i}} \mathcal{X}_T\right) \;+\; \mathbf{P}^1 \times \left(\Phi(\mathcal{T}^{+\alpha_i})\bigcap \Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}\right),$$

and

$$\mathcal{T} * \alpha_i \;=\; \mathcal{T}_{\alpha_i-1,\alpha_i}\bigcup \mathcal{T}^{+\alpha_i},$$

it suffices to show that for each $T\in\mathcal{T}_{\alpha_i-1,\alpha_i}$, the fibre of $\mathcal{X}_T$ at 0 is

$$\Psi(T)\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}.$$

Let $T\in\mathcal{T}_{\alpha_i-1,\alpha_i}$. If the first row of T has length exceeding $l+1$, then $\Psi(T;L_t)=\Psi(T)$, so $\mathcal{X}_T$ is the constant family over $\mathbf{P}^1$ with fibre $\Psi(T)\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}$. Now suppose the first row of T has length $b\leq l+1$. Then, for $t\neq 0$, $\Psi(T;L_t)=\Omega(F_b\cap L_t, H_T)$. Since $\lim_{t\to 0}F_b\bigcap L_t = F_{b+1}$, we see that $\Psi(T)=\Omega(F_{b+1},H_T)$ is the fibre at 0 of the family over $\mathbf{P}^1$ whose fibre at $t\neq 0$ is $\Psi(T;L_t)$. Since $\Psi(T)\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}$ is generically transverse, there is an open subset $U_T\subset\mathbf{P}^1$ such that for $t\in U_T\setminus\{0\}$, the intersection $\Psi(T;L_t)\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}$ is generically transverse. This shows that the fibre at 0 of $\mathcal{X}_T$ is $\Psi(T)\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}$, which completes the proof of Part 1 of Theorem D.

6.3. **Proof of Theorem D, Part 2.** Let $0\leq s<\alpha_i$ and suppose $X_0\in\mathcal{G}_{i,s+1}$. Then

$$X_0 \;=\; \left[\Phi(\mathcal{T}_{s,\alpha_i};N)+\Phi(\mathcal{T}^{+(s+1)})\bigcap\Omega_N\right]\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m},$$

where N has codimension α_i-s in $\mathbf{P}^n$ and meets $\mathcal{F}$ $(l,s+1)$-properly, and this intersection is generically transverse. We make a useful calculation.

6.4. **Lemma.** *Let $L_0=N\cap H_{l+s+1}$, a hyperplane in N. Then*

$$\Phi(\mathcal{T}^{+s}(\alpha_i-s);N)+\Phi(\mathcal{T}^{+(s+1)})\bigcap\Omega_N \;=\; \Phi(\mathcal{T}^{+s})\bigcap\Omega_{L_0}.$$

Since $\mathcal{T}_{s,\alpha_i}=\mathcal{T}_{s-1,\alpha_i}\bigcup\mathcal{T}^{+s}(\alpha_i-s)$, this Lemma shows that

$$X_0 \;=\; \left[\Phi(\mathcal{T}_{s-1,\alpha_i};N)+\Phi(\mathcal{T}^{+s})\bigcap\Omega_{L_0}\right]\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}.$$

We use Lemma 6.4 to complete the proof of Theorem D, and prove it at the end of the section. Let $N_\bullet$ be any complete flag in $N/F_{l+\alpha_i-s+2}$ refining the images of the partial flag

$$F_{l+\alpha_i-s+1}\;\subset\; N\cap F_l\;\subset\;\cdots\;\subset\; N\cap F_{\lfloor\frac{l+s}{2}\rfloor+1}\;\subset\; L_0.$$

Let L_∞ be any hyperplane of N which meets $\mathcal{F}$ (l,s)-properly. Application of Lemma 6.1 to the flag $N_\bullet$ in $N/F_{l+\alpha_i-s+2}$ yields a pencil L_t of hyperplanes of N, each containing $F_{l+\alpha_i-s+2}$, such that for $\lfloor\frac{l+s}{2}\rfloor+1\leq j\leq l$ and $t\neq 0$, L_t meets $N\cap F_j$ properly, with $\lim_{t\to 0}L_t\cap N\cap F_j = N\cap F_{j+1}$. Since $L_t\cap N\cap F_j = L_t\cap F_j$ for j in this range, L_t meets F_j properly, as N meets $\mathcal{F}$ $(l,s+1)$-properly.

Let $\mathcal{X}$ be the subscheme of $\mathbf{P}^1\times\mathbf{G}_1\mathbf{P}^n$ whose fibre at a non-zero $t\in\mathbf{P}^1$ is

$$X_t = \left[\Phi(\mathcal{T}_{s-1,\alpha_i};L_t)+\Phi(\mathcal{T}^{+s})\bigcap\Omega_{L_t}\right]\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}.$$

We claim that X_0 is the fibre of $\mathcal{X}$ at 0; Part 2 follows from this fact in much the same manner as Part 1 followed from the analogous fact in §6.2. We show X_0 is the fibre of $\mathcal{X}$ at 0 by examining each component separately. For $T\in\mathcal{T}_{s-1,\alpha_i}$ let $\mathcal{X}_T$ be the subscheme of $\mathbf{P}^1\times\mathbf{G}_1\mathbf{P}^n$ whose fibre at $t\neq 0$ is

$$(\mathcal{X}_T)_t \;=\; \Psi(T;L_t)\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}.$$

Arguing as at the end of §6.2, we conclude that $\Psi(T;N)\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}$ is the fibre of $\mathcal{X}_T$ at 0. For $S\in\mathcal{T}^{+s}$, let $\mathcal{X}_S$ be the subscheme of $\mathbf{P}^1\times\mathbf{G}_1\mathbf{P}^n$ whose fibre at t is

$$(\mathcal{X}_S)_t \;=\; \Psi(S)\bigcap\Omega_{L_t}\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}.$$

Arguing as at the end of §6.2, we conclude that $\Psi(S)\bigcap\Omega_{L_0}\bigcap\Omega_{K_{i+1}}\bigcap\cdots\bigcap\Omega_{K_m}$ is the fibre of $\mathcal{X}_S$ at 0. Since $\mathcal{X}=\sum_{T\in\mathcal{T}_{s-1,\alpha_i}}\mathcal{X}_T+\sum_{S\in\mathcal{T}^{+s}}\mathcal{X}_S$, we see that the fibre of $\mathcal{X}$ at 0 is X_0. ∎

6.5. **Proof of Lemma 6.4.** Let $T\in\mathcal{T}^{+s}$. We show

$$\Psi(T)\bigcap\Omega_{L_0}\;=\;\Psi(T(\alpha_i-s))+\Psi(T^{+1})\bigcap\Omega_N.$$

Summing over $T\in\mathcal{T}^{+s}$ will complete the proof. We treat the case when T is rectangular separately from the general case.

Suppose T is rectangular. Then $\Psi(T)=\mathbf{G}_1H_T$, so by Lemma 2.3(2), $\Psi(T)\bigcap\Omega_{L_0}$ is equal to $\Omega(H_T\cap L_0,H_T)$, and this intersection is generically transverse only if L_0 meets H_T properly. Since T is a rectangular tableau in $\mathcal{T}^{+s}$, $s\leq l$ and $l+s=2k$ is even, so $\lfloor\frac{l+s+1}{2}\rfloor = k\leq l$. Then N meets F_k properly, as N meets $\mathcal{F}$ $(l,s+1)$-properly. Since T is rectangular, Lemma 2.9(4) implies $H_T\cap H_{l+s+1}=F_{k+1}$. It follows that H_T meets $L_0=N\cap H_{l+s+1}$ properly, and $H_T\cap L_0 = N\cap F_{k+1}$. Since $H_T=H_{T(\alpha_i-s)}$ and $k+\alpha_i-s=\lfloor\frac{l+s}{2}\rfloor+\alpha_i-s$ is the length of the first row of $T(\alpha_i-s)$, we have

$$\Psi(T)\bigcap\Omega_{L_0}\;=\;\Omega(N\cap F_{\lfloor\frac{l+s}{2}\rfloor+1},H_{T(\alpha_i-s)})\;=\;\Psi(T(\alpha_i-s);N).$$

Now suppose T is not rectangular. Let b be the length of the first row of T, so $b \geq \lfloor \frac{l+s+1}{2} \rfloor$. Then $H_T \cap H_{l+s+1} = H_{T^{+1}}$. Since $T^{+1} \in \mathcal{T}^{+(s+1)}$, N meets $H_{T^{+1}}$ properly, so H_T meets L_0 properly. Lemma 2.4 with $F = F_{b+1}$, $P = H_T$, and $H = H_{l+s+1}$ implies

$$\Omega(F_{b+1}, H_T) \bigcap \Omega_{L_0} \;=\; \Omega(F_{b+1} \cap N, H_T) + \Omega(F_{b+1}, H_T \cap H_{l+s+1}) \bigcap \Omega_N.$$

But this is $\Psi(T(\alpha_i - s); N) + \Psi(T^{+1}) \bigcap \Omega_N$, which completes the proof. ∎

7. An algebra of tableaux

The Schubert classes σ_λ of a Grassmann variety form an integral basis for its Chow ring. Thus there exist integral constants $c^\lambda_{\mu\,\nu}$ defined by the identity:

$$\sigma_\mu \cdot \sigma_\nu = \sum_\lambda c^\lambda_{\mu\,\nu} \sigma_\lambda.$$

In 1934, Littlewood and Richardson [12] gave a conjectural formula for these constants, which was proven in 1978 by Thomas [16].

Lascoux and Schützenberger [11] showed how to construct the ring of symmetric functions as a subalgebra of a non-commutative associative ring called the plactic algebra whose additive group is the free abelian group Λ with basis the set of (non-standard) Young tableaux. Each tableau T of shape λ determines a monomial summand of the Schur function s_λ.

Evaluating s_λ at Chern roots of the dual to the tautological bundle of the Grassmannian gives the Schubert class σ_λ. Non-symmetric monomials in these Chern roots are not defined, so individual Young tableaux are not expected to appear in the geometry of Grassmannians. In this context, the crucial use we made of the Schubert varieties $\Psi(T)$ is surprising.

A feature of our methods is the correspondence between an iterative construction of the set $\alpha_1 * \cdots * \alpha_m$ and the rational curves in the proof of Theorem D. This suggests an alternate non-commutative, associative product $\circ$ on Λ. The resulting algebra has surjections to the ring of symmetric functions and to Chow rings of Grassmannians.

Additional combinatorial preliminaries for this section may be found in [14]. Here, partitions λ, μ, and ν may have any number of rows. Suppose T and U are, respectively, a tableau of shape μ and a skew tableau of shape λ/μ. Let $T \bigcup U$ be the tableau of shape λ whose first $|\mu|$ entries comprise T, and remaining entries comprise U, with each increased by $|\mu|$. For tableaux S and T where the shape of S is λ, define

$$S \circ T = \sum S \bigcup U,$$

the sum taken over all ν and all skew tableaux U of shape ν/λ Knuth equivalent to T. Figure 3 shows an example. This product is related to the operation $*$ of §2.5: Let Y_α be the unique standard tableau of shape $(\alpha, 0)$. If T has at most 2 rows, then $T * \alpha$ is the set of summands of $T \circ Y_\alpha$ with at most two rows.

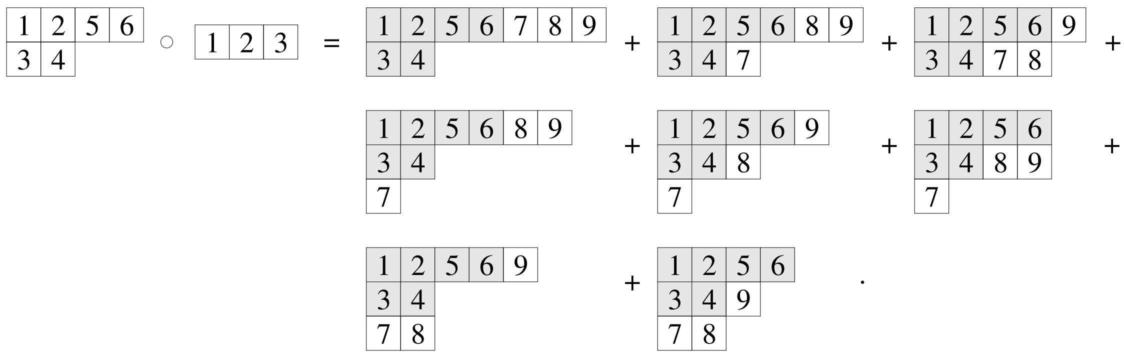
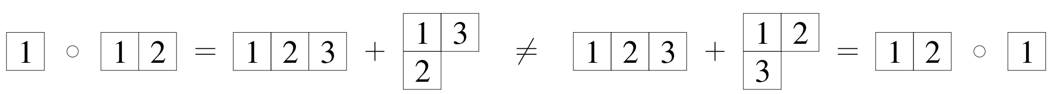


FIGURE 3. The composition $\circ$

Theorem F. *The product* $\circ$ *determines an associative non-commutative* $\mathbf{Z}$*-algebra structure on* Λ *with unit the empty tableau* $\emptyset$*. Moreover,* $\circ$ *is not the plactic product.*

Proof: In the plactic algebra, the product of two tableaux is always a third, showing $\circ$ is not the plactic product. For any tableau T, we have $\emptyset \circ T = T \circ \emptyset = T$. Note



so $\circ$ is non-commutative. To show associativity, let R, S, and T be tableaux. Then

$$R \circ (S \circ T) = \sum R \bigcup W,$$

the sum taken over W Knuth equivalent to $S \bigcup V$, where V is Knuth equivalent to T. Let U' be the first $|S|$ entries in W, and V' the last $|T|$ entries, each decreased by $|S|$, thus,

$$R \circ (S \circ T) = \sum R \bigcup U' \bigcup V',$$

the sum taken over U' Knuth equivalent to S and V' to T, which is $(R \circ S) \circ T$. ∎

Let $m < n$. For a tableau T of shape λ, let $\phi(T)$ be the Schur function s_λ. Define $\phi_{m,n}(T)$ to be 0 if $\lambda_1 + m \geq n$ or $\lambda_{m+1} \neq 0$ and σ_λ otherwise. Then ϕ and $\phi_{m,n}$ are additive surjections from Λ to, respectively, the algebra of symmetric functions and $A^*\mathbf{G}_m\mathbf{P}^n$.

Theorem G. *The maps ϕ and $\phi_{m,n}$ are $\mathbf{Z}$-algebra homomorphisms.*

Proof: For any tableaux S and T of shape ν, and arbitrary partitions λ and μ, there is a natural bijection (given by Haiman's dual equivalence [6]) between the set of tableaux with shape λ/μ Knuth equivalent to S and those Knuth equivalent to T, and this common number is $c^\lambda_{\mu\,\nu}$. Thus ϕ is an algebra homomorphism. It follows that $\phi_{m,n}$ is as well. ∎

8. Enumerative Geometry and Arrangements Over Finite Fields

A main result of this paper, Theorem C, shows that any Schubert-type enumerative problem concerning lines in projective space may be solved over $\mathbf{R}$. By 'solved' over a field k, we mean there are flags in $\mathbf{P}^n_k$ determining Schubert varieties which meet transversally in finitely many points, all of which are defined over k. We describe two families of enumerative problems and, for each problem in each family, we determine the fields over which it may be solved. We also consider the problem of finding arrangements over finite fields.

8.1. **The n lines meeting four $(n-1)$-planes in $\mathbf{P}^{2n-1}$.** Given three pairwise non-intersecting $(n-1)$-planes L_1, L_2, and L_3 in $\mathbf{P}^{2n-1}$, there are coordinates $x_1, \ldots, x_{2n}$ for $\mathbf{P}^{2n-1}$ such that L_1, L_2, and L_3 are defined by the equations:

$$\begin{aligned} L_1 &: \ x_1 = x_2 = \cdots = x_n = 0 \\ L_2 &: \ x_{n+1} = \cdots = x_{2n} = 0 \\ L_3 &: \ x_1 - x_{n+1} = \cdots = x_n - x_{2n} = 0 \end{aligned}$$

One may check that $\Omega_{L_1} \bigcap \Omega_{L_2} \bigcap \Omega_{L_3}$ is a transverse intersection, and that if $\Sigma_{1,n-1} \subset \mathbf{P}^{2n-1}$ is the union of the lines meeting each of L_1, L_2, and L_3, then $\Sigma_{1,n-1}$ is the image of the standard Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^{n-1}$ into $\mathbf{P}^{2n-1}$ ([7]):

$$\psi : [a,b] \times [y_1, \ldots, y_n] \longmapsto [ay_1, \ldots, ay_n, by_1, \ldots, by_n].$$

The lines meeting L_1, L_2, and L_3 are the images of $\mathbf{P}^1 \times \{p\}$, for $p \in \mathbf{P}^{n-1}$. The Segre variety $\Sigma_{1,n-1}$ has degree n, so a general $(n-1)$-plane L_4 meets $\Sigma_{1,n-1}$ in n distinct points, each determining a line meeting $L_1, \ldots, L_4$. These lines, $\ell_1, \ldots, \ell_n$, meet L_1 in distinct points which span L_1. If these lines are defined over k, then we may change coordinates so that each ℓ_j is the span of the standard basis elements e_j and e_{n+j}.

For $1 \leq j \leq n$, let $p_j = [\alpha_j, \beta_j] \in \mathbf{P}^1_k$ be the first coordinate of $\psi^{-1}(\ell_j \cap L_4)$. Then

$$L_4 \quad : \quad \beta_1 x_1 - \alpha_1 x_{n+1} = \cdots = \beta_n x_n - \alpha_n x_{2n} = 0.$$

Also, $p_1, \ldots, p_n$ are distinct; otherwise $L_4 \cap \Sigma_{1,n-1}$ contains a line.

Thus, if this enumerative problem may be solved over k, then k has at least $n-1$ elements. Conversely, suppose k has at least $n-1$ elements and let $p_1, \ldots, p_n$ be distinct elements in $\mathbf{P}^1_k$ and define $L_1, \ldots, L_4$ and $\ell_1, \ldots, \ell_n$ as above. Then $\ell_1, \ldots, \ell_n$ are precisely the lines which meet each of $L_1, \ldots, L_4$.

8.2. **The n lines meeting a fixed line and $n+1$ $(n-1)$-planes in $\mathbf{P}^{n+1}$.** A line ℓ and $(n-1)$-planes $K_1, \ldots, K_n$ in $\mathbf{P}^{n+1}$ are in linear general position if for every $p \in \ell$, the hyperplanes $\Gamma_i(p) = \langle p, K_i \rangle$, for $1 \leq i \leq n$, meet in a line. In this case, the union

$$S_{1,n-1} = \bigcup_{p \in \ell} \Bigl(\Gamma_1(p) \cap \cdots \cap \Gamma_n(p) \Bigr)$$

is a rational normal surface scroll. Moreover, the lines meeting each of ℓ, $K_1, \ldots, K_n$ are precisely those lines $\lambda(p) = \Gamma_1(p) \cap \cdots \cap \Gamma_n(p)$ for $p \in \ell$.

Since $S_{1,n-1}$ has degree n, a general $(n-1)$-plane K_{n+1} meets $S_{1,n-1}$ in n distinct points, each determining a line $\lambda(p)$ which meets $\ell, K_1, \ldots, K_{n+1}$. If k is finite with q elements, there are only $q+1$ lines $\lambda(p)$ defined over k. Thus it is necessary that $q \geq n-1$ to solve this problem over k. We show this condition suffices.

All rational normal surface scrolls are projectively equivalent, ([7], §9), so we may assume that $S_{1,n-1}$ has the following standard form. Let $x_1, x_2, y_1, \ldots, y_n$ be coordinates for $\mathbf{P}^{n+1}$ where ℓ has equation $y_1 = \cdots = y_n = 0$. Then for $p = [a, b, 0, \ldots, 0] \in \ell$, we have that $\lambda(p)$ is the linear span of p and the point $[0, 0, a^{n-1}, a^{n-2}b, \ldots, ab^{n-2}, b^{n-1}]$.

Let $p_1, \ldots, p_n \in \mathbf{P}^1$ be distinct, and let $F = \sum_{i=0}^n A_i b^i a^{n-i}$ be a form on $\mathbf{P}^1$ vanishing at $p_1, \ldots, p_n$. Define K_{n+1} by the vanishing of the two linear forms

$$\Lambda_1 \ : \ x_2 - y_1 \qquad \Lambda_2 \ : \ A_0 x_1 + A_1 y_1 + \cdots + A_n y_n.$$

The intersection of $S_{1,n-1}$ and the hyperplane defined by Λ_1 is the rational normal curve

$$\psi : [a, b] \longmapsto [a^n, a^{n-1}b, a^{n-1}b, a^{n-2}b^2, \ldots, ab^{n-1}, b^n].$$

Since $\psi^*(\Lambda_2)$ is the form F, we see that $\lambda(p_1), \ldots, \lambda(p_n)$ are the lines meeting each of $\ell, K_1, \ldots, K_{n+1}$. Thus, if k has at least $n-1$ elements, this enumerative problem may be solved over k.

These two families are the only non-trivial examples of Schubert-type enumerative problems for which we know an explicit description of their solutions. Each of these problems can be solved over any field k where $\#\mathbf{P}^1_k$ exceeds the number of solutions. In particular, they may be solved over $\mathbf{Q}$. It would be interesting to find exact solutions to other enumerative problems in order to test whether these observations hold more generally.

8.3. **Arrangements over finite fields.** In §2.6 we remarked it is possible to construct arrangements over some finite fields. Here we show how. Recall that an arrangement $\mathcal{F}$ in $\mathbf{P}^n$ is a collection of $2n-3$ distinct hyperplanes $H_2, \ldots, H_{2n-2}$ in $\mathbf{P}^n$ such that for $m = 2, \ldots, n$, the following two conditions hold:

(A.1) $F_m := H_{\{2,3,\ldots,2m-2\}}$ has codimension m, and if $m \neq n$, then $F_m \subset H_{2m-1}$ but $F_m \not\subset H_{2m}$. Thus $F_{m+1} = F_m \cap H_{2m}$, and if $A \subset \{2, \ldots, 2m-1\}$, then $H_A \not\subset H_{2m}$.

(A.2) If $A \subset \{2, \ldots, 2m-2\}$, and $F_m \neq H_A$, then $H_A \not\subset H_{2m-1}$.

We characterize subsets A of $\{2, \ldots, 2n-2\}$ which give distinct subspaces H_A. This is used to estimate the order of a field k sufficient to construct an arrangement.

8.4. **Lemma.** *Let $\mathcal{F}$ be an arrangement and A a subset of $\{2, \ldots, 2n-2\}$ with $\dim H_A = n-m$. Then there exists $B = \{b_1 < \cdots < b_m\} \subset \{2, \ldots, 2n-2\}$ with $H_B = H_A$, where B satisfies*

(A.3) *Either $b_i \geq 2i$, for $i = 1, \ldots, m$, or else $b_1 = 2$, $b_2 = 3$, and there is some*

$$2 \leq j \leq m \quad \textit{with} \begin{cases} b_i = 2i-2 & \textit{for } 2 < i \leq j \\ b_i \geq 2i & \textit{for } j < i \end{cases} .$$

Moreover, if B' is another subset satisfying (A.3), *then* $\dim H_{B'} = n-m$ *and* $H_B = H_{B'}$ *if and only if* $B = B'$.

Proof: Let $A \subset \{2, \ldots, 2n-2\}$ and suppose $A = \{a_1 < \cdots < a_k\}$. Discarding, if necessary, elements a_i of A such that $H_{\{a_1,\ldots,a_{i-1}\}} \subset H_{a_i}$, we may assume that $m = k$ and, for $1 \leq i \leq m$, $\dim H_{\{a_1,\ldots,a_i\}} = n-i$. If $a_i \geq 2i$ for all i, set $B = A$.

Otherwise, let j be the largest index with $a_j < 2j$. Since $\{a_1, \ldots, a_j\}$ is a subset of $\{2, \ldots, 2j-1\}$, (A.1) implies $F_j = H_{\{2,\ldots,2j-1\}} \subset H_{\{a_1,\ldots,a_j\}}$.

But $\dim F_j = n-j = \dim H_{\{a_1,\ldots,a_j\}}$, which shows $F_j = H_{\{a_1,\ldots,a_j\}}$. Define $b_1,\ldots,b_m$ by $b_1 = 2$, $b_2 = 3$, and for $i > 2$,

$$b_i \;=\; \begin{cases} 2i-2 & \text{if } 2 < i \leq j \\ a_i & \text{if } j < i \end{cases}.$$

The $H_B = H_A$, as an easy induction and (A.1) show $H_{\{b_1,\ldots,b_j\}} = F_j$.

If $B' \subset \{2,\ldots,2n-2\}$ satisfies (A.3), then arguing as in the proof of Lemma 2.9(3) shows $H_B = H_{B'} \Rightarrow B = B'$. ∎

We estimate the size of a finite field k sufficient to construct an arrangement.

Theorem H. *There exists an arrangement in $\mathbf{P}^n_k$ if the order of k is at least*

$$\frac{(2n-4)!}{(n-2)!(n-1)!} \;+\; \sum_{i=0}^{n-4} \frac{(2i)!}{i!(i+1)!}.$$

Proof: Consider the problem of inductively constructing an arrangement in $\mathbf{P}^n_k$. Let $2 \leq m \leq n-2$ and suppose we have found $H_2,\ldots,H_{2m-1}$ satisfying (A.1) and (A.2). This defines $F_m = H_2 \cap \cdots \cap H_{2m-1}$. Let H_{2m} be any hyperplane in $\mathbf{P}^n_k$ not containing F_m. Set $F_{m+1} = F_m \cap H_{2m}$. Then, we must find a hyperplane $H_{2m+1} \subset \mathbf{P}^n_k/F_{m+1}$ with $H_B \not\subset H_{2m+1}$, for all $B \subset \{2,\ldots,2m-2\}$ satisfying the condition (A.3) of Lemma 8.4. We investigate when this is possible. Let $\mathcal{S}_m$ be the set of all B satisfying (A.3), with $B \subset \{2,\ldots,2m-2\}$.

Let $\check{\mathbf{P}}^m$ be the set of hyperplanes in $\mathbf{P}^n$ defined over k which contain F_{m+1}. Every codimension m subspace H_B containing F_{m+1} determines a hyperplane $\check{H}_B$ in $\check{\mathbf{P}}^m$ consisting of those hyperplanes in $\mathbf{P}^n$ containing H_B. Moreover, $\check{H}_B$ is defined over k whenever H_B is defined over k. Thus there exists a hyperplane H_{2m+1} in $\mathbf{P}^n_k/F_{m+1}$ with $H_B \not\subset H_{2m+1}$ for all $B \in \mathcal{S}_m$, if, as sets of k-valued points,

$$X \;=\; \check{\mathbf{P}}^m \setminus \bigcup_{B\in\mathcal{S}_m} \check{H}_B \;\neq\; \emptyset.$$

We estimate $\#\bigcup_{B\in\mathcal{S}_m} \check{H}_B$, to show that the hypotheses imply $X \neq \emptyset$. We claim that for each $m = 0,\ldots,n-2$,

$$s_m \;:=\; \#\mathcal{S}_m \;=\; \frac{(2m)!}{m!(m+1)!} \;+\; \sum_{i=0}^{m-2} \frac{(2i)!}{i!(i+1)!}.$$

Granting this, suppose k has $q \geq s_{n-2}$ elements. Since $\check{\mathbf{P}}^m$ has $(q^{m+1}-1)/(q-1)$ elements and each $\check{H}_B$ has $(q^m-1)/(q-1)$ elements, X is non-empty if $q^{m+1}-1>(q^m-1)s_m$. This holds because

$$\left\lfloor\frac{q^{m+1}-1}{q^m-1}\right\rfloor \geq q \geq s_{n-2} \geq s_m,$$

by our assumption on $q=\#k$.

To enumerate $\mathcal{S}_m$, let $B=\{b_1<\cdots<b_m\}\in\mathcal{S}_m$. Then there is some $j\in\{0,2,\ldots,m\}$ such that $b_i\geq 2i$ for $i>j$, but $b_i<2i$ for $i\leq j$. If we set $c_i=b_{j+i}-2j$ for $i=1,\ldots,m-j$, then $c_1,\ldots,c_{m-j}$ is the second row of a Young tableau of shape $(m-j,m-j)$. This is because for each i, $2i\leq c_i\leq 2(m-j)$. Conversely, if $c_1,\ldots,c_{m-j}$ is the second row of a tableau of shape $(m-j,\, m-j)$, then

$$\{2<3<\cdots<2j-2<c_1+2j<\cdots<c_{m-j}+2j\}\in\mathcal{S}_m.$$

Let $\mathcal{T}_s$ be the set of tableaux of shape (s,s). These arguments show there is a bijection

$$\mathcal{S}_m \longleftrightarrow \mathcal{T}_m\cup\mathcal{T}_{m-2}\cup\mathcal{T}_{m-3}\cup\ldots\cup\mathcal{T}_0.$$

By the hook length formula of Frame, Robinson, and Thrall [3], $\#\mathcal{T}_s=\frac{(2s)!}{s!(s+1)!}$. Thus $\#\mathcal{S}_m=\frac{(2m)!}{m!(m+1)!}+\sum_{i=0}^{m-2}\frac{(2i)!}{i!(i+1)!}$. ∎

This result is not the best possible, as we used a crude estimate to count the set X. For $\mathbf{P}^4$, Theorem H gives $\#k\geq 3$ and for $\mathbf{P}^5$, $\#k\geq 7$. However, the arrangements in §2.8 are defined over smaller fields. The arrangement in $\mathbf{P}^4$ is defined over the field with two elements, while the arrangement in $\mathbf{P}^5$ may be defined over the field with four elements: If $\mathbf{F}_2$ is the field with 2 elements so that $\mathbf{F}_4=\mathbf{F}_2[x]/(x^2+x+1)$ is the field with 4 elements, then changing the coefficients 2 and 3 in (1) of §2.8 to x and $x+1$, gives an arrangement in $\mathbf{P}^5$ defined over $\mathbf{F}_4$.

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