A MONOID FOR THE GRASSMANNIAN BRUHAT ORDER

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Abstract. Structure constants for the multiplication of Schubert polynomials by Schur symmetric polynomials are related to the enumeration of chains in a new partial order on $\mathcal{S}_\infty$, the Grassmannian Bruhat order. Here we present a monoid $\mathcal{M}$ related to this order. We develop a notion of reduced sequences for $\mathcal{M}$ and show that $\mathcal{M}$ is analogous to the nil-Coxeter monoid for the weak order on $\mathcal{S}_\infty$.

1. Introduction

Let $\mathcal{S}_\infty$ denote the infinite symmetric group consisting of permutations of $\{1, 2, \ldots\}$ which fix all but finitely many numbers. In their approach to the Schubert calculus for flag manifolds, Lascoux and Schützenberger [7, 8, 9, 10] defined Schubert polynomials $\mathfrak{S}_u \in \mathbb{Z}[x_1, x_2, \ldots]$, a homogeneous basis indexed by permutations $u \in \mathcal{S}_\infty$. By construction, the degree of $\mathfrak{S}_u$ is the length, $\ell(u)$, of u. We refer the reader to [11] for an interesting detailed account of Schubert polynomials.

It is a famous open problem to understand the multiplicative structure constants for the Schubert polynomials. From algebraic geometry, the structure constants c^w_{uv} defined by the identity

$$\mathfrak{S}_u \mathfrak{S}_v \;=\; \sum_{w \in \mathcal{S}_\infty} c^w_{uv} \mathfrak{S}_w$$

are positive integers, and in some special cases they are the Littlewood-Richardson coefficients. A combinatorial construction for the c^w_{uv} is not known.

It is believed that c^w_{uv} counts the number of chains from u to w in the Bruhat order which satisfy conditions imposed by v [2]. In particular, if v is a Grassmannian permutation with descent in k, then one can restrict the chains to a suborder: the k-Bruhat order $\leq_k$ on $\mathcal{S}_\infty$ [9, 13, 2]. In [2], a study of $\leq_k$ leads to a new partial order $\preceq$ on $\mathcal{S}_\infty$ which we call the *Grassmannian Bruhat* order. This order is ranked and has the property that a nonempty interval $[u, w]_k$ in a k-Bruhat order is isomorphic to the interval $[1, wu^{-1}]_\preceq$ in the Grassmannian Bruhat order (independent of k). As a special case, every interval in Young's lattice is an interval in this Grassmannian Bruhat order. The Grassmannian Bruhat order is by definition linked to the structure of the flag manifolds considered as a module over the ring of symmetric polynomials, but this order is combinatorially interesting on its own. The aim of this paper is to present a monoid $\mathcal{M}$ that describes the chain structure of this order.

In Section 3, we sketch the main features of the Grassmannian Bruhat order $\preceq$ but the detailed background is found in [2]. We recall here the definition of the order $\preceq$ and its rank

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function $\ell_{\mathbf{u}}$. For $\zeta \in \mathcal{S}_\infty$, let $up(\zeta) = \{j : \zeta^{-1}(j) < j\}$ and let $dw(\zeta) = \{j : \zeta^{-1}(j) > j\}$. Set $\ell_{\mathbf{u}}(\zeta)$ to be

$$\begin{aligned}&|\{(i,j) \in up(\zeta) \times dw(\zeta) : i > j\}| - |\{(\zeta(i),\zeta(j)) \in up(\zeta) \times dw(\zeta) : i > j\}|\\ &\quad -|\{(\zeta(i),\zeta(j)) \in up(\zeta)^{\times 2} : i < j \text{ and } \zeta(i) > \zeta(j)\}|\\ &\quad -|\{(\zeta(i),\zeta(j)) \in dw(\zeta)^{\times 2} : i < j \text{ and } \zeta(i) > \zeta(j)\}|.\end{aligned}$$

Definition 1.1 (Grassmannian Bruhat Order on $\mathcal{S}_\infty$)**.** *$\eta \preceq \zeta$ if and only if*

(1) $\alpha \leq \eta(\alpha) \leq \zeta(\alpha)$ *for* $\alpha \in \zeta^{-1}\big(up(\zeta)\big)$,
(2) $\alpha \geq \eta(\alpha) \geq \zeta(\alpha)$ *for* $\alpha \in \zeta^{-1}\big(dw(\zeta)\big)$,
(3) $\big(\eta(\alpha) < \eta(\beta) \implies \zeta(\alpha) < \zeta(\beta)\big)$ *for* $\alpha < \beta \in \zeta^{-1}(up(\zeta))$ *or* $\alpha < \beta \in \zeta^{-1}(dw(\zeta))$.

We consider the monoid $\mathcal{M}$ that has a $\mathbf{0}$ and generators $\mathbf{u}_{\alpha\beta}$ indexed by integers $0 < \alpha < \beta$, subject to the relations:

$$\begin{array}{lrcll}
(1) & \mathbf{u}_{\beta\gamma}\mathbf{u}_{\gamma\delta}\mathbf{u}_{\alpha\gamma} & \equiv & \mathbf{u}_{\beta\delta}\mathbf{u}_{\alpha\beta}\mathbf{u}_{\beta\gamma}, & \text{if } \alpha<\beta<\gamma<\delta,\\
(2) & \mathbf{u}_{\alpha\gamma}\mathbf{u}_{\gamma\delta}\mathbf{u}_{\beta\gamma} & \equiv & \mathbf{u}_{\beta\gamma}\mathbf{u}_{\alpha\beta}\mathbf{u}_{\beta\delta}, & \text{if } \alpha<\beta<\gamma<\delta,\\
(3) & \mathbf{u}_{\alpha\beta}\mathbf{u}_{\gamma\delta} & \equiv & \mathbf{u}_{\gamma\delta}\mathbf{u}_{\alpha\beta}, & \text{if } \beta<\gamma \text{ or } \alpha<\gamma<\delta<\beta,\\
(4) & \mathbf{u}_{\alpha\gamma}\mathbf{u}_{\beta\delta} & \equiv & \mathbf{u}_{\beta\delta}\mathbf{u}_{\alpha\gamma} \ \equiv\ \mathbf{0}, & \text{if } \alpha\leq\beta<\gamma\leq\delta,\\
(5) & \mathbf{u}_{\beta\gamma}\mathbf{u}_{\alpha\beta}\mathbf{u}_{\beta\gamma} & \equiv & \mathbf{u}_{\alpha\beta}\mathbf{u}_{\beta\gamma}\mathbf{u}_{\alpha\beta} \ \equiv\ \mathbf{0}, & \text{if } \alpha<\beta<\gamma.
\end{array} \tag{1.1}$$

The relation between $\mathcal{M}$ and the order $\preceq$ on $\mathcal{S}_\infty$ is obtained via a faithful representation of $\mathcal{M}$ as linear operators on the group algebra $\mathbb{Q}\mathcal{S}_\infty$. Let $(\alpha\ \ \beta) \in \mathcal{S}_\infty$ be the transposition that interchanges α and β. We define the linear operator $\hat{\mathbf{u}}_{\alpha\beta}$ by

$$\begin{array}{rccl}
\hat{\mathbf{u}}_{\alpha\beta}: & \mathbb{Q}\mathcal{S}_\infty & \longrightarrow & \mathbb{Q}\mathcal{S}_\infty,\\
& \zeta & \longmapsto & \begin{cases}(\alpha\ \ \beta)\zeta & \text{if } \ell_{\mathbf{u}}\big((\alpha\ \ \beta)\zeta\big) = \ell_{\mathbf{u}}(\zeta)+1,\\ 0 & \text{otherwise.}\end{cases}
\end{array} \tag{1.2}$$

Theorem 1.2.

(a) *The map $\ell_{\mathbf{u}} : \mathcal{S}_\infty \to \mathbb{N}$ is well defined by $\ell_{\mathbf{u}}(\zeta) = \ell(\zeta u) - \ell(u)$ for any u and k such that $u \leq_k \zeta u$.*
(b) *The operators $\hat{\mathbf{u}}_{\alpha\beta}$ satisfy the relations (1.1), and a composition of operators is characterized by its value at the identity. That is $\hat{\mathbf{u}}_{\alpha'_m\beta'_m} \cdots \hat{\mathbf{u}}_{\alpha'_1\beta'_1} = \hat{\mathbf{u}}_{\alpha_n\beta_n} \cdots \hat{\mathbf{u}}_{\alpha_1\beta_1}$ if and only if $\hat{\mathbf{u}}_{\alpha'_m\beta'_m} \cdots \hat{\mathbf{u}}_{\alpha'_1\beta'_1}1 = \hat{\mathbf{u}}_{\alpha_n\beta_n} \cdots \hat{\mathbf{u}}_{\alpha_1\beta_1}1$.*
(c) *For $\mathbf{x} = \mathbf{u}_{\alpha_n\beta_n} \cdots \mathbf{u}_{\alpha_2\beta_2}\mathbf{u}_{\alpha_1\beta_1} \in \mathcal{M}$, the map $\mathbf{x} \mapsto \hat{\mathbf{x}} = \hat{\mathbf{u}}_{\alpha_n\beta_n} \cdots \hat{\mathbf{u}}_{\alpha_2\beta_2}\hat{\mathbf{u}}_{\alpha_1\beta_1}$ is a faithful representation of $\mathcal{M}$.*
(d) *The map $\mathcal{M} \to \mathcal{S}_\infty \cup \{\mathbf{0}\}$, well defined by $\mathbf{x} \mapsto \hat{\mathbf{x}}1$, is a bijection.*
(e) *The Grassmannian Bruhat order $\preceq$ on $\mathcal{S}_\infty$ is ranked by $\ell_{\mathbf{u}}$. We have $\eta \preceq \zeta$ if and only if there exists $\mathbf{x} \in \mathcal{M}$ such that $\zeta = \hat{\mathbf{x}}\eta$. The order $\preceq$ satisfies the property: $[u, \zeta u]_k \cong [1, \zeta]_\preceq$ whenever $u \leq_k \zeta u$. In particular $[\eta, \zeta]_\preceq \cong [1, \zeta\eta^{-1}]_\preceq$ whenever $\eta \preceq \zeta$.*
(f) *The set $R_{\mathbf{u}}(\zeta) = \{\hat{\mathbf{x}} : \hat{\mathbf{x}}1 = \zeta\}$ is in bijection with the set of all maximal chains in $[1, \zeta]_\preceq$.*

We call the elements of $R_{\mathbf{u}}(\zeta)$ the $\mathbf{u}$-reduced sequences of ζ. Parts (a) and (e) of Theorem 1.2 were obtained in §3.2 of [2]. We have included them for completeness. In Section 3, we show the remaining parts. In Section 2, we emphasize the parallel between Theorem 1.2 and a similar classical results on the weak order of $\mathcal{S}_\infty$ and the nil-Cotexer monoid.

Recall [11] that the Schur polynomial $S_\lambda(x_1, x_2, \ldots, x_k) = \mathfrak{S}_{v(\lambda,k)}$ for a unique Grassmannian permutation $v(\lambda, k)$. In Theorem E of [2], we have shown that if $c^w_{uv(\lambda,k)} \neq 0$, then $c^w_{uv(\lambda,k)}$ depends only on λ and $\zeta = wu^{-1}$. We can thus define constants c^ζ_λ such that $c^w_{uv(\lambda,k)} = c^{wu^{-1}}_\lambda$ whenever $u \leq_k w$. We have (*cf.* Prop. 1.1 [2])

$$|R_{\mathbf{u}}(\zeta)| = \sum_\lambda f^\lambda c^\zeta_\lambda, \tag{1.3}$$

where f^λ is the number of standard Young tableaux of shape λ. In Section 4 we give a description of the constant $c^w_{uv(\lambda,k)}$ using elements of $R_{\mathbf{u}}(\zeta)$. This description will be helpful in some subsequent work [3, 4].

2. Orders and monoids on $\mathcal{S}_\infty$

Let $\ell(u)$ denote the length of a permutation $u \in \mathcal{S}_\infty$. The weak order $\leq_{wk}$ on $\mathcal{S}_\infty$ is the transitive closure of the following cover relation: for $u, w \in \mathcal{S}_\infty$, we say that w covers u in the weak order if $\ell(w) = \ell(u) + 1$ and wu^{-1} is a simple transposition $(\alpha\ \alpha{+}1)$. Maximal chains from the identity to $w \in \mathcal{S}_\infty$ correspond to reduced sequences for w. The nil-Coxeter monoid $\mathcal{N}$ plays a role [8] in studying reduced sequences. The monoid $\mathcal{N}$ has a $\mathbf{0}$ and generators $\mathbf{u}_i$ indexed by integers $i > 0$, subject to the nil-Coxeter relations:

$$\begin{array}{rcll} \mathbf{u}_\alpha \mathbf{u}_{\alpha+1} \mathbf{u}_\alpha & \equiv & \mathbf{u}_{\alpha+1} \mathbf{u}_\alpha \mathbf{u}_{\alpha+1}, & \\ \mathbf{u}_\alpha \mathbf{u}_\beta & \equiv & \mathbf{u}_\beta \mathbf{u}_\alpha, & \text{if } |\alpha - \beta| > 1, \\ \mathbf{u}_\alpha \mathbf{u}_\alpha & \equiv & \mathbf{0}. & \end{array} \tag{2.1}$$

There is a faithful representation of $\mathcal{N}$ as linear operators on the group algebra $\mathbb{Q}\mathcal{S}_\infty$. For this, consider the linear map $\hat{\mathbf{u}}_\alpha : \mathbb{Q}\mathcal{S}_\infty \to \mathbb{Q}\mathcal{S}_\infty$ defined by

$$\zeta \quad \longmapsto \quad \begin{cases} (\alpha\ \ \alpha{+}1)\zeta & \text{if } \ell\big((\alpha\ \ \alpha{+}1)\zeta\big) = \ell(\zeta) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition is a reformulation of well known results of J. Tits about reduced sequences of a permutation and the weak order. See [11] for proofs.

Proposition 2.1.

(a) *The map* $\ell : \mathcal{S}_\infty \to \mathbb{N}$ *is well defined.*

(b) *The operators* $\hat{\mathbf{u}}_\alpha$ *satisfy the relations (2.1), and a composition of operators is characterized by its value at the identity. That is* $\hat{\mathbf{u}}_{\alpha_n} \cdots \hat{\mathbf{u}}_{\alpha_1} = \hat{\mathbf{u}}_{\beta_m} \cdots \hat{\mathbf{u}}_{\beta_1}$ *if and only if* $\hat{\mathbf{u}}_{\alpha_n} \cdots \hat{\mathbf{u}}_{\alpha_1} 1 = \hat{\mathbf{u}}_{\beta_m} \cdots \hat{\mathbf{u}}_{\beta_1} 1$.

(c) *For* $\mathbf{x} = \mathbf{u}_{\alpha_n} \cdots \mathbf{u}_{\alpha_2} \mathbf{u}_{\alpha_1} \in \mathcal{N}$, *the map* $\mathbf{x} \mapsto \hat{\mathbf{x}} = \hat{\mathbf{u}}_{\alpha_n} \cdots \hat{\mathbf{u}}_{\alpha_2} \hat{\mathbf{u}}_{\alpha_1}$ *is a faithful representation of* $\mathcal{N}$.

(d) *The map* $\mathcal{N} \to \mathcal{S}_\infty \cup \{\mathbf{0}\}$, *well defined by* $\mathbf{x} \mapsto \hat{\mathbf{x}}1$, *is a bijection.*

(e) *The weak order* $\leq_{wk}$ *on* $\mathcal{S}_\infty$ *is ranked by* ℓ. *We have* $u \leq_{wk} w$ *if and only if there exists* $\mathbf{x} \in \mathcal{N}$ *such that* $w = \hat{\mathbf{x}}u$. *Also* $[\eta, \zeta]_{wk} \cong [1, \zeta\eta^{-1}]_{wk}$ *whenever* $\eta \leq_{wk} \zeta$.

(f) *The set* $R(w) = \{\hat{\mathbf{x}} : \hat{\mathbf{x}}1 = w\}$ *is in bijection with the set of all maximal chains in* $[1, w]_{wk}$. *The elements of* $R(w)$ *are the reduced sequences of* w.

At this point we note the striking resemblance between Theorem 1.2 and Proposition 2.1. The proof of Proposition 2.1 relies on the understanding of reduced sequences. For Theorem

1.2, the order $\preceq$ is new and its chains have not been studied previously. We develop the elementary theory of reduced sequences for $\preceq$.

We note that not all orders on $\mathcal{S}_\infty$ have such a simple monoid. In particular, the Bruhat order $\leq$ on $\mathcal{S}_\infty$ has no known monoid. Recall that w covers u in the Bruhat order if $\ell(w) = \ell(u)+1$ and wu^{-1} is a transposition $(\alpha\ \beta)$. In fact, very little is known about the problem of chain enumeration for the Bruhat order. We believe that a monoid for the Bruhat order would not satisfy conditions as simple as those of Theorem 1.2 and Proposition 2.1.

The monoid structure for the weak order was a key factor in the following results. Under the nil-Coxeter-Knuth relations

$$\begin{gathered}
\mathbf{u}_\alpha\mathbf{u}_{\alpha+1}\mathbf{u}_\alpha \equiv \mathbf{u}_{\alpha+1}\mathbf{u}_\alpha\mathbf{u}_{\alpha+1},\\
\mathbf{u}_\beta\mathbf{u}_\gamma\mathbf{u}_\alpha \equiv \mathbf{u}_\beta\mathbf{u}_\alpha\mathbf{u}_\gamma \quad\text{and}\quad \mathbf{u}_\alpha\mathbf{u}_\gamma\mathbf{u}_\beta \equiv \mathbf{u}_\gamma\mathbf{u}_\alpha\mathbf{u}_\beta, \qquad \text{if } \alpha<\beta<\gamma,\\
\mathbf{u}_\alpha\mathbf{u}_\alpha \equiv \mathbf{0},
\end{gathered} \tag{2.2}$$

the set of all reduced sequences $R(w)$ for a permutation $w \in \mathcal{S}_\infty$ is refined into classes, called Coxeter-Knuth cells, indexed by some semi-standard tableaux. The cardinality of a cell is the number of standard tableaux of the same shape as the cell's index [5, 8, 14]. This decomposition suggests an action of the symmetric group on $R(w)$. The symmetric function corresponding to such an action is the function F_w introduced by Stanley in [14]. Equation (1.3) suggests the possibility of similar structure for the monoid $\mathcal{M}$ and relations (1.1).

3. k-Bruhat orders and the monoid $\mathcal{M}$

Monk's rule [11] determines the multiplicative structure of Schubert polynomials:

$$\mathfrak{S}_u(x_1+x_2+\cdots+x_k) = \sum_{\substack{a\leq k<b\\ \ell(u(a\,b))=\ell(u)+1}} \mathfrak{S}_{u(a\,b)}.$$

Successive applications of this give

$$\mathfrak{S}_u(x_1+x_2+\cdots+x_k)^n = \sum_{\substack{w\,\in\,\mathcal{S}_\infty\\ \ell(w)=\ell(u)+n}} \gamma(u,w,k)\mathfrak{S}_w,$$

where $\gamma(u,w,k)$ counts the sequences of transpositions $(a_1\,b_1)$, $(a_2\,b_2)$, $\ldots$, $(a_n\,b_n)$ such that $w = u(a_1\,b_1)(a_2\,b_2)\cdots(a_n\,b_n)$ and, for all r, we have $a_r \leq k < b_r$ with

$$\ell\big(u(a_1\,b_1)(a_2\,b_2)\cdots(a_{r-1}\,b_{r-1})\big) = \ell\big(u(a_1\,b_1)(a_2\,b_2)\cdots(a_r\,b_r)\big)+1.$$

On the other hand,

$$(x_1+x_2+\cdots+x_k)^n = \sum_\lambda f^\lambda S_\lambda(x_1,x_2,\ldots,x_k),$$

where $S_\lambda(x_1,x_2,\ldots,x_k)$ is the Schur polynomial indexed by a partition λ of n. There is a unique Grassmannian permutation $v(\lambda,k)$ such that the Schubert polynomial $\mathfrak{S}_{v(\lambda,k)}$ is equal to the Schur polynomial $S_\lambda(x_1,x_2,\ldots,x_k)$ [11]. Hence

$$\mathfrak{S}_u(x_1+x_2+\cdots+x_k)^n = \sum_\lambda f^\lambda \mathfrak{S}_u\mathfrak{S}_{v(\lambda,k)} = \sum_w\left(\sum_\lambda f^\lambda c^w_{u\,v(\lambda,k)}\right)\mathfrak{S}_w,$$

and we have

$$\sum_{\lambda} f^{\lambda} c^{w}_{u\,v(\lambda,k)} = \gamma(u,w,k).$$

This suggests that we should study the partial order defined by the relation: $u \leq_k w$ if and only if $\gamma(u,w,k) > 0$. Equivalently, this is the partial order with covering relation given by the index of summation in Monk's rule. We call this suborder of the Bruhat order the *k-Bruhat order*. We denote by $[u,w]_k$ the interval from u to w in the k-Bruhat order. Hence $\gamma(u,w,k)$ is the number of maximal chains in $[u,w]_k$.

These cover relations give some invariants of the k-Bruhat order. For example, consider the following maximal chain in the 3-Bruhat order:

$$(3,1,5,2,6,4) \leq_3 (3,1,6,2,5,4) \leq_3 (3,2,6,1,5,4) \leq_3 (3,5,6,1,2,4).$$

Here and after, we use a finite list $\big(w(1), w(2), \ldots, w(n)\big)$ to denote any permutation $w \in \mathcal{S}_n \subset \mathcal{S}_\infty$. In this example, the first three entries of the permutations do not decrease and the other entries do not increase. Also, the second and third entries remain in the same relative order for all permutations in the chain. This leads to a characterization of the k-Bruhat order based on such invariants.

Proposition 3.1 (Theorem A of [2]). *For $u, w \in \mathcal{S}_\infty$, $u \leq_k w$ if and only if*

(1) $u(i) \leq w(i)$ *for* $i \leq k$,
(2) $u(i) \geq w(i)$ *for* $i > k$,
(3) $(u(i) < u(j) \implies w(i) < w(j))$ *for* $i < j \leq k$ *or* $k < i < j$.

The sufficiency of these conditions follows from the existence of a specific maximal chain in the interval $[u,w]_k$. We call it the CM-chain of $[u,w]_k$.

Definition 3.2 (CM-chain). *For $u <_k w$, the CM-chain of the interval $[u,w]_k$ is recursively defined as follows:*

- *If $\ell(w) = \ell(u)+1$ then the unique chain $u <_k w$ is the CM-chain of $[u,w]_k$.*
- *If $\ell(w) > \ell(u)+1$, let $a \leq k < b$ be the unique integers such that*
  I *$u(a) < w(a)$ and $w(a) = \max\{w(j) : j \leq k, u(j) < w(j)\}$,*
  II *$u(b) > u(a) \geq w(b)$ and $w(b) = \min\{w(j) : j > k, u(j) > u(a) \geq w(j)\}$.*
  Let $u_1 = u(a\,b)$. The CM-chain of $[u,w]_k$ is
  $$u = u_0 <_k u_1 <_k u_2 <_k \cdots <_k u_n = w,$$
  where $u_1 <_k u_2 <_k \cdots <_k u_n$ is the CM-chain of $[u_1,w]_k$.

It is not obvious that conditions **I** and **II** define unique integers $a \leq k < b$. We refer the reader to §3.1 of [2] for a complete proof of this fact. The symmetry in the conditions (1)-(3) of Proposition 3.1 implies the following lemma.

Lemma 3.3. *Let m be any integer such that $u, w \in \mathcal{S}_m$. Let ω_0 denote the longest element $(m, m-1, \ldots, 1)$ of $\mathcal{S}_m$. Then the map $\Omega_m : \mathcal{S}_m \to \mathcal{S}_m$ defined by $\Omega_m(u) = \omega_0 u \omega_0$ is an order preserving involution. That is*

$$u \leq_k w \qquad \Longleftrightarrow \qquad \Omega_m(u) \leq_{m-k} \Omega_m(w).$$

Lemma 3.3 suggests the definition of another specific maximal chain in the interval $[u,w]_k$ image of the CM-chain under the map Ω.

Definition 3.2$'$ (DCM-chain) *The DCM-chain is obtained as in Definition 3.2, replacing* **I** *and* **II** *by:*

I$'$ $u(b) > w(b)$ *and* $w(b) = \min\{w(j) : j > k, u(j) > w(j)\}$,

II$'$ $u(a) < u(b) \leq w(a)$ *and* $w(a) = \max\{w(j) : j \leq k, u(j) < u(b) \leq w(j)\}$.

For example, if $u = (2,1,6,4,3,5)$ and $w = (4,5,6,1,2,3)$, the first step of the procedure for the CM-chain of $[u,w]_3$ gives us $(a,b) = (2,4)$. The full chain is given below, written from bottom to top.

$(4,5,6,1,2,3)$	$(4,5,6,1,2,3)$	$(4,5,6,1,2,3)$
$(3,5,6,1,2,4)$	$(4,3,6,1,2,5)$	$(3,5,6,1,2,4)$
$(2,5,6,1,3,4)$	$(4,1,6,3,2,5)$	$(3,4,6,1,2,5)$
$(2,4,6,1,3,5)$	$(3,1,6,4,2,5)$	$(2,4,6,1,3,5)$
$(2,1,6,4,3,5)$	$(2,1,6,4,3,5)$	$(2,1,6,4,3,5)$
CM-Chain	A Maximal Chain	DCM-Chain

Consider a maximal maximal chain of $[u,w]_k$,

$$u = u_0 <_k u_1 <_k u_2 <_k \cdots <_k u_n = w,$$

where $u_{i+1} = u_i(a_i\, b_i)$. If this chain is the CM-chain, then $w(a_i) > w(a_j)$, or $w(a_i) = w(a_j)$ and $w(b_i) < w(b_j)$ for all $1 \leq i < j \leq n$. Our first objective is to generate all the maximal chains of $[u,w]_k$.

Proposition 3.4 (Theorem E of [2])**.** *For* $u \leq_k w$ *and* $u' \leq_{k'} w'$, *if* $wu^{-1} = w'(u')^{-1}$, *then* $v \mapsto vu^{-1}u'$ *induces* $[u,w]_k \cong [u',w']_{k'}$.

Proposition 3.5 (Theorem 3.1.5 of [2])**.** *Let* $w = \big(j_1, j_2, \ldots, j_k, \ldots\big)$ *where, to the right of* j_k, *we put the complement of* $up(\zeta)$ *in increasing order. We have that* $[\zeta^{-1}w, w]_k$ *is nonempty.*

With the above two propositions the function $\ell_{\mathbf{u}}$ becaumes clearer. The number k in Proposition 3.5 is the smallest possible for which $[u,w]_k$ is nonempty and $w = \zeta u$. The length difference $\ell(w) - \ell(u)$ is the same for all nonempty $[u,w]_k$ such that $w = \zeta u$. With this in mind, we can see that $\ell_{\mathbf{u}}(\zeta) = \ell(w) - \ell(u)$ for any nonempty $[u,w]_k$ such that $w = \zeta u$. Using Propositions 3.4 and 3.5, we see that $\eta \preceq \zeta$ if and only if there exists u and k such that $u \leq_k \eta u \leq_k \zeta u$. It follows from the definition that the order $\preceq$ is ranked by $\ell_{\mathbf{u}}$ and $[1, \zeta\eta^{-1}]_{\preceq} \cong [\eta, \zeta]_{\preceq}$ via the map $\xi \mapsto \xi\eta$.

The operators $\hat{\mathbf{u}}_{\alpha\beta}$ in (1.2) are defined so that $\hat{\mathbf{u}}_{\alpha\beta}\eta = \zeta$ if and only if ζ covers η in $\preceq$. In particular, nonzero compositions $\hat{\mathbf{x}} = \hat{\mathbf{u}}_{\alpha_n\beta_n} \cdots \hat{\mathbf{u}}_{\alpha_2\beta_2}\hat{\mathbf{u}}_{\alpha_1\beta_1}$ such that $\hat{\mathbf{x}}\eta = \zeta$ correspond bijectively to maximal chains in $[\eta,\zeta]_{\preceq}$:

$$\eta \preceq \hat{\mathbf{u}}_{\alpha_1\beta_1}\eta \preceq \hat{\mathbf{u}}_{\alpha_2\beta_2}\hat{\mathbf{u}}_{\alpha_1\beta_1}\eta \preceq \cdots \preceq \hat{\mathbf{x}}\eta = \zeta$$

We note that the isomorphism $[1, \zeta\eta^{-1}]_{\preceq} \cong [\eta, \zeta]_{\preceq}$ implies

$$\hat{\mathbf{x}}\eta = \zeta \qquad \Longleftrightarrow \qquad \hat{\mathbf{x}}1 = \zeta\eta^{-1}. \tag{3.1}$$

Hence the operator $\hat{\mathbf{x}}$ is completely defined by its value at 1.

The isomorphism $[1, wu^{-1}]_{\preceq} \cong [u,w]_k$ given by $\eta \mapsto \eta u$, induces an isomorphism on chains. Given a maximal chain

$$u = u_0 <_k u_1 <_k u_2 <_k \cdots <_k u_n = w \tag{3.2}$$

of $[u,w]_k$, we adopt the following conventions.

- Let $a_i \leq k < b_i$ be such that $u_{i+1} = u_i(a_i\, b_i)$.

- Let $\alpha_i = u_{i-1}(a_i)$ and $\beta_i = u_{i-1}(b_i)$. Hence $u_i = (\alpha_i\, \beta_i)u_{i-1}$.

Under the isomorphism above, this defines a unique (nonzero) composition

$$\hat{\mathbf{x}} = \hat{\mathbf{u}}_{\alpha_n\beta_n} \cdots \hat{\mathbf{u}}_{\alpha_2\beta_2}\hat{\mathbf{u}}_{\alpha_1\beta_1} \tag{3.3}$$

such that $wu^{-1} = \hat{\mathbf{x}}1$. Conversely, given a nonzero composition as in (3.3) such that $wu^{-1} = \hat{\mathbf{x}}1$, we define a unique maximal chain as in (3.2) where $u_i = (\hat{\mathbf{u}}_{\alpha_i\beta_i} \cdots \hat{\mathbf{u}}_{\alpha_1\beta_1}1)u$. This correspondence is used to encode maximal chains for the rest of the paper. Via this identification, we will refer to a nonzero composition $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}1 = wu^{-1}$ as a maximal chain of $[u, w]_k$.

In the next theorem we will show that every maximal chain of an interval is obtained from the CM-chain via the relation (1.1). For example, let $\zeta = (5, 4, 2, 1, 3)$. Proposition 3.5 gives $u = (2, 1, 4, 3, 5) \leq_2 (4, 5, 1, 2, 3) = \zeta u$. From Definition 3.2, the CM-chain is $\hat{\mathbf{u}}_{34}\hat{\mathbf{u}}_{23}\hat{\mathbf{u}}_{45}\hat{\mathbf{u}}_{14}$.

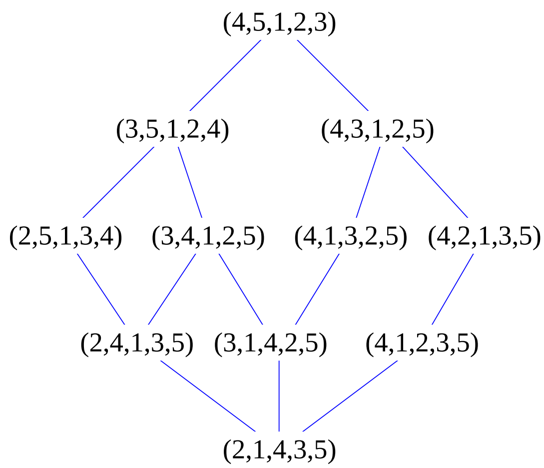

FIGURE 1. The interval $[(2, 1, 4, 3, 5),\ (4, 5, 1, 2, 3)]_2$.

Now if we apply the relations (1)-(3) of (1.1) to the CM-chain we get:

$$\hat{\mathbf{u}}_{34}\underline{\hat{\mathbf{u}}_{23}\hat{\mathbf{u}}_{45}}\hat{\mathbf{u}}_{14} \equiv \hat{\mathbf{u}}_{34}\hat{\mathbf{u}}_{45}\underline{\hat{\mathbf{u}}_{23}\hat{\mathbf{u}}_{14}} \equiv \underline{\hat{\mathbf{u}}_{34}\hat{\mathbf{u}}_{45}\hat{\mathbf{u}}_{14}}\hat{\mathbf{u}}_{23} \equiv \hat{\mathbf{u}}_{35}\underline{\hat{\mathbf{u}}_{13}\hat{\mathbf{u}}_{34}\hat{\mathbf{u}}_{23}} \equiv \hat{\mathbf{u}}_{35}\hat{\mathbf{u}}_{23}\hat{\mathbf{u}}_{12}\hat{\mathbf{u}}_{24}.$$

These are all the maximal chains of the interval $[u, w]_k$ as depicted in Figure 1. The first two equivalences are instances of the relation (3) of (1.1), the last two are instances of relations (1) and (2), respectively. The second chain is the DCM-chain.

Theorem 3.6. *If $u \leq_k w$, then any two maximal chains in $[u, w]_k$ are connected by a series of relations (1)-(3) of (1.1). Moreover, it is never possible to apply any of the relations (4) or (5) of (1.1) to a maximal chain.*

Proof We first show that any of the relations (1)-(3) of (1.1) that can be applied to a maximal chain

$$\hat{\mathbf{x}} = \hat{\mathbf{u}}_{\alpha_n\beta_n} \cdots \hat{\mathbf{u}}_{\alpha_2\beta_2}\hat{\mathbf{u}}_{\alpha_1\beta_1} \tag{3.4}$$

in $[u, w]_k$ results in another maximal chain. Moreover, the relations (4) and (5) can never be applied to this chain. Given the maximal chain (3.4), let $u_i = (\hat{\mathbf{u}}_{\alpha_i\beta_i} \cdots \hat{\mathbf{u}}_{\alpha_1\beta_1}1)u$ be as before, for $0 \leq i \leq n$. Then since $u_{i-1} \leq_k u_i$ is a cover,

(i) $u_i = (\alpha_i\ \beta_i)u_{i-1} = u_{i-1}(a_i\ b_i)$ with $a_i \leq k < b_i$.
(ii) If $\alpha_i < \gamma < \beta_i$, then $u_{i-1}^{-1}(\gamma) < a_i$ or $b_i < u_{i-1}^{-1}(\gamma)$.

Consider applying the relations (1.1) to a segment of length two in the chain (3.4). We may assume that the segment is $\hat{\mathbf{u}}_{\alpha_2\beta_2}\hat{\mathbf{u}}_{\alpha_1\beta_1}$. Suppose $\{\alpha_1,\beta_1\}\cap\{\alpha_2,\beta_2\}=\emptyset$, and assume $\alpha_1<\alpha_2$, as the other case is symmetric. There are three possible relative orders for the numbers $\alpha_1,\beta_1,\alpha_2$ and β_2. We consider each in turn. If $\alpha_1<\alpha_2<\beta_1<\beta_2$, the situation in relation (4) with strict inequalities, then condition (ii) for $i=1$ implies $a_2=u_0^{-1}(\alpha_2)<a_1$, and for $i=2$ implies $a_1=u_1^{-1}(\beta_1)<a_2$, a contradiction. Now suppose $\alpha_1<\beta_1<\alpha_2<\beta_2$ or $\alpha_1<\alpha_2<\beta_2<\beta_1$. An example of each case is found as a square in Figure 1. Then (i) and (ii) impose no additional conditions on a_1,a_2,b_1 and b_2, so $u_0\leq_k u_0(a_2\ b_2)\leq_k u_0(a_2\ b_2)(a_1\ b_1)=u_2$.

Suppose one of the relations (1) or (2) of (1.1) applies to a segment of length three. Again an example of each case is found as a hexagon in Figure 1. Both arguments are similar, so suppose that (1) applies. We have $\alpha<\beta<\gamma<\delta$ and the segment is $\hat{\mathbf{u}}_{\beta\gamma}\hat{\mathbf{u}}_{\gamma\delta}\hat{\mathbf{u}}_{\alpha\gamma}$. By condition (ii), the numbers α,β,γ, and δ appear in u in one of the following two orders

$$\big(\ldots,\beta,\ldots,\alpha,\ldots,\gamma,\ldots,\delta,\ldots\big) \quad \text{or} \quad \big(\ldots,\beta,\ldots,\alpha,\ldots,\delta,\ldots,\gamma,\ldots\big).$$

The argument in both case are similar. In the first case, the chain is

$$\begin{matrix}\big(\ldots,\gamma,\ldots,\delta,\ldots,\alpha,\ldots,\beta,\ldots\big)\\ \big(\ldots,\beta,\ldots,\delta,\ldots,\alpha,\ldots,\gamma,\ldots\big)\\ \big(\ldots,\beta,\ldots,\gamma,\ldots,\alpha,\ldots,\delta,\ldots\big)\\ \big(\ldots,\beta,\ldots,\alpha,\ldots,\gamma,\ldots,\delta,\ldots\big)\end{matrix}$$

It is clear that

$$\begin{matrix}\big(\ldots,\gamma,\ldots,\delta,\ldots,\alpha,\ldots,\beta,\ldots\big)\\ \big(\ldots,\gamma,\ldots,\beta,\ldots,\alpha,\ldots,\delta,\ldots\big)\\ \big(\ldots,\gamma,\ldots,\alpha,\ldots,\beta,\ldots,\delta,\ldots\big)\\ \big(\ldots,\beta,\ldots,\alpha,\ldots,\gamma,\ldots,\delta,\ldots\big)\end{matrix}$$

is also a chain. This is represented by $\hat{\mathbf{u}}_{\beta\delta}\hat{\mathbf{u}}_{\alpha\beta}\hat{\mathbf{u}}_{\beta\gamma}$, completing this case. To conclude our first objective, we notice that the fourth relation, with equalities, or the fifth relation, are clearly not possible for k-Bruhat orders, by Proposition 3.1 (1) and (2).

We now show that any two maximal chains in $[u,w]_k$ are connected by successive uses of the relations (1.1). It suffices to show that any maximal chain $\hat{\mathbf{x}}$ is connected to the CM-chain. For this we proceed by induction on n. If $n=1$, then there is a unique maximal chain. Let $n>1$ and assume that the theorem holds for all intervals $[u',w']_{k'}$ such that $\ell(w')-\ell(u')<n$. That is, we may assume that $\hat{\mathbf{x}}=\hat{\mathbf{y}}\hat{\mathbf{u}}_{\alpha_1\beta_1}$ where $\mathbf{y}$ is any maximal chain. If a_1,b_1 satisfy the conditions **I** and **II** of Definition 3.2 then choosing $\hat{\mathbf{y}}$ to be the CM-chain of $[u_1,w]_k$ completes the proof since then $\hat{\mathbf{x}}$ is the CM-chain of $[u,w]_k$. If condition **I** fails, then $w(a_1)$ is not maximal with $u(a_1)<w(a_1)$. In this case assume that $\hat{\mathbf{y}}$ is the CM-chain of $[u_1,w]$ so that $w(a_2)>w(a_1)$. We have two sub-cases to consider:

Case 1a: $\{\alpha_1,\beta_1\}\cap\{\alpha_2,\beta_2\}=\emptyset$. We can use relation (3) of (1.1) and get

$$\hat{\mathbf{x}}\equiv\hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1}\hat{\mathbf{u}}_{\alpha_2\beta_2}.$$

The hypothesis on $\mathbf{y}$ and $w(a_2)>w(a_1)$ implies that $\hat{\mathbf{u}}_{\alpha_2\beta_2}$ is the first step of the CM-chain of $[u,w]_k$. We can use our induction hypothesis on $[\hat{\mathbf{u}}_{\alpha_2\beta_2}u,w]_k$ and get $\hat{\mathbf{x}}\equiv\hat{\mathbf{z}}\hat{\mathbf{u}}_{\alpha_2\beta_2}$, the CM-chain of $[u,w]_k$.

Case 1b: $\alpha_2<\beta_2=\alpha_1<\beta_1$. Since $\mathbf{y}$ is the CM-chain of $[u_1,w]_k$, we have

$$\beta_2=\alpha_3<\beta_3=\alpha_4<\cdots<\beta_{m-1}=\alpha_m,$$

for $m \geq 3$, where $\beta_m = w(a_2) > w(a_1) \geq \beta_1$. Let $3 \leq s \leq m$ be such that $\alpha_s < \beta_1 < \beta_s$. We can apply the relations (1.1) and get

$$\begin{aligned}
\hat{\mathbf{x}} &= \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_m\beta_m}\cdots\hat{\mathbf{u}}_{\alpha_s\alpha_s}\cdots\hat{\mathbf{u}}_{\alpha_2\beta_2}\hat{\mathbf{u}}_{\alpha_1\beta_1}\\
&\equiv \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_m\beta_m}\cdots\hat{\mathbf{u}}_{\alpha_{s+1}\alpha_{s+1}}\hat{\mathbf{u}}_{\alpha_s\beta_1}\hat{\mathbf{u}}_{\beta_1\beta_s}\hat{\mathbf{u}}_{\alpha_2\beta_1}\hat{\mathbf{u}}_{\alpha_{s-1}\beta_{s-1}}\cdots\hat{\mathbf{u}}_{\alpha_3\beta_3}\\
&\equiv \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_m\beta_m}\cdots\hat{\mathbf{u}}_{\alpha_{s+1}\alpha_{s+1}}\hat{\mathbf{u}}_{\alpha_s\beta_1}\hat{\mathbf{u}}_{\alpha_{s-1}\beta_{s-1}}\cdots\hat{\mathbf{u}}_{\alpha_3\beta_3}\hat{\mathbf{u}}_{\beta_1\beta_s}\hat{\mathbf{u}}_{\alpha_2\beta_1}\\
&\equiv \hat{\mathbf{z}}\hat{\mathbf{u}}_{\alpha_2\beta_1}.
\end{aligned}$$

where, by the induction hypothesis, $\hat{\mathbf{z}}$ is the CM-chain of $[\hat{\mathbf{u}}_{\alpha_1\beta_2}u, w]_k$. Here $\hat{\mathbf{u}}_{\alpha_2\beta_1}$ is the first step in the CM-chain of $[u, w]_k$. Hence $\hat{\mathbf{x}} \equiv \hat{\mathbf{z}}\hat{\mathbf{u}}_{\alpha_2\beta_1}$, the CM-chain of $[u, w]_k$.

If condition **I** holds but condition **II** fails, then $w(b_1)$ is not minimal. In this case assume that $\mathbf{y}$ is the DCM-chain of $[u_1, w]$. Here, we must have that $w(b_2) < w(b_1)$ and again we have two sub-cases to consider:

Case 2a: $\{\alpha_1, \beta_1\} \cap \{\alpha_2, \beta_2\} = \emptyset$. We can use the relation (3) of (1.1) and the induction hypothesis to get

$$\hat{\mathbf{x}} \equiv \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1}\hat{\mathbf{u}}_{\alpha_2\beta_2} \equiv \hat{\mathbf{z}}\hat{\mathbf{u}}_{\alpha_2\beta_2},$$

where $\hat{\mathbf{z}}$ is the CM-chain of $[\hat{\mathbf{u}}_{\alpha_2\beta_2}u, w]_k$. If $\hat{\mathbf{u}}_{\alpha_2\beta_2}$ is the first step in the CM-chain of $[u, w]_k$ we are done. If not, then condition $\mathbf{I}'$ on $\hat{\mathbf{u}}_{\alpha_2\beta_2}$ implies that only condition **I** can fail in $\hat{\mathbf{z}}\hat{\mathbf{u}}_{\alpha_2\beta_2}$ and we are back to cases 1a or 1b.

Case 2b: $\alpha_1 < \beta_1 = \alpha_2 < \beta_2$. Since $\mathbf{y}$ is the DCM-chain of $[u_1, w]_k$, we have

$$\alpha_2 = \beta_3 > \alpha_3 = \beta_4 > \cdots > \alpha_{m-1} = \beta_m,$$

for $m \geq 3$, where $\alpha_m = w(b_2) > w(b_1) \geq \alpha_1$. Let $3 \leq s \leq m$ be such that $\beta_s > \alpha_1 > \alpha_s$. We can apply the relations (1.1) and get

$$\begin{aligned}
\hat{\mathbf{x}} &= \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_m\beta_m}\cdots\hat{\mathbf{u}}_{\alpha_s\alpha_s}\cdots\hat{\mathbf{u}}_{\alpha_2\beta_2}\hat{\mathbf{u}}_{\alpha_1\beta_1}\\
&\equiv \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_m\beta_m}\cdots\hat{\mathbf{u}}_{\alpha_{s+1}\alpha_{s+1}}\hat{\mathbf{u}}_{\alpha_1\beta_s}\hat{\mathbf{u}}_{\alpha_s\beta_1}\hat{\mathbf{u}}_{\alpha_1\beta_2}\hat{\mathbf{u}}_{\alpha_{s-1}\beta_{s-1}}\cdots\hat{\mathbf{u}}_{\alpha_3\beta_3}\\
&\equiv \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_m\beta_m}\cdots\hat{\mathbf{u}}_{\alpha_{s+1}\alpha_{s+1}}\hat{\mathbf{u}}_{\alpha_1\beta_s}\hat{\mathbf{u}}_{\alpha_{s-1}\beta_{s-1}}\cdots\hat{\mathbf{u}}_{\alpha_3\beta_3}\hat{\mathbf{u}}_{\alpha_s\beta_1}\hat{\mathbf{u}}_{\alpha_1\beta_2}\\
&\equiv \hat{\mathbf{z}}\hat{\mathbf{u}}_{\alpha_1\beta_2},
\end{aligned} \tag{3.5}$$

where $\hat{\mathbf{z}}$ is the CM-chain of $[\hat{\mathbf{u}}_{\alpha_1\beta_2}u, w]_k$. If $\hat{\mathbf{u}}_{\alpha_1\beta_2}$ is the first step in the CM-chain of $[u, w]_k$, then we are done. If not, then condition $\mathbf{I}'$ on $\hat{\mathbf{u}}_{\alpha_1\beta_2}$ implies that only the condition **I** can fail in $\hat{\mathbf{z}}\hat{\mathbf{u}}_{\alpha_2\beta_2}$ and again we are back to cases 1a or 1b. □

We now complete the characterization of compositions $\hat{\mathbf{x}} = \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1}$ corresponding to maximal chains for some $[u, w]_k$. If $\hat{\mathbf{x}}$ corresponds to a maximal chain in $[u, w]_k$, then $wu^{-1} = \hat{\mathbf{x}}1$. Hence $w = \zeta u$ for $\zeta = \hat{\mathbf{x}}1 = wu^{-1}$. Conversely, Proposition 3.5 shows that for any $\zeta \in \mathcal{S}_\infty$ we can find u and w such that $w = \zeta u$ and $[u, w]_k$ is nonempty for some k. In the following, we say that a composition $\hat{\mathbf{x}} = \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1}$ is $\mathbf{u}$-reduced if $\hat{\mathbf{x}}1 \neq \mathbf{0}$. Theorem 3.6 gives us a way of generating all $\mathbf{u}$-reduced sequences for $\zeta \in \mathcal{S}_\infty$; they are all connected via the relations (1)-(3) of (1.1). To complete our study, we characterize the compositions $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}} = \mathbf{0}$.

Theorem 3.7. *Let $\hat{\mathbf{x}} = \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1}$ be a composition. If $\hat{\mathbf{x}}1 = \mathbf{0}$, then $\hat{\mathbf{x}} \equiv \mathbf{0}$ modulo the relations (1.1).*

Proof We proceed by induction on n. When $n = 2$, $\hat{\mathbf{x}}1 = \mathbf{0}$ implies that relation (4) applies to $\hat{\mathbf{x}}$. Suppose $n \geq 3$ and the theorem holds for all compositions of length $< n$. Let $\hat{\mathbf{y}} = \hat{\mathbf{u}}_{\alpha_{n-1}\beta_{n-1}}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1}$ and we may assume that $\hat{\mathbf{y}}1 = \tau \neq \mathbf{0}$.

We first characterize those w such that $\tau^{-1}w \leq_k w$, for some k. Let $up(\tau)$ and $dw(\tau)$ be defined as above, and let $fix(\tau)$ be the set of fixed points of τ. By Proposition 3.1, $u = \tau^{-1}w \leq_k w$ if and only if

- $up(\tau) \subseteq \{w(i) : 1 \leq i \leq k\} \subseteq up(\tau) \cup fix(\tau)$,
- for $i < j \leq k$ or $k < i < j$, if $u(i) < u(j)$ then $w(i) < w(j)$.

The second condition implies that if $\alpha < \gamma$ are in $up(\tau) \cup fix(\tau)$ and $\tau^{-1}(\alpha) > \tau^{-1}(\gamma)$, then $\max\{w^{-1}(\alpha), w^{-1}(\gamma)\} \leq k$ implies $w^{-1}(\alpha) < w^{-1}(\gamma)$. Similarly, if $\gamma < \beta$ are in $dw(\tau) \cup fix(\tau)$ and $\tau^{-1}(\gamma) > \tau^{-1}(\beta)$ then $k < \min\{w^{-1}(\beta), w^{-1}(\gamma) \leq k\}$ implies $w^{-1}(\gamma) < w^{-1}(\beta)$. With this and the definition of $\preceq$, we see that $\ell_{\mathbf{u}}\big((\alpha_n, \beta_n)\tau\big) \neq \ell_{\mathbf{u}}(\tau) + 1$ implies one of the following holds:

(a) $\alpha_n \in dw(\tau)$,
(b) $\beta_n \in up(\tau)$,
(c) $\alpha_n < \gamma < \beta_n$ where $\tau^{-1}(\alpha_n) > \tau^{-1}(\gamma)$, or $\tau^{-1}(\gamma) > \tau^{-1}(\beta_n)$.

We complete the proof by showing that each case (a), (b), or (c) implies $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{y}} \equiv \mathbf{0}$ modulo the relations (1.1).
If (a) holds: By Theorem 3.6 we may assume that $\hat{\mathbf{y}}$ is any maximal chain. Let $\hat{\mathbf{y}} = \hat{\mathbf{z}}\hat{\mathbf{u}}_{\alpha_1\beta_1}$. Note that if $\alpha_n \in dw\big(\hat{\mathbf{z}}1\big)$ then the induction hypothesis applies and we are done. We can thus assume that $\alpha_n = \alpha_1$. But this must be true for any maximal chain $\hat{\mathbf{y}}$. Since $\alpha_1 = \min\big(dw(\tau)\big)$ for the DCM-chain, we have $\alpha_n = \min\big(dw(\tau)\big)$. Now let $\hat{\mathbf{y}}$ be the CM-chain, and consider its initial segment $\hat{\mathbf{u}}_{\alpha_m\beta_m}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1}$ where $\beta_1 = \alpha_2 < \beta_2 = \alpha_3 < \cdots < \beta_{m-1} = \alpha_m$ and $\beta_m = \max\big(up(\tau)\big)$. If $\big|up(\tau)\big| > 1$, then $m < n-1$. Consider the next operator $\hat{\mathbf{u}}_{\alpha_{m+1}\beta_{m+1}}$. Since $\alpha_1 = \min\big(dw(\tau)\big)$, we have $\alpha_1 < \alpha_m + 1$, and since $\beta_m = \max\big(up(\tau)\big)$, we have $\beta_{m+1} < \beta_m$. Thus we may apply a sequence of the relations (1)-(3) of (1.1), as in (3.5), to obtain $\hat{\mathbf{y}} \equiv \hat{\mathbf{z}}'\hat{\mathbf{u}}_{\alpha_{m+1}\beta'}$ for some $\hat{\mathbf{z}}'$ and β'. Since $\alpha_n = \alpha_1 \in dw\big(\hat{\mathbf{z}}'1\big)$, the induction hypothesis applies to conclude $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{z}}' \equiv \mathbf{0}$. Thus we may assume that (a) holds and $\big|up(\tau)\big| = 1$. That is, $\beta_1 = \alpha_2 < \beta_2 = \alpha_3 < \cdots < \beta_{n-2} = \alpha_{n-1}$ and $\alpha_n = \alpha_1$. If $\beta_n < \alpha_{n-1}$ or $\beta_n > \beta_{n-1}$ then we apply relation (3) to obtain $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{y}} \equiv \hat{\mathbf{u}}_{\alpha_{n-1}\beta_{n-1}}\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{u}}_{\alpha_{n-2}\beta_{n-2}}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1} \equiv \hat{\mathbf{u}}_{\alpha_{n-1}\beta_{n-1}}\hat{\mathbf{y}}'$, and $\hat{\mathbf{y}}' \equiv \mathbf{0}$ by the induction hypothesis. If $\beta_n = \alpha_{n-1}$ then we may apply relation (2) to obtain $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{x}} \equiv \hat{\mathbf{u}}_{\alpha_{n-2}\beta_{n-2}}\hat{\mathbf{u}}_{\alpha_n\alpha_{n-2}}\hat{\mathbf{u}}_{\alpha_{n-2}\beta_{n-1}}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1}$, which is equivalent to $\mathbf{0}$ as before. Finally if $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{u}}_{\alpha_{n-1}\beta_{n-1}} \equiv \mathbf{0}$
If (b) holds: This case is similar to (a), the map Ω_n from Lemma 3.3 can be used to interchange the roles of conditions (a) and (b).
If (c) holds: Assume that $\tau^{-1}(\alpha_n) > \tau^{-1}(\gamma)$. The other case, $\tau^{-1}(\gamma) > \tau^{-1}(\beta_n)$, is argued in a similar fashion using the map Ω_n. We may also assume that (a) does not hold, hence we have $\tau^{-1}(\gamma) < \tau^{-1}(\alpha_n) \leq \alpha_n < \gamma < \beta_n$ and, in particular, $\gamma \in up(\tau)$. Let γ be minimal with these properties. We may assume that $\hat{\mathbf{y}}$ is the CM-chain and we let $\hat{\mathbf{y}} = \hat{\mathbf{u}}_{\alpha_{n-1}\beta_{n-1}}\hat{\mathbf{z}}$ and $\hat{\mathbf{z}}1 = \sigma \in \mathcal{S}_\infty$. In this case $\beta_{n-1} = \min\big(up(\tau)\big) \leq \gamma$. If $\beta_{n-1} < \gamma$ then the minimality of γ implies $\beta_{n-1} \leq \alpha_n$. We have a four sub-cases:

(i) If $\beta_{n-1} = \gamma$ and $\alpha_{n-1} \leq \alpha_n$, then $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{u}}_{\alpha_{n-1}\beta_{n-1}} \equiv \mathbf{0}$ is an instance of relation (4) of (1.1).
(ii) If $\beta_{n-1} = \gamma$ and $\alpha_{n-1} > \alpha_n$, then $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{y}} \equiv \hat{\mathbf{u}}_{\alpha_{n-1}\beta_{n-1}}\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{z}}$. Since $\tau^{-1}(\gamma) < \alpha_n$ and $\mathbf{x}$ is the CM-chain, we must have $\beta_{n-2} = \alpha_{n-1}$. So $\alpha_n < \alpha_{n-1} = \beta_{n-2} < \gamma < \beta_n$ and $\sigma^{-1}(\beta_{n-2}) = \tau^{-1}(\gamma) < \alpha_n$. By the induction hypothesis $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{y}} \equiv \mathbf{0}$.

(iii) If $\beta_{n-1} < \alpha_n$, then $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{y}} \equiv \hat{\mathbf{u}}_{\alpha_{n-1}\beta_{n-1}}\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{z}}$ where $\sigma^{-1}(\beta_{n-2}) = \tau^{-1}(\gamma)$. The induction hypothesis applies and again $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{z}} \equiv \mathbf{0}$.

(iv) If $\beta_{n-1} = \alpha_n$, then since $\hat{\mathbf{y}}$ is the CM-chain, the minimality of γ implies that $\beta_m = \gamma < \beta_n$ for some $1 \le m \le n-2$, with

$$\alpha_n = \beta_{n-1} > \alpha_{n-1} = \beta_{n-2} > \cdots > \alpha_{m+2} = \beta_{m+1} > \alpha_{m+1}.$$

For some $1 \le s \le m$ we also have

$$\gamma = \beta_m > \alpha_m = \beta_{m-1} > \cdots > \alpha_{s+1} = \beta_s,$$

where $\alpha_s = \tau^{-1}(\gamma) < \tau^{-1}(\alpha_n) = \alpha_{m+1}$. If $s > 1$ we may appeal to the induction hypothesis and get $\hat{\mathbf{u}}_{\alpha_n\beta_n}\hat{\mathbf{y}} \equiv \mathbf{0}$. Thus we may assume that $s = 1$. Also, since $\alpha_1 < \alpha_{m+1} < \beta_{m+1} \le \alpha_n < \gamma = \beta_m$ we may apply relations (1)-(3) as in (3.5) to obtain

$$\hat{\mathbf{u}}_{\alpha_{m+1}\beta_{m+1}}\hat{\mathbf{u}}_{\alpha_m\beta_m}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1} \equiv \hat{\mathbf{u}}_{\alpha'_m\beta'_m}\cdots\hat{\mathbf{u}}_{\alpha'_1\beta'_1}\hat{\mathbf{u}}_{\alpha_{m+1}\beta_{m+1}},$$

where $\gamma = \beta_m = \beta'_m$, $\beta'_{m-1} = \alpha'_m$, $\beta'_{m-2} = \alpha'_{m-1}$, …, $\beta'_1 = \alpha'_2$ and $\alpha'_1 = \alpha_1$. Hence we can use the induction hypothesis on $\hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_{m+2}\beta_{m+2}}\hat{\mathbf{u}}_{\alpha'_m\beta'_m}\cdots\hat{\mathbf{u}}_{\alpha'_1\beta'_1}$, to obtain

$$\hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_{m+2}\beta_{m+2}}\hat{\mathbf{u}}_{\alpha'_m\beta'_m}\cdots\hat{\mathbf{u}}_{\alpha'_1\beta'_1} \equiv \mathbf{0},$$

and this concludes our proof. □

Proof [of Theorem 1.2]

(a) This is a direct consequence of Proposition 3.4 and Proposition 3.5.
(b) Theorem 3.6 and Theorem 3.7 imply that the operators $\hat{\mathbf{u}}_{\alpha\beta}$ satisfy the relations (1.1). Equation (3.1) gives the characterization part.
(c) This is a consequence of (b), Theorem 3.6, and Theorem 3.7.
(d) Injection is from part (b) and (c). Surjection is given by Proposition 3.5.
(e) Follows from the definitions of $\preceq$ and $\hat{\mathbf{u}}_{\alpha\beta}$.
(f) This is a direct consequence (a)-(f) above. □

4. A description of c_λ^ζ.

We give a description of the constants c_λ^ζ appearing in Equation (1.3) using the elements of $\mathcal{M}$. This will be useful in some subsequent work [3, 4]. It can also be used by the interested reader to derive combinatorial proofs of many of the geometrical identities of [2].

Recall that the Schur polynomial $S_\lambda(x_1, x_2, \ldots, x_k)$ equals $\mathfrak{S}_{v(\lambda,k)}$ for a unique Grassmannian permutation $v(\lambda, k)$. We have

$$\mathfrak{S}_u\mathfrak{S}_{v(\lambda,k)} = \sum_w c^w_{uv(\lambda,k)}\mathfrak{S}_w. \tag{4.1}$$

First we consider a special case of (4.1). The Schubert polynomial $\mathfrak{S}_{v((n),k)} = h_n(x_1, x_2, \ldots, x_k)$ is the homogeneous symmetric polynomial on k variables. Lascoux and Schützenberger [7] formulated a Pieri-type formula for $\mathfrak{S}_u\mathfrak{S}_{v((n),k)}$. In [1], proven in [13], we have reformulated this rule. Using Theorem 1.2, we can state it here as follows:

$$\mathfrak{S}_u\mathfrak{S}_{v((n),k)} = \sum_{\substack{\hat{\mathbf{x}} = \hat{\mathbf{u}}_{\alpha_n\beta_n}\cdots\hat{\mathbf{u}}_{\alpha_1\beta_1} \not\equiv \mathbf{0} \\ \alpha_1<\alpha_2<\cdots<\alpha_n}} \mathfrak{S}_{(\hat{\mathbf{x}}1)u}. \tag{4.2}$$

There are now other proofs of (4.2), some of them are combinatorial [12, 15]. Let $p = (p_1, p_2, \ldots, p_r)$ be a sequence of r integers such that $p_1 + p_2 + \cdots + p_r = n$. We say that a $\mathbf{u}$-composition $\hat{\mathbf{x}} = \hat{\mathbf{u}}_{\alpha_n\beta_n} \cdots \hat{\mathbf{u}}_{\alpha_1\beta_1}$ *weakly fits* p if

$$\begin{aligned} \alpha_1 < \alpha_2 < \cdots < \alpha_{p_1}, \\ \alpha_{p_1+1} < \alpha_{p_1+2} < \cdots < \alpha_{p_1+p_2}, \\ \vdots \\ \alpha_{n-p_r+1} < \alpha_{n-p_r+2} < \cdots < \alpha_n, \end{aligned}$$

and for all i, we have $p_i \geq 0$. Let $\mathcal{H}_p(\zeta) = \{\hat{\mathbf{x}} \in R_{\mathbf{u}}(\zeta) : \zeta = \hat{\mathbf{x}}1 \textit{ and } \hat{\mathbf{x}} \textit{ weakly fits } p\}$. Note that $\mathcal{H}_p(\zeta) = \emptyset$ if some $p_i < 0$.

Remark 4.1. *From (4.2), $\mathcal{H}_p(wu^{-1})$ is the coefficient of $\mathfrak{S}_w$ in the product*

$$\mathfrak{S}_u \mathfrak{S}_{v((p_1),k)} \mathfrak{S}_{v((p_2),k)} \cdots \mathfrak{S}_{v((p_r),k)}$$

when all $p_i > 0$.

Now consider the Jacobi identity [11]: for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ a partition of n,

$$\mathfrak{S}_{v(\lambda,k)} = S_\lambda(x_1, x_2, \ldots, x_k) = \det\left(h_{\lambda_i+j-i}(x_1, x_2, \ldots, x_k)\right)_{1\leq i,j\leq r}, \tag{4.3}$$

where $h_0(x_1, x_2, \ldots, x_k) = 1$, $h_n(x_1, x_2, \ldots, x_k) = \mathfrak{S}_{v((n),k)}$ for $n > 0$, and $h_n = 0$ for $n < 0$. For $\sigma \in \mathcal{S}_r$, let $\lambda_\sigma = \big(\lambda_\sigma(1), \lambda_\sigma(2), \ldots, \lambda_\sigma(r)\big)$, where $\lambda_\sigma(i) = \lambda_{\sigma(i)} + i - \sigma(i)$. Denote by $\epsilon(\sigma)$ the sign of the permutation $\sigma \in \mathcal{S}_r$. Expanding the determinant (4.3) in (4.1), and using (4.2), we get

$$\begin{aligned} \mathfrak{S}_u \mathfrak{S}_{v(\lambda,k)} &= \sum_{\sigma\in\mathcal{S}_r} \epsilon(\sigma) \mathfrak{S}_u \mathfrak{S}_{v((\lambda_\sigma(1)),k)} \mathfrak{S}_{v((\lambda_\sigma(2)),k)} \cdots \mathfrak{S}_{v((\lambda_\sigma(r)),k)} \\ &= \sum_{w\in\mathcal{S}_\infty} \left(\sum_{\sigma\in\mathcal{S}_r} \epsilon(\sigma) \left|\mathcal{H}_{\lambda_\sigma}(wu^{-1})\right| \right) \mathfrak{S}_w. \end{aligned}$$

Thus

$$c^w_{uv(\lambda,k)} = \sum_{\sigma\in\mathcal{S}_r} \epsilon(\sigma) \left|\mathcal{H}_{\lambda_\sigma}(wu^{-1})\right|.$$

This is a consequence of Theorem 1.2. From this we deduce the following proposition.

Proposition 4.2.

(1) *$c^w_{uv(\lambda,k)} = 0$ if $u \not\leq_k w$, and*
(2) *if $u \leq_k w$ then $c^w_{uv(\lambda,k)}$ depends only on λ and wu^{-1}.*

Hence, $c^\zeta_\lambda := c^w_{uv(\lambda,k)}$ is well defined for any $u \leq_k w$ with $\zeta = wu^{-1}$. We have

Theorem 4.3. $c^\zeta_\lambda = \sum_{\sigma\in\mathcal{S}_r} \epsilon(\sigma) \left|\mathcal{H}_{\lambda_\sigma}(\zeta)\right|$.

Let us illustrate Theorem 4.3 on an example. Let $\zeta = (2, 5, 4, 1, 6, 3)$. Using Proposition 3.5 we have $(3, 1, 2, 5, 6, 4) = u \leq_4 \zeta u = (4, 2, 5, 6, 3, 1)$. In Figure 2, we have drawn the interval $[u, \zeta u]_4$ and we have labeled each covering edge in the interval by the index α of the corresponding $\hat{\mathbf{u}}_{\alpha\beta}$. Here we have removed the commas and parentheses to represent the permutations in a more compact form. Note that there are 14 maximal chains in this interval.

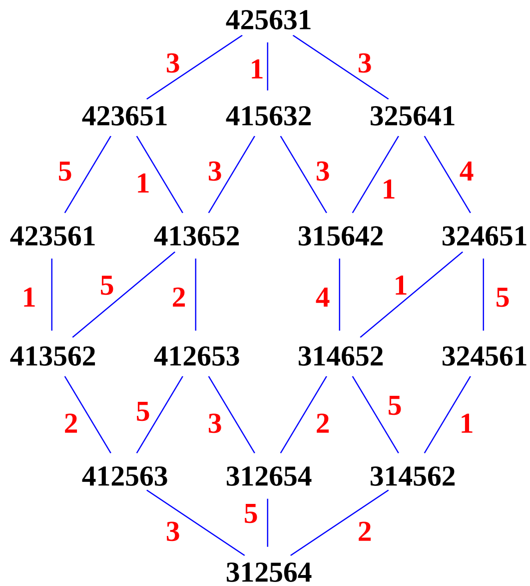

FIGURE 2. The interval $[(3,1,2,5,6,4),\ (4,2,5,6,3,1)]_4$.

Theorem 4.3 gives us that

$$\begin{aligned} c^{\zeta}_{(2,2,1)} &= \left|\mathcal{H}_{(2,2,1)}(\zeta)\right| - \left|\mathcal{H}_{(1,3,1)}(\zeta)\right| - \left|\mathcal{H}_{(2,0,3)}(\zeta)\right| + \left|\mathcal{H}_{(1,0,4)}(\zeta)\right| \\ &\quad + \left|\mathcal{H}_{(-1,3,3)}(\zeta)\right| - \left|\mathcal{H}_{(-1,2,4)}(\zeta)\right|. \end{aligned}$$

The sets $\mathcal{H}_{(-1,3,3)}(\zeta)$ and $\mathcal{H}_{(-1,2,4)}(\zeta)$ are both empty since the indices contains a negative component. Looking at Figure 2, we find $\mathcal{H}_{(2,2,1)}(\zeta) = \{\hat{\mathbf{u}}_{12}\hat{\mathbf{u}}_{35}\hat{\mathbf{u}}_{23}\hat{\mathbf{u}}_{56}\hat{\mathbf{u}}_{34},\ \hat{\mathbf{u}}_{34}\hat{\mathbf{u}}_{45}\hat{\mathbf{u}}_{12}\hat{\mathbf{u}}_{56}\hat{\mathbf{u}}_{24}\}$ and $\mathcal{H}_{(1,3,1)}(\zeta) = \mathcal{H}_{(2,0,3)}(\zeta) = \emptyset$. Hence $c^{\zeta}_{(2,2,1)} = 2$. Now for $\lambda = (2,1,1,1)$ and $\sigma \in \mathcal{S}_4$, the sequences λ_σ that do not contains a negative component are $(2,1,1,1)$, $(2,1,0,2)$, $(2,0,2,1)$, $(2,0,0,3)$, $(0,3,1,1)$, $(0,3,0,2)$, $(0,0,4,1)$ and $(0,0,0,5)$. For our example, we have $\left|\mathcal{H}_{(2,1,1,1)}(\zeta)\right| = 5$, $\left|\mathcal{H}_{(2,1,0,2)}(\zeta)\right| = 2$, $\left|\mathcal{H}_{(2,0,2,1)}(\zeta)\right| = 2$ and all the others are empty. Hence $c^{\zeta}_{(2,1,1,1)} = 5 - 2 - 2 = 1$. Using (1.3) for this example, we get $c^{\zeta}_{\lambda} = 0$ for the other λ, since $14 = 5 * 2 + 4 * 1$ is the total number of maximal chains.

Most of the geometrical identities of [2] can now be proven combinatorially using Theorem 4.3, but some of them are still very surprising. For example, Theorem H of [2] states that for $\gamma = (1,2,3,\ldots,n)$ and ζ in $\mathcal{S}_n$, $c^{\zeta}_{\lambda} = c^{\gamma\zeta\gamma^{-1}}_{\lambda}$. We do not know how to show this combinatorially. Here, Equation (1.3) implies that $\left|R_{\mathbf{u}}(\zeta)\right| = \left|R_{\mathbf{u}}(\gamma\zeta\gamma^{-1})\right|$. This suggests the existence of a bijection $\varphi\colon R_{\mathbf{u}}(\zeta) \longrightarrow R_{\mathbf{u}}(\gamma\zeta\gamma^{-1})$. Note that the two Posets $[1,\zeta]_{\preceq}$ and $[1,\gamma\zeta\gamma^{-1}]_{\preceq}$ are not necessarily isomorphic. For example let $\zeta = (2,4,1,3)$, the interval $[1,\gamma\zeta\gamma^{-1}]_{\preceq}$ is a hexagon and $[1,\zeta]_{\preceq}$ is not, it is a *kite*. On the other hand, since the Jacobi identity (4.3) is invertible, the equality $c^{\zeta}_{\lambda} = c^{\gamma\zeta\gamma^{-1}}_{\lambda}$ implies that $\left|\mathcal{H}_p(\zeta)\right| = \left|\mathcal{H}_p(\gamma\zeta\gamma^{-1})\right|$ for any p. Is it possible to construct the bijection φ such that $\varphi\big(\mathcal{H}_p(\zeta)\big) = \mathcal{H}_p(\gamma\zeta\gamma^{-1})$?

We should point out that Theorem 4.3 needs to be improved. It is a useful combinatorial description of the c^{ζ}_{λ} but it is unsatisfactory. It would be more elegant to have a formula that does not involve signs. There are still many open questions about the Grassmannian Bruhat order. We shall conclude with a list of them:

(i) As suggested by Equation (1.3), can we find a representation of the symmetric group $\mathcal{S}_{\ell_{\mathbf{u}}(\zeta)}$ on $\mathbb{Q}R_{\mathbf{u}}(\zeta)$ with character given by $\sum_\lambda c^\zeta_\lambda \chi^\lambda$?

(ii) Can we find a partition of $R_{\mathbf{u}}(\zeta)$ similar to the one discussed after Eq. (2.2)?

(iii) Can we describe the polynomial $P_n(t) = \displaystyle\sum_{\zeta\in\mathcal{S}_n} t^{\ell_{\mathbf{u}}(\zeta)}$?

(iv) What are the properties of the partial order $\preceq$. e.g. What is its Möbius function? Is any interval Cohen-Macauley? (We should mention here that the intervals contain *hexagons* in general, hence they are not shellable in the classical sense.)

(v) Is it possible to find a faithful representation of $\mathcal{M}$ as operators on the polynomial ring $\mathbb{Z}[x_1, x_2, x_3, \ldots]$?

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