PIERI-TYPE FORMULAS FOR MAXIMAL ISOTROPIC GRASSMANNIANS VIA TRIPLE INTERSECTIONS

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Abstract. We give an elementary proof of the Pieri-type formula in the cohomology ring of a Grassmannian of maximal isotropic subspaces of an orthogonal or symplectic vector space. This proof proceeds by explicitly computing a triple intersection of Schubert varieties. The multiplicities (which are powers of 2) in the Pieri-type formula are seen to arise from the intersection of a collection of quadrics with a linear space.

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Introduction

We give an elementary geometric proof of Pieri-type formulas in the cohomology rings of Grassmannians of maximal isotropic subspaces of orthogonal or symplectic vector spaces. For this, we explicitly compute a triple intersection of Schubert varieties, where one is a special Schubert variety. Previously, Sertöz [16] had studied such triple intersections in orthogonal Grassmannians, but was unable to determine the intersection multiplicities.

The multiplicities here (0 or powers of 2) arise as the intersection multiplicity of a linear subspace (defining the special Schubert variety) with a collection of quadrics and linear subspaces (determined by the other two Schubert varieties). This is similar to the triple intersection proof of the classical Pieri formula (*cf.* [9]) where the multiplicities (0 or 1) count the points in the intersection of linear subspaces.

These Pieri-type formulas are due to Hiller and Boe [8], who used the Chevalley formula [2]. Another proof, using the Leibniz formula for divided differences, was given by Pragacz and Ratajski [13]. These formulas have important geometric applications. Using them Pragacz [12] established Giambelli-type formulas for the above Grassmanians. This led to a solution of some classical enumerative problems (see [6] for a summary of this activity).

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In Section 1, we give the basic definitions, state the Pieri-type formulas, and give an outline of the proof. In Section 2, we describe the intersection of two Schubert varieties, which we use in Section 3 to complete the proof. While we work in the cohomology ring of a complex variety, our arguments hold for the Chow ring [4] of the same variety defined over any algebraically closed field not of characteristic 2.

1. The Grassmannian of maximal isotropic subspaces

For more details on the geometry and cohomology of these spaces, see [6]. Let U be a complex vector space equipped with a non-degenerate bilinear form β, either symmetric or alternating. A subspace H of U is *isotropic* if the restriction of β to H is identically zero. Isotropic subspaces have dimension at most half that of U. The Grassmannian of maximal isotropic subspaces of U is the set of all isotropic subspaces of U of maximal dimension. These spaces are quite different in the three cases of β alternating, β symmetric and dimension U odd, or β symmetric and dimension U even. In this third case, the Grassmannian has two connected components, each isomorphic to the Grassmannian of maximal isotropic subspaces in a generic hyperplane of U. Indeed, the quadric hypersurface in $\mathbb{P}^{2n+1}$ contains two families of n-planes [7]—each a component of the isotropic Grassmannian—and either family restricts to the family of $(n-1)$-planes on the quadric in a generic hyperplane section.

We thus consider two cases: Either β is symmetric on a vector space V of dimension $2n+1$ or else β is alternating on a vector space W of dimension $2n$. Write B_n or $B(V)$ for the Grassmannian of maximal isotropic subspaces of V, and C_n or $C(W)$ for the Grassmannian of maximal isotropic subspaces of W. The orthogonal group $SO_{2n+1}\mathbb{C} = \mathrm{Aut}(V, \beta)$ acts transitively on B_n with the stabilizer P_0 of a point a maximal parabolic subgroup associated to the short root, hence $B_n = SO_{2n+1}\mathbb{C}/P_0$. Similarly, $C_n = Sp_{2n}\mathbb{C}/P_0$, the quotient of the symplectic group by a maximal parabolic subgroup P_0 associated to the long root.

Both B_n and C_n are smooth complex manifolds of dimension $\binom{n+1}{2}$. While not isomorphic if $n > 1$, they have identical decompositions into Schubert cells. For an integer j, let $\overline{\jmath}$ denote $-j$. Choose bases $\{e_{\overline{n}}, \ldots, e_n\}$ of V and $\{f_{\overline{n}}, \ldots, f_n\}$ of W for which

$$\beta(e_i, e_j) = \begin{cases} 1 & \text{if } i = \overline{\jmath} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta(f_i, f_j) = \begin{cases} j/|j| & \text{if } i = \overline{\jmath} \\ 0 & \text{otherwise} \end{cases}.$$

Thus $\beta(e_1, e_0) = \beta(f_{\overline{2}}, f_1) = 0$ and $\beta(e_0, e_0) = \beta(f_{\overline{1}}, f_1) = -\beta(f_1, f_{\overline{1}}) = 1$.

Schubert varieties are determined by sequences

$$\mu : \ n \geq \mu_1 > \mu_2 > \cdots > \mu_n \geq \overline{n}$$

whose set of absolute values $\{|\mu_1|, \dots, |\mu_n|\}$ equals $\{1, 2, \dots, n\}$. Let $\mathbb{SY}_n$ denote this set of sequences. The Schubert variety X_μ of B_n is

$$\{H \in B_n \mid \dim(H \cap \langle e_{\mu_j}, \dots, e_n\rangle) \geq j \text{ for } 1 \leq j \leq n\}$$

and the Schubert variety Y_μ of C_n

$$\{H \in C_n \mid \dim(H \cap \langle f_{\mu_j}, \dots, f_n\rangle) \geq j \text{ for } 1 \leq j \leq n\}.$$

Both X_μ and Y_μ have codimension $|\mu| := \mu_1 + \cdots + \mu_k$, where $\mu_k > 0 > \mu_{k+1}$. Given $\lambda, \mu \in \mathbb{SY}_n$, we see that

$$X_\mu \supset X_\lambda \iff Y_\mu \supset Y_\lambda \iff \mu_j \leq \lambda_j \text{ for } 1 \leq j \leq n.$$

Define the *Bruhat order* on $\mathbb{SY}_n$ by $\mu \leq \lambda$ if $\mu_j \leq \lambda_j$ for $1 \leq j \leq n$. Note that $\mu \leq \lambda$ if and only if $\mu_j \leq \lambda_j$ for those j with $0 < \mu_j$.

Example 1.1. Suppose $n = 4$. Then $X_{3\,2\,\overline{1}\,\overline{4}}$ consists of those $H \in B_4$ such that

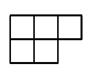
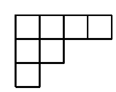
$$\dim(H \cap \langle e_3, e_4\rangle) \geq 1,\ \dim(H \cap \langle e_2, e_3, e_4\rangle) \geq 2,$$
$$\text{and } \dim(H \cap \langle e_{\overline{1}}, \dots, e_4\rangle) \geq 3.$$

We also have $3\,2\,\overline{1}\,\overline{4} < 3\,2\,1\,\overline{4} < 4\,3\,1\,\overline{2}$ while $3\,2\,1\,\overline{4}$ and $4\,1\,\overline{2}\,\overline{3}$ are incomparable.

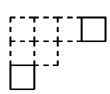
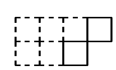
Define $P_\lambda := [X_\lambda]$, the cohomology class Poincaré dual to the fundamental cycle of X_λ in the homology of B_n. Likewise set $Q_\lambda := [Y_\lambda]$. Since Schubert varieties are closures of cells from a decomposition into (real) even-dimensional cells, these *Schubert classes* $\{P_\lambda\}$, $\{Q_\lambda\}$ form bases for integral cohomology:

$$H^*B_n = \bigoplus_\lambda P_\lambda \cdot \mathbb{Z} \qquad \text{and} \qquad H^*C_n = \bigoplus_\lambda Q_\lambda \cdot \mathbb{Z}.$$

Each $\lambda \in \mathbb{SY}_n$ determines and is determined by its diagram, also denoted λ. The diagram of λ is a left-justified array of $|\lambda|$ boxes with λ_j boxes in the jth row, for $\lambda_j > 0$. Thus

$3\,2\,\overline{1}\,\overline{4} \longleftrightarrow$  and $4\,2\,1\,\overline{3} \longleftrightarrow$ .

The Bruhat order corresponds to inclusion of diagrams. Given $\mu \leq \lambda$, let λ/μ be their set-theoretic difference. For instance,

$4\,2\,1\,\overline{3}/3\,2\,\overline{1}\,\overline{4} \longleftrightarrow$  and $4\,3\,\overline{1}\,\overline{2}/3\,2\,\overline{1}\,\overline{4} \longleftrightarrow$ .

Two boxes are connected if they share a vertex or an edge; this defines *components* of λ/μ. We say λ/μ is a *skew row* if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_n$ equivalently, if λ/μ has at most one box in each column. Thus $4\,2\,1\,\overline{3}/3\,2\,\overline{1}\,\overline{4}$ is a skew row, but $3\,2\,\overline{1}\,\overline{4}/1\,\overline{2}\,\overline{3}\,\overline{4}$ is not.

The *special Schubert class* $p_m \in H^*B_n$ ($q_m \in H^*C_n$) is the class whose diagram consists of a single row of length m. Hence, $p_2 = P_{2\,\overline{1}\,\overline{3}\,\overline{4}}$. A *special Schubert variety* X_K (Y_K) is the collection of all maximal isotropic subspaces which meet a fixed isotropic subspace K nontrivially. If $\dim K = n+1-m$, then $[X_K] = p_m$ and $[Y_K] = q_m$. When λ/μ is a skew row, let $\delta(\lambda/\mu)$ count the components of the diagram λ/μ and $\varepsilon(\lambda/\mu)$ count the components of λ/μ which do not meet the first column.

Theorem 1.2 (Pieri-type Formula)**.** *For any* $\mu \in \mathbb{SY}_n$ *and* $1 \leq m \leq n$*,*

1. $P_\mu \cdot p_m \;=\; \sum 2^{\delta(\lambda/\mu)-1} P_\lambda$ *and*
2. $Q_\mu \cdot q_m \;=\; \sum 2^{\varepsilon(\lambda/\mu)} Q_\lambda$,

both sums over all λ *with* $|\lambda| - |\mu| = m$ *and* λ/μ *a skew row.*

Example 1.3. For example,

$$\begin{aligned} P_{3\,2\,\overline{1}\,\overline{4}} \cdot p_2 &= 2\cdot P_{4\,2\,1\,\overline{3}} + P_{4\,3\,\overline{1}\,\overline{2}} \qquad \text{and} \\ Q_{3\,2\,\overline{1}\,\overline{4}} \cdot q_2 &= 2\cdot Q_{4\,2\,1\,\overline{3}} + 2\cdot Q_{4\,3\,\overline{1}\,\overline{2}}, \end{aligned}$$

as $4\,2\,1\,\overline{3}/3\,2\,\overline{1}\,\overline{4}$ has two components, one meeting the first column, and $4\,3\,\overline{1}\,\overline{2}/3\,2\,\overline{1}\,\overline{4}$ has one component, which does not meet the first column.

Define λ^c by $\lambda^c_j := \overline{\lambda_{n+1-j}}$. Let [pt] be the class dual to a point. The Schubert basis is self-dual with respect to the intersection pairing: If $|\lambda| = |\mu|$, then

$$P_\mu \cdot P_{\lambda^c} \;=\; Q_\mu \cdot Q_{\lambda^c} \;=\; \begin{cases} [\mathrm{pt}] & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

Define the Schubert variety X'_{λ^c} to be

$$\{H \in B_n \mid \dim(H \cap \langle e_{\overline{n}}, \ldots, e_{\lambda_j}\rangle) \geq n+1-j \text{ for } 1 \leq j \leq n\}.$$

This is a translate of X_{λ^c} by an element of $SO_{2n+1}\mathbb{C}$. We similarly define Y'_{λ^c}. For any λ, μ, $X_\mu \cap X'_{\lambda^c}$ is a (dimensionally) proper intersection [11]. This is because if X_μ and X'_{λ^c} are *any* Schubert varieties in general position, then there is a basis for V such that these varieties and the form β are as given. The analogous facts hold for the varieties Y'_{λ^c}.

To establish Theorem 1.2, it suffices to compute degrees of the zero dimensional schemes

$$X_\mu \cap X'_{\lambda^c} \cap X_K \quad \text{and} \quad Y_\mu \cap Y'_{\lambda^c} \cap Y_K,$$

where K is a general isotropic $(n+1-m)$-plane and $|\lambda| = |\mu| + m$.

We only do the (more difficult) orthogonal case of Theorem 1.2 in full, and indicate the differences for the symplectic case. We first determine when $X_\mu \cap X'_{\lambda^c}$ is non-empty. Let $\mu, \lambda \in \mathbb{SY}_n$. Then $H \in X_\mu \cap X'_{\lambda^c}$ implies

$\dim(H \cap \langle e_{\mu_j}, \ldots, e_{\lambda_j} \rangle) \geq 1$, for every $1 \leq j \leq n$. Hence $\mu \leq \lambda$ is necessary for $X_\mu \cap X'_{\lambda^c}$ to be nonempty. In fact, if $|\mu| = |\lambda|$, then

$$X_\mu \cap X'_{\lambda^c} \;=\; \begin{cases} \langle e_{\lambda_1}, \ldots, e_{\lambda_n} \rangle & \text{if } \lambda = \mu \\ \emptyset & \text{otherwise,} \end{cases}$$

and the intersection is transverse (see Lemma 3.3), which establishes (1).

Suppose $\mu \leq \lambda$ in $\mathbb{SY}_n$. For each component d of λ/μ, let $\mathrm{col}(d)$ be the indices of the columns of d and of the column just to the left of d, which is 0 if d meets the first column. For each component d of λ/μ, define a quadratic form β_d:

$$\beta_d \;:=\; \sum_{\substack{\overline{n} \leq j \leq n \\ |j| \in \mathrm{col}(d)}} x_j x_{\overline{j}},$$

where $x_{\overline{n}}, \ldots, x_n$ are coordinates for V dual to the basis $e_{\overline{n}}, \ldots, e_n$. For each *fixed point* of λ/μ (j such that $\lambda_j = \mu_j$), define the linear form $\alpha_j := x_{\overline{\lambda_j}}$. If no component meets the first column, then 0 is a fixed point of λ/μ and we set $\alpha_0 := x_0$. Let $Z_{\lambda/\mu}$ be the common zero locus of these forms α_j and β_d. In Section 2, we prove:

Lemma 1.4. *Suppose $\mu \leq \lambda$ and $H \in X_\mu \cap X'_{\lambda^c}$. Then $H \subset Z_{\lambda/\mu}$.*

For $\mu \leq \lambda \in \mathbb{SY}_n$, let $\delta(\lambda/\mu)$ count the components of λ/μ.

Theorem 1.5. *Let $\mu, \lambda \in \mathbb{SY}_n$ and suppose K is a general isotropic $(n+1-m)$-plane with $|\mu| + m = |\lambda|$. Then*

$$X_\mu \cap X'_{\lambda^c} \cap X_K$$

is non-empty only if λ/μ is a skew row. Moreover, if λ/μ is a skew row, then $K \cap Z_{\lambda/\mu}$ consists of $2^{\delta(\lambda/\mu)-1}$ isotropic lines, counted with multiplicity.

Proof. If φ counts the fixed points of λ/μ and $\delta = \delta(\lambda/\mu)$, then we have the following equation (Lemma 2.1):

$$n + 1 \;=\; \varphi + \delta + \#\text{columns of } \lambda/\mu. \tag{2}$$

Thus, if $m = |\lambda| - |\mu|$, then $\varphi + \delta \geq n + 1 - m$, with equality only when λ/μ is a skew row.

For each $0 \leq i \leq n$, there is a unique form among the α_j, β_d in which one of the coordinates $x_i, x_{\overline{i}}$ appear. Thus $Z_{\lambda/\mu}$ is defined in $\mathbb{P}(V)$ by β, the α_j, and any $\delta - 1$ of the β_d. Hence $Z_{\lambda/\mu}$ has codimension $\varphi + \delta - 1$ in the set of isotropic points, a $SO_{2n+1}\mathbb{C}$-orbit. We see that a general isotropic $(n+1-m)$-plane K meets $Z_{\lambda/\mu}$ non-trivially only if λ/μ is a skew row, as this intersection is proper [11]. In that case, $K \cap Z_{\lambda/\mu}$ (in $\mathbb{P}(V)$) is zero-dimensional of degree $2^{\delta-1}$, as it is defined on K by $\delta - 1$ quadratic forms and φ linear forms. ∎

Proof of Theorem 1.2. Suppose $\lambda, \mu \in \mathbb{SY}_n$ with $|\lambda| - |\mu| = m > 0$. Let K be a general isotropic $(n+1-m)$-plane in V. We compute the degree of

$$X_\mu \cap X'_{\lambda^c} \cap X_K. \tag{3}$$

By Theorem 1.5, this is non-empty only if λ/μ is a skew row. Suppose that is the case. Theorem 3.1 asserts that a general isotropic line in $Z_{\lambda/\mu}$ is contained in a unique $H \in X_\mu \cap X'_{\lambda^c}$. By Theorem 1.5, $K \cap Z_{\lambda/\mu}$ is $2^{\delta(\lambda/\mu)-1}$ isotropic lines (counted with multiplicity), we see that (3) has degree $2^{\delta(\lambda/\mu)-1}$. This completes the proof of Theorem 1.2. ∎

Example 1.6. Let $n = 4$ and $m = 2$, so that $n + 1 - m = 3$. The local coordinates for $X_{3\,2\,\overline{1}\,\overline{4}} \cap X'_{(4\,2\,1\,\overline{3})^c}$ described in Lemma 3.3 show that, for any $x, z \in \mathbb{C}$, the row span H of the matrix with rows g_i and columns e_j

	$e_{\overline{4}}$	$e_{\overline{3}}$	$e_{\overline{2}}$	$e_{\overline{1}}$	e_0	e_1	e_2	e_3	e_4
g_1	·	·	·	·	·	·	·	$-x$	1
g_2	·	·	·	·	·	·	1	·	·
g_3	·	·	·	1	$2z$	$-2z^2$	·	·	·
g_4	x	1	·	·	·	·	·	·	·

is a generic maximal isotropic subspace in $X_{3\,2\,\overline{1}\,\overline{4}} \cap X'_{(4\,2\,1\,\overline{3})^c}$. We write '·' in place of the entries of 0. Suppose K is the row span of the matrix with rows v_i

	$e_{\overline{4}}$	$e_{\overline{3}}$	$e_{\overline{2}}$	$e_{\overline{1}}$	e_0	e_1	e_2	e_3	e_4
v_1	·	1	·	1	·	·	1	·	1
v_2	1	1	·	1	2	-2	1	-1	1
v_3	·	·	1	·	·	-1	·	·	·

Then K is an isotropic 3-plane, and the forms

$$\begin{aligned} \beta_0 &= 2x_{\overline{1}}x_1 + x_0^2 \\ \beta_d &= x_{\overline{4}}x_4 + x_{\overline{3}}x_3 \\ \alpha_2 &= x_{\overline{2}} \end{aligned}$$

define the 2 isotropic lines $\langle v_1\rangle$ and $\langle v_2\rangle$ in K. Lastly, for $i = 1, 2$, there is a unique $H_i \in X_{3\,2\,\overline{1}\,\overline{4}} \cap X'_{(4\,2\,1\,\overline{3})^c}$ with $v_i \in H_i$. In these coordinates,

$$H_1 \ :\ x = z = 0 \quad \text{and} \quad H_2 \ :\ x = z = 1,$$

which shows

$$\#\left(X_{3\,2\,\overline{1}\,\overline{4}} \cap X'_{(4\,2\,1\,\overline{3})^c} \cap X_K\right) \ =\ 2,$$

the coefficient of $P_{4\,2\,1\,\overline{3}}$ in the product $P_{3\,2\,\overline{1}\,\overline{4}} \cdot p_2$ of Example 1.3.

In the symplectic case, β is not a form, $\alpha_0 = x_0$ does not arise, only components d which do not meet the first column give quadratic forms β_d, and the analysis of Lemma 3.2 (2) in Section 3 is simpler.

2. The intersection of two Schubert varieties

We study the intersection of two Schubert varieties. Theorem 2.3 expresses $X_\mu \cap X'_{\lambda^c}$ as a product whose factors correspond to components of λ/μ, and each factor is itself an intersection of two Schubert varieties. These factors are described in Lemmas 2.4 and 2.5, and in Corollary 2.7.

The first step towards Theorem 2.3 is the following combinatorial lemma.

Lemma 2.1. *Let φ count the fixed points and δ the components of λ/μ. Then we have*

$$n+1 \;=\; \varphi+\delta+\#\text{columns of } \lambda/\mu, \tag{2}$$

and $\lambda_{j+1} < \mu_j$ precisely when $|\mu_j|$ is an empty column of λ/μ.

Proof. Let $0 \le l \le n$. We claim that either l indexes a column of λ/μ or else it does not, and in that case, either $l+1$ indexes a column of λ/μ or else l is a fixed point of λ/μ. This proves (2) as the numbers l which do not index a column but $l+1$ does are in bijection with the components of λ/μ.

The case when $l=0$ is our definition of a fixed point.

Suppose $l > 0$ is an empty column of λ/μ. Then there is no i with $\mu_i < l \le \lambda_i$. Let μ_j be the part of μ with $|\mu_j| = l$. If $\mu_j = l$, then $\mu_{j+1} < \mu_j = l$ and so $\lambda_{j+1} < \mu_j = l$ as well. Then either $\mu_j < \lambda_j$ so $l+1$ is a column of λ/μ or else $\mu_j = \lambda_j$ is a fixed point of λ/μ.

Suppose now that $\mu_j = -l$. Let a be the largest index with $l < \mu_a$. We show that $\lambda_j = -l$, which will complete the proof. First, if a part λ_i of λ equals l, then we must have $\mu_i < l = \lambda_i$, contradicting l being an empty column of λ/μ. This shows $\lambda_{a+1} < l$ and also that there is a part λ_i of λ with $\lambda_i = -l$. Since $\lambda, \mu \in \mathbb{SY}_n$, we must have $\{1,\ldots,l\} = \{|\mu_{a+1}|,\ldots,|\mu_j|\} = \{|\lambda_{a+1}|,\ldots,|\lambda_i|\}$. This shows that $j = a+l = i$. ∎

Let d_0 be the component of λ/μ meeting the first column (if any). Define mutually orthogonal subspaces V_φ, V_0, and V_d, for each component d of λ/μ not meeting the first column ($d \ne d_0$) as follows:

$$\begin{aligned}
V_\varphi &:= \langle e_{\mu_j}, e_{\overline{\mu_j}} \mid \mu_j = \lambda_j \rangle,\\
V_0 &:= \langle e_0, e_l, e_{\bar l} \mid l \in \mathrm{col}(d_0) \rangle,\\
V_d^- &:= \langle e_{\bar l} \mid l \in \mathrm{col}(d) \rangle,\\
V_d^+ &:= \langle e_l \mid l \in \mathrm{col}(d) \rangle,
\end{aligned}$$

and set $V_d := V_d^- \oplus V_d^+$. Then

$$V \;=\; V_\varphi \oplus V_0 \oplus \bigoplus_{d \ne d_0} V_d.$$

For each fixed point $\mu_j = \lambda_j$ of λ/μ, define the linear form $\alpha_j := x_{\overline{\mu_j}}$. For each component d of λ/μ, let the quadratic form β_d be the restriction of the form β to V_d. Composing with the projection of V to V_d gives a quadratic form (also written β_d) on V. If there is no component meeting the first column, define $\alpha_0 := x_0$ and call 0 a fixed point of λ/μ. If $d \neq d_0$, then the form β_d identifies V_d^+ and V_d^- as dual vector spaces. For $H \subset V_d^-$, let $H^\perp \subset V_d^+$ be its annihilator.

Lemma 2.2. *Let $H \in X_\mu \cap X'_{\lambda^c}$. Then*

(1) $H \cap V_\varphi = \langle e_{\mu_j} \mid \mu_j = \lambda_j \rangle$.
(2) $\dim(H \cap V_0) = \#col(d_0) - 1$.
(3) *For all components d of λ/μ which do not meet the first column,*

$$\begin{aligned} \dim(H \cap V_d^+) &= \#rows\ of\ d, \\ \dim(H \cap V_d^-) &= \#col(d) - \#rows\ of\ d, \end{aligned}$$

and $\left(H \cap V_d^-\right)^\perp = H \cap V_d^+$.

Proof of Lemma 2.2. Let $H \in X_\mu \cap X'_{\lambda^c}$. Suppose $\lambda_{j+1} < \mu_j$ so that $|\mu_j|$ is an empty column of λ/μ. Then the definition of Schubert variety implies

$$H = H \cap \langle e_{\overline{n}}, \ldots, e_{\lambda_{j+1}} \rangle \oplus H \cap \langle e_{\mu_j}, \ldots, e_n \rangle.$$

Suppose $d \neq d_0$. If the rows of d are $j, \ldots, k$, then

$$\begin{aligned} H \cap V_d^+ &= H \cap \langle e_{\mu_k}, \ldots, e_{\lambda_j} \rangle \\ &= H \cap \langle e_{\overline{n}}, \ldots, e_{\lambda_j} \rangle \cap \langle e_{\mu_k}, \ldots, e_n \rangle, \end{aligned}$$

and so has dimension at least $k - j + 1$.

Similarly, if $l, \ldots, m$ are the indices i with $\overline{\lambda_j} \leq \mu_i, \lambda_i \leq \overline{\mu_k}$, then $H \cap V_d^-$ has dimension at least $m-l+1$. Hence $\frac{1}{2}\dim V_d = \#\mathrm{col}(d) = k+m-l-j+2$, as $\lambda_j, \ldots, \lambda_k, \overline{\lambda_l}, \ldots, \overline{\lambda_m}$ are the columns of d.

Since H is isotropic, $\dim H_d^+ + \dim H_d^- \leq \#\mathrm{col}(d)$, which proves the first part of (3). Moreover, $H \cap V_d^+ \subset \left(H \cap V_d^-\right)^\perp$ as H is isotropic, and equality follows by counting dimensions.

Similar arguments prove the other statements. ∎

For $H \in X_\mu \cap X'_{\lambda^c}$, define $H_\varphi := H \cap V_\varphi$, $H_0 := H \cap V_0$, and $H_d^\pm := H \cap V_d^\pm$.

Proof of Lemma 1.4. Note that $H_\varphi \subset V_\varphi$ is the zero locus of the linear forms α_j, H_0 is isotropic in V_0, and, for each component d of λ/μ not meeting the first column, $H_d := H_d^+ \oplus H_d^-$ is isotropic in V_d. It follows from Lemma 2.2 that the forms α_j, β_d vanish on $H_\varphi \oplus H_0 \oplus \bigoplus_{d \neq d_0} H_d$. Dimension-counting shows that this sum equals H, which proves the lemma. ∎

As the spaces V_φ, V_0, and the V_d are mutually orthogonal, the decomposition $H = H_\varphi \oplus H_0 \oplus \bigoplus_{d\neq d_0} H_d$ is an orthogonal direct sum. Also, $X_\mu \cap X'_{\lambda^c}$ is an irreducible variety, as it has an algebraic stratification with a unique stratum of largest dimension [3].

Theorem 2.3. *Suppose λ/μ is a skew row. With the definitions given above, the map*

$$\{H_0 \mid H \in X_\mu \cap X'_{\lambda^c}\} \;\times \prod_{d\neq d_0} \{H_d \mid H \in X_\mu \cap X'_{\lambda^c}\} \longrightarrow\; X_\mu \cap X'_{\lambda^c}$$

defined by

$$(H_0, \ldots, H_d, \ldots) \longmapsto \langle H_\varphi, H_0, \ldots, H_d, \ldots\rangle$$

is an isomorphism of algebraic varieties.

Proof. By the previous discussion, this map is an injection. For surjectivity, note that both sides are irreducible and have the same dimension. Indeed, $\dim(X_\mu \cap X'_{\lambda^c}) = |\lambda| - |\mu|$, the number of boxes in λ/μ. Lemmas 2.4 and 2.5 show that each factor has dimension equal to the number of boxes in the corresponding component. ∎

Suppose there is a component d_0 meeting the first column. Let l be the largest column in d_0, and define $\lambda(0), \mu(0) \in \mathbb{SY}_l$ as follows: Let j be the first row of d_0 so that $l = \lambda_j$. Then, since d_0 is a component, for each $j \leq i < j+l-1$, we have $\lambda_{i+1} \geq \mu_i$ and $l = \overline{\mu_{j+l-1}}$. Set

$$\begin{aligned} \mu(0) &:= \mu_j > \cdots > \mu_{j+l-1} \\ \lambda(0) &:= \lambda_j > \cdots > \lambda_{j+l-1} \end{aligned}$$

Define $\lambda(0)^c$ by $\lambda(0)^c_p = \overline{\lambda(0)_{l+1-p}}$. The following lemma is straightforward.

Lemma 2.4. *With the above definitions,*

$$\{H_0 \mid H \in X_\mu \cap X'_{\lambda^c}\} \;\simeq\; X_{\mu(0)} \cap X'_{\lambda(0)^c}$$

as subvarieties of $B_l \simeq B(V_0)$, and $\lambda(0)/\mu(0)$ has a unique component meeting the first column and no fixed points.

We similarly identify $\{H_d \mid H \in X_\mu \cap X'_{\lambda^c}\}$ as an intersection $X_{\mu(d)} \cap X'_{\lambda(d)^c}$ of Schubert varieties in $B_{\#\text{columns of } d} \simeq B(\langle e_0, V_d\rangle)$. Let $j, \ldots, k$ be the rows of d and $l, \ldots, m$ be the indices i with $\overline{\lambda_j} \leq \mu_i, \lambda_i \leq \overline{\mu_k}$, as in the proof of Lemma 2.2. Let $p = \#$columns of d and define $\lambda(d), \mu(d) \in \mathbb{SY}_p$ as follows. Set $a = \mu_k$, and define

$$\begin{aligned} \mu(d) &:= \mu_j - a + 1 > \cdots > \quad 1 \quad > \mu_l + a - 1 > \cdots > \mu_m + a - 1 \\ \lambda(d) &:= \lambda_j - a + 1 > \cdots > \lambda_k - a + 1 > \lambda_l + a - 1 > \cdots > \lambda_m + a - 1 \end{aligned}$$

Define $\lambda(d)^c$ by $\lambda(d)^c_j = \overline{\lambda(d)_{p+1-j}}$. The following lemma is straightforward.

Lemma 2.5. *With these definitions,*

$$\{H_d \mid H \in X_\mu \cap X'_{\lambda^c}\} \;\simeq\; X_{\mu(d)} \cap X'_{\lambda(d)^c}$$

as subvarieties of $B_p \simeq B(\langle e_0, V_d\rangle)$ and $\lambda(d)/\mu(d)$ has a unique component not meeting the first column and no non-zero fixed points.

Suppose now that $\mu, \lambda \in \mathbb{SY}_n$ where λ/μ has a unique component d not meeting the first column and no non-zero fixed points. Suppose λ has k rows. A consequence of Lemma 2.2 is that the map $H^+_d \mapsto \langle H^+_d, (H^+_d)^\perp\rangle$ gives an isomorphism

$$\{H^+_d \mid H \in X_\mu \cap X'_{\lambda^c}\} \;\xrightarrow{\sim}\; X_\mu \cap X'_{\lambda^c}. \tag{4}$$

We identify the domain of this map, a subvariety of the (classical) Grassmannian $G_k(V^+)$ of k-planes in $V^+ := \langle e_1, \ldots, e_n\rangle$. See [10, 7, 5] for basics on the Grassmannian. Schubert subvarieties $\Omega_\sigma, \Omega'_{\sigma^c}$ of $G_k(V^+)$ are indexed by partitions $\sigma \in \mathbb{Y}_k$, that is, integer sequences $\sigma = (\sigma_1, \ldots, \sigma_k)$ with $n-k \geq \sigma_1 \geq \cdots \geq \sigma_k \geq 0$. For $\sigma \in \mathbb{Y}_k$ define $\sigma^c \in \mathbb{Y}_k$ by $\sigma^c_j = n-k-\sigma_{k+1-j}$. For $\sigma, \tau \in \mathbb{Y}_k$, define

$$\begin{aligned} \Omega_\tau &:= \{H \in G_k(V^+) \mid \dim(H \cap \langle e_{k+1-j+\tau_j}, \ldots, e_n\rangle) \geq j,\ 1 \leq j \leq k\} \\ \Omega'_{\sigma^c} &:= \{H \in G_k(V^+) \mid \dim(H \cap \langle e_1, \ldots, e_{j+\sigma_{k+1-j}}\rangle) \geq j,\ 1 \leq j \leq k\}. \end{aligned}$$

Let $\lambda, \mu \in \mathbb{SY}_n$ with $\mu \leq \lambda$, and suppose $\mu_k > 0 > \mu_{k+1}$. Define partitions σ and τ in $\mathbb{Y}_k$ (which depend upon λ and μ) by

$$\begin{aligned} \tau &:= \mu_1 - k \geq \cdots \geq \mu_k - 1 \geq 0 \\ \sigma &:= \lambda_1 - k \geq \cdots \geq \lambda_k - 1 \geq 0. \end{aligned}$$

Lemma 2.6. *Let $\mu \leq \lambda \in \mathbb{SY}_n$, and define $\sigma, \tau \in \mathbb{Y}_k$, and k as above. If $H \in X_\mu \cap X'_{\lambda^c}$, then $H \cap V^+ = \langle e_1, \ldots, e_n\rangle$ contains a k-plane $L \in \Omega_\tau \cap \Omega'_{\sigma^c}$.*

Proof. Suppose first that $H \in X_\mu$ satisfies $\dim(H \cap \langle e_{1+\mu_{k+1}}, \ldots, e_n\rangle) = k$. Since $\mu_k > 0 > \mu_{k+1}$, we must have that $L := H \cap V^+$ has dimension k as L lies between two spaces

$$H \cap \langle e_{\mu_k}, \ldots, e_n\rangle \;\subset\; L \;\subset\; H \cap \langle e_{1+\mu_{k+1}}, \ldots, e_n\rangle$$

each of dimension k. Moreover, $L \in \Omega_\tau$ since for $1 \leq j \leq k$, $k+1-j+\tau_j = \mu_j$ and $L \cap \langle e_{\mu_j}, \ldots, e_n\rangle = H \cap \langle e_{\mu_j}, \ldots, e_n\rangle$, which has dimension at least j. If $H \in X'_{\lambda^c}$, then similar arguments show $L \in \Omega'_{\sigma^c}$. The lemma follows as such H are dense in $X_\mu \cap X'_{\lambda^c}$. ∎

Corollary 2.7. *Suppose λ/μ has a unique component not meeting the first column and no non-zero fixed points and let σ, τ, and k be defined as in Lemma 2.6. We have:*

$$\{H_d^+ \mid H \in X_\mu \cap X'_{\lambda^c}\} \;=\; \Omega_\tau \cap \Omega'_{\sigma^c},$$

as subvarieties of $G_k(V^+)$.

Remark 2.8. The symplectic analogs of Lemma 2.5 and Corollary 2.7, which are identical save for the necessary replacement of Y for X and C_p for B_p, show an interesting connection between the geometry of $C(W)$ and $B(V)$. Namely, suppose λ/μ has no component meeting the first column. Then the projection map $V \twoheadrightarrow W$ defined by

$$e_i \;\longmapsto\; \begin{cases} 0 & \text{if } i = 0 \\ f_i & \text{otherwise} \end{cases}$$

and its left inverse $W \hookrightarrow V$ defined by $f_j \mapsto e_j$ induce isomorphisms

$$X_\mu \cap X'_{\lambda^c} \;\stackrel{\sim}{\longleftrightarrow}\; Y_\mu \cap Y'_{\lambda^c}.$$

3. Pieri-type intersections of Schubert varieties

Let λ/μ be a skew row and set $Z_{\lambda/\mu}$ be the zero locus of the forms α_j and β_d of Section 2. In Section 1, we deduced Theorem 1.2 from the following theorem.

Theorem 3.1. *Let λ/μ be a skew row, $Z_{\lambda/\mu}$ be as above, and $\langle v\rangle$ a general line in $Z_{\lambda/\mu}$. Then $X_\mu \cap X'_{\lambda^c} \cap X_{\langle v\rangle}$ is a singleton.*

Proof. Let $\mathcal{Q}_0$ be the cone of isotropic points in V_0 and $\mathcal{Q}_d$ the cone of isotropic points in V_d for $d \neq d_0$. These are the zero loci of the forms β_0 and β_d, respectively. Thus

$$Z_{\lambda/\mu} \;=\; H_\varphi \oplus \mathcal{Q}_0 \oplus \bigoplus_{d \neq d_0} \mathcal{Q}_d$$

and so a general non-zero vector v in $Z_{\lambda/\mu}$ has the form

$$v \;=\; \sum_{\mu_j = \lambda_j} a_j e_{\mu_j} \;+\; v_0 \;+ \sum_{d \neq d_0} v_d,$$

where $a_j \in \mathbb{C}^\times$ and $v_0 \in \mathcal{Q}_0$, $v_d \in \mathcal{Q}_d$ are general vectors.

Thus, if $H \in X_\mu \cap X'_{\lambda^c} \cap X_{\langle v\rangle}$, then $v_0 \in H_0$ and $v_d \in H_d$. By Theorem 2.3, H is determined by H_0 and the H_d, thus it suffices to prove that H_0 and the H_d are uniquely determined by the vectors v_0, v_d. By Lemmas 2.4 and 2.5, this is just the case of the theorem when λ/μ has a single component, which is Lemma 3.2 below. ▪

Lemma 3.2. *Suppose $\lambda, \mu \in \mathbb{SY}_n$ where λ/μ is a skew row with a unique component and no non-zero fixed points. The $Z_{\lambda/\mu} = \mathcal{Q}$, the set of isotropic points in V, and*

1. *If λ/μ does not meet the first column and $v \in \mathcal{Q}$ is a general vector, then $X_\mu \cap X'_{\lambda^c} \cap X_{\langle v\rangle}$ is a singleton.*
2. *If λ/μ meets the first column and $v \in \mathcal{Q}$ is general, then $X_\mu \cap X'_{\lambda^c} \cap X_{\langle v\rangle}$ is a singleton.*

Proof of (1). Recall that $V^+ = \langle e_1, \ldots, e_n\rangle$ and $V^- = \langle e_{\overline{n}}, \ldots, e_{\overline{1}}\rangle$. Let $v \in \mathcal{Q}$ be a general vector. Since $\mathcal{Q} \subset V^+ \oplus V^-$, $v = v^+ \oplus v^-$ with $v^+ \in V^+$ and $v^- \in V^-$. Suppose $\mu_k > 0 > \mu_{k+1}$. Consider the set

$$\{H^+ \in G_k(V^+) \mid v \in H^+ \oplus \left(H^+\right)^\perp\} \;=\; \{H^+ \mid v^+ \in H^+ \subset (v^-)^\perp\}.$$

This is a Schubert variety $\Omega''_{h(n-k,k)}$ of G_kV^+, where $h(n-k,k)$ is the partition of hook shape with a single row of length $n-k$ and a single column of length k.

Under the isomorphisms of (4) and Lemma 2.5, and with the identification of Corollary 2.7, we see that

$$X_\mu \cap X'_{\lambda^c} \cap X_{\langle v\rangle} \;\simeq\; \Omega_\tau \cap \Omega'_{\sigma^c} \cap \Omega''_{h(n-k,k)},$$

where σ, τ are as defined in the paragraph preceding Lemma 2.6. For $\rho \in \mathbb{Y}_k$, let $S_\rho := [\Omega_\rho]$ be the cohomology class Poincaré dual to the fundamental cycle of Ω_ρ in $H^*G_kV^+$. The multiplicity we wish to compute is

$$\deg(S_\tau \cdot S_{\sigma^c} \cdot S_{h(n-k,k)}). \tag{5}$$

By the classical Pieri formula (as $S_{h(n-k,k)} = S_{n-k} \cdot S_{1^{k-1}}$), we see that (5) is 1 as σ/τ has exactly one box in each diagonal. To see this, note that the transformation $\mu, \lambda \longmapsto \tau, \sigma$ takes columns of λ/μ to diagonals of σ/τ. ∎

Our proof of Lemma 3.2 (2) uses an explicit system of local coordinates for $X_\mu \cap X'_{\lambda^c}$ in the special case where λ/μ is a skew row with a unique component meeting the first column, and the further restriction that a component λ_{k+1} of λ is 1. We shall see that this is no restriction, as either λ or μ^c must have a part equal to 1, for such λ, μ.

Let λ/μ be as in Lemma 3.2 (2), and suppose $\lambda_{k+1}=1$. For $x_0,\ldots,x_{n-1}$,$y_2,\ldots,y_n\in\mathbb{C}$, define isotropic vectors $g_j\in V$ as follows:

$$g_j \ := \ \begin{cases} e_{\lambda_j}+\displaystyle\sum_{i=\mu_j}^{\lambda_j-1} x_i\, e_i & j\leq k \\ -2x_0^2e_1+2x_0e_0+e_{\overline{1}}+\displaystyle\sum_{i=\mu_{k+1}}^{\overline{2}} y_{\overline{\imath}}e_i & j=k+1 \\ e_{\lambda_j}+\displaystyle\sum_{i=\mu_j}^{\lambda_j-1} y_{\overline{\imath}}\, e_i & j>k+1 \end{cases} . \tag{6}$$

Lemma 3.3. *Let $\lambda,\mu\in\mathbb{SY}_n$ where λ/μ is a skew row meeting the first column with no fixed points and one part of λ equal to 1, say $\lambda_{k+1}=1$. This forces $\mu_k>0>\mu_{k+1}$. Define $\tau,\sigma\in\mathbb{Y}_k$, and k as for Lemma 2.6 and also $g_1,\ldots,g_n$ as in (6). Then*

1. *For any $x_1,\ldots,x_{n-1}\in\mathbb{C}$, we have $\langle g_1,\ldots,g_k\rangle\ \in\ \Omega_\tau\cap\Omega'_{\sigma^c}$.*
2. *For any $x_0,\ldots,x_{n-1}\in\mathbb{C}$ with $x_{\overline{\mu_{k+1}}},\ldots,x_{\overline{\mu_{n-1}}}\neq 0$, the condition that $H:=\langle g_1,\ldots,g_n\rangle$ is isotropic determines a unique $H\in X_\mu\cap X'_{\lambda^c}$.*

Moreover, these coordinates parameterize dense subsets of the intersections, and the intersections are transverse along these subsets.

Proof. Statement 1 is immediate from the definitions. For 2, note that $\langle g_1,\ldots,g_n\rangle$ is isotropic if and only if

$$\beta(g_i,g_j)\ =\ 0 \quad\text{for}\quad i\leq k<j.$$

Observe that for $i\leq k<j$,

$$\beta(g_i,g_j)\ \not\equiv\ 0 \quad\Longleftrightarrow\quad [\mu_i,\lambda_i]\cap[\overline{\lambda_j},\overline{\mu_j}]\neq\emptyset.$$

Suppose $\beta(g_i,g_j)\not\equiv 0$. If we order the variables $x_0<\cdots<x_{n-1}<y_2<\cdots<y_n$, then the lexicographically leading term of $\beta(g_i,g_j)$ will be

$$\begin{array}{rl} y_{\lambda_i} & \text{if } \lambda_i\in[\overline{\lambda_j},\overline{\mu_j}], \\ y_{\overline{\mu_j}}x_{\overline{\mu_j}} & \text{if } \lambda_i\notin[\overline{\lambda_j},\overline{\mu_j}],\ \text{ so } \mu_i<\overline{\mu_j}<\lambda_i, \quad\text{or} \\ y_n=y_{\overline{\mu_n}} & \text{if } i=1,\ j=n. \end{array}$$

Since $\{2,\ldots,n\}=\{\lambda_2,\ldots,\lambda_{k-1},\overline{\mu_k},\ldots,\overline{\mu_n}\}$, each y_l appears in the leading term of a unique $\beta(g_i,g_j)$ with $i\leq k<j$, showing there are $n-1$ nontrivial equations $\beta(g_i,g_j)=0$, and that these determine $y_2,\ldots,y_n$ uniquely in terms of the x_i when $x_{\overline{\mu_{k+1}}},\ldots,x_{\overline{\mu_{n-1}}}\neq 0$.

These coordinates parameterize an n-dimensional subset of $X_\mu\cap X'_{\lambda^c}$. Since $X_\mu\cap X'_{\lambda^c}$ is irreducible of dimension n [3], this subset is dense. To complete the proof, observe that the equations $\beta(g_i, g_j) = 0$ define a reduced scheme in the set of parameters $x_0, \ldots, x_{n-1}$, $y_2, \ldots, y_n$. ▪

Example 3.4. Let $\lambda = 6\,5\,3\,1\,\overline{2}\,\overline{4}$ and $\mu = 5\,3\,1\,\overline{2}\,\overline{4}\,\overline{6}$ so $k = 3$. We display the vectors g_i in a matrix

	$e_{\overline{6}}$	$e_{\overline{5}}$	$e_{\overline{4}}$	$e_{\overline{3}}$	$e_{\overline{2}}$	$e_{\overline{1}}$	e_0	e_1	e_2	e_3	e_4	e_5	e_6
g_1	·	·	·	·	·	·	·	·	·	·	·	x_5	1
g_2	·	·	·	·	·	·	·	·	·	x_3	x_4	1	·
g_3	·	·	·	·	·	·	·	x_1	x_2	1	·	·	·
g_4	·	·	·	·	y_2	1	$2x_0$	$-2x_0^2$	·	·	·	·	·
g_5	·	·	y_4	y_3	1	·	·	·	·	·	·	·	·
g_6	y_6	y_5	1	·	·	·	·	·	·	·	·	·	·

Then there are 5 non-zero equations $\beta(g_i, g_j) = 0$ with $i \leq 3 < j$:

$$\begin{aligned} 0 &= \beta(g_3, g_4) &=& \ y_2x_2 + x_1 \\ 0 &= \beta(g_3, g_5) &=& \ y_3 + x_2 \\ 0 &= \beta(g_2, g_5) &=& \ y_4x_4 + y_3x_3 \\ 0 &= \beta(g_2, g_6) &=& \ y_5 + x_4 \\ 0 &= \beta(g_1, g_6) &=& \ y_6 + y_5x_5 \end{aligned}$$

Solving, we obtain:

$$y_2 = -x_1/x_2,\ y_3 = -x_2,\ y_4 = -y_3x_3/x_4,\ y_5 = -x_4,\ \text{and } y_6 = -y_5x_5.$$

Proof of Lemma 3.2 (2). Suppose $\lambda, \mu \in \mathbb{SY}_n$ where λ/μ is a skew row with a single component meeting the first column and no fixed points. Let v be a general isotropic vector and consider the condition that $v \in H$ for $H \in X_\mu \cap X'_{\lambda^c}$. Let $\sigma, \tau \in \mathbb{Y}_k$ be defined as in the paragraph preceding Lemma 2.6. We first show that there is a unique $L \in \Omega_\tau \cap \Omega_{\sigma^c}$ with $L \subset H$, and then argue that H is unique.

The conditions on μ and λ imply that $\mu_n = \overline{n}$ and $\mu_j = \lambda_{j+1}$ for $j < n$. We further suppose that $\lambda_{k+1} = 1$, so that the last row of λ/μ has length 1. This is no restriction, as the isomorphism of V defined by $e_j \mapsto e_{\overline{j}}$ sends $X_\mu \cap X'_{\lambda^c}$ to $X_{\lambda^c} \cap X'_{(\mu^c)^c}$ and one of λ/μ or μ^c/λ^c has last row of length 1.

Let v be a general isotropic vector. Scale v so that its $e_{\overline{1}}$-component is 1. Let $2z$ be its e_0-component, then necessarily its e_1-component is $-2z^2$. Let $v^- \in V^-$ be the projection of v to V^-. Similarly define $v^+ \in V^+$. Set $v' := v^+ + 2z^2e_1$, so that $\beta(v^-, v') = 0$ and

$$v \ = \ v^- + 2z(e_0 - ze_1) + v'.$$

Let $H \in X_\mu \cap X'_{\lambda^c}$, and suppose that $v \in H$. In the notation of Lemma 2.6, let $L \in \Omega_\tau \cap \Omega_{\sigma^c}$ be a k-plane in $H \cap V^+$. If H is general, in that

$$\dim(H \cap \langle e_{\overline{n}}, \ldots, e_{\lambda_{k+2}} \rangle) = \dim(H \cap \langle e_{\overline{n}}, \ldots, e_0 \rangle) = n - k - 1,$$

then $\langle L, e_1 \rangle$ is the projection of H to V^+. As $v \in H$, we have $v^+ \in \langle L, e_1 \rangle$. Since $L \subset v^\perp \cap V^+ = (v^-)^\perp$, we see that $v' \in L$, and hence

$$v' \in L \subset (v^-)^\perp.$$

As in the proof of part (1), there is a (necessarily unique) such $L \in \Omega_\tau \cap \Omega_{\sigma^c}$ if and only if σ/τ has a unique box in each diagonal. But this is the case, as the transformation $\mu, \lambda \longrightarrow \tau, \sigma$ takes columns of λ/μ (greater than 1) to diagonals of σ/τ.

To complete the proof, we use the local coordinates for $X_\mu \cap X'_{\lambda^c}$ and $\Omega_\tau \cap \Omega_{\sigma^c}$ of Lemma 3.3. Since v is general, we may assume that the k-plane $L \in \Omega_\tau \cap \Omega_{\sigma^c}$ determined by $v' \in L \subset (v^-)^\perp$ has non-vanishing coordinates $x_{\overline{\mu_{k+1}}}, \ldots, x_{\overline{\mu_{n-1}}}$, so that there is an $H \in X_\mu \cap X'_{\lambda^c}$ in this system of coordinates with $L = H \cap V^+$.

Such an H is determined up to a choice of coordinate x_0. The requirement that $v \in H$ forces the projection $\langle e_{\overline{1}} + 2x_0 e_0 \rangle$ of H to $\langle e_{\overline{1}}, e_0 \rangle$ to contain $e_{\overline{1}} + 2ze_0$, the projection of v to $\langle e_{\overline{1}}, e_0 \rangle$. Hence $x_0 = z$, and it follows that there is at most one $H \in X_\mu \cap X'_{\lambda^c}$ with $v \in H$. Let $g_1, \ldots, g_n$ be the vectors (6) determined by the coordinates $x_1, \ldots, x_{n-1}$ for L with $x_0 = z$. We claim $v \in H := \langle g_1, \ldots, g_n \rangle$.

Indeed, since $v' \in L$ and $v^- \in L^\perp = \langle g_{k+1} - 2z(e_0 - ze_1), g_{k+2}, \ldots, g_n \rangle$, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with

$$v^- + v' = \alpha_1 g_1 + \cdots + \alpha_{k+1}(g_{k+1} - 2z(e_0 - ze_1)) + \cdots + \alpha_n g_n.$$

We must have $\alpha_{k+1} = 1$, since the $e_{\overline{1}}$-component of both v and g_{k+1} is 1. It follows that

$$v = \sum_{i=1}^{n} \alpha_i g_i \quad \in H. \qquad \blacksquare$$

Remarks. It would be interesting to continue this program to give triple intersection proofs of Pieri-type formulas in all Grassmannians of classical groups. This would give *new* formulas and complement the work of Pragacz and Ratajski [13, 14, 15]. In general, there are two distinct types of special Schubert classes and our methods work best with one type. Pragacz and Ratajski gave Pieri-type formulas in these Grassmannians for the other type.

These explicit methods are similar to those used to prove the Pieri-type formula for classical flag varieties [17] and for isotropic flag varieties [1].

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