

# MAXIMALLY INFLECTED REAL RATIONAL CURVES

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ABSTRACT. We introduce and begin the topological study of real rational plane curves, all of whose inflection points are real. The existence of such curves is a corollary of results in the real Schubert calculus, and their study has consequences for the important Shapiro and Shapiro conjecture in the real Schubert calculus. We establish restrictions on the number of real nodes of such curves and construct curves realizing the extreme numbers of real nodes. These constructions imply the existence of real solutions to some problems in the Schubert calculus. We conclude with a discussion of maximally inflected curves of low degree.

## INTRODUCTION

In 1876, Harnack [15] established the bound of  $g + 1$  for the number of ovals of a smooth curve of genus  $g$  in  $\mathbb{R}\mathbb{P}^2$  and showed this bound is sharp by constructing curves with this number of ovals. Since then such curves with maximally many ovals, or  $M$ -curves, have been primary objects of interest in the topological study of real algebraic plane curves (part of the 16th problem of Hilbert) [44, 42]. We introduce and begin the study of maximally inflected curves, which may be considered to be analogs of  $M$ -curves among rational curves, in the sense that, like the classical maximal condition, maximal inflection implies non-trivial restrictions on the topology. The case of a maximally inflected curve that we are most interested in is a rational plane curve of degree  $d$ , all of whose  $3(d-2)$  inflection points are real. More generally, consider a parameterization of a rational curve of degree  $d$  in  $\mathbb{P}^r$  with  $d > r$ ; a map from  $\mathbb{P}^1$  to  $\mathbb{P}^r$  of degree  $d$ . Such a map is said to be *ramified* at a point  $s \in \mathbb{P}^1$  if its first  $r$  derivatives do not span  $\mathbb{P}^r$ . Over  $\mathbb{C}$ , there always will be  $(r+1)(d-r)$  such points of ramification, counted with multiplicity. A *maximally inflected curve* is, by definition, a parameterized real rational curve, all of whose ramification points are real.

We first address the question of existence of maximally inflected curves. A conjecture of B. Shapiro and M. Shapiro in the real Schubert calculus [39] would imply the existence of maximally inflected curves, for every possible placement of the ramification points. A. Eremenko and A. Gabrielov [9] proved the conjecture of Shapiro and Shapiro in the special case when  $r = 1$ . In that case, a maximally inflected curve is a real rational function, all of whose critical points are real.

When  $r > 1$  and for special types of ramification (including flexes and cusps of plane curves) and when the ramification points are clustered very near to one another, there do exist maximally inflected curves, by a result in the real Schubert calculus [38]. When the degree  $d$  is even and the curve has only simple flexes, but arbitrarily placed, the existence

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of such curves follows from the results of Eremenko and Gabrielov on the degree of the Wronski map [8].

Suppose  $r = 2$ , the case of plane curves. Consider a maximally inflected plane curve of degree  $d$  with the maximal number  $3(d-2)$  of real flexes. By the genus formula, the curve has at most  $\binom{d-1}{2}$  ordinary double points. We deduce from the Klein [21] and Plücker [26] formulas that the number of real nodes of such a curve is however at most  $\binom{d-2}{2}$ . Then we show that this bound is sharp. We use E. Shustin's theorem on combinatorial patchworking for singular curves [33] to construct maximally inflected curves that realize the lower bound of 0 real nodes, and another theorem of Shustin concerning deformations of singular curves [36] to construct maximally inflected curves with the upper bound of  $\binom{d-2}{2}$  real nodes. We also classify such curves in degree 4, and discuss some aspects of the classification of quintics.

The connection between the Schubert calculus and rational curves in projective space (linear series on  $\mathbb{P}^1$ ) originated in work of G. Castelnuovo [4] on  $g$ -nodal rational curves. This led to the use of Schubert calculus in Brill-Noether theory (see Chapter 5 of [16] for an elaboration). In a closely related vein, the Schubert calculus features in the local study of flattenings of curves in singularity theory [31, 19, 1]. In turn, the theory of limit linear series of D. Eisenbud and J. Harris [6, 7] provides essential tools to show reality of the special Schubert calculus [38], which gives the existence of many types of maximally inflected curves. The constructions we give using patchworking and gluing show the existence of real solutions to some problems in the Schubert calculus. In particular, they show the existence of maximally inflected plane curves with given type of ramification when the degree  $d$  is odd, extending the results of Eremenko-Gabrielov on the degree of the Wronski map [8].

This paper is organized as follows. In Section 1 we describe the connection between the Schubert calculus and maps from  $\mathbb{P}^1$  with specified points of ramification. In Section 2, we show the existence of some special types of maximally inflected curves and discuss the conjecture of Shapiro and Shapiro in this context. We restrict our attention to maximally inflected plane curves in Section 3, establishing bounds on the number of solitary points and real nodes. In Section 4, we construct maximally inflected curves with extreme numbers of real nodes and in Section 5, we discuss the classification of quartics. In Section 6, we consider some aspects of the classification of quintics.

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## 1. SINGULARITIES OF PARAMETERIZED CURVES

**1.1. Notations and conventions.** Given non-zero vectors  $v_1, v_2, \dots, v_n$  in  $\mathbb{A}^{r+1}$ , we denote their linear span in the projective space  $\mathbb{P}^r$  by  $\langle v_1, v_2, \dots, v_n \rangle$ . A subvariety  $X$  of  $\mathbb{P}^r$  is *nondegenerate* if its linear span is  $\mathbb{P}^r$ , equivalently, if it does not lie in a hyperplane. A rational map between varieties is denoted by a broken arrow  $X \dashrightarrow Y$ .

Let  $d > r$ . The center of a (surjective) linear projection  $\mathbb{P}^d \dashrightarrow \mathbb{P}^r$  is a  $(d-r-1)$ -dimensional linear subspace of  $\mathbb{P}^d$ . This center determines the projection up to projective transformations of  $\mathbb{P}^r$ . Consequently, we identify the Grassmannian  $Grass_{d-r-1}\mathbb{P}^d$  of  $(d-r-1)$ -dimensional linear subspaces of  $\mathbb{P}^d$  with the space of linear projections  $\mathbb{P}^d \dashrightarrow \mathbb{P}^r$  (modulo projective transformations of  $\mathbb{P}^r$ ).

A flag  $F_\bullet$  in  $\mathbb{P}^d$  is a sequence

$$F_\bullet : F_0 \subset F_1 \subset \cdots \subset F_d = \mathbb{P}^d$$

of linear subspaces with  $\dim F_i = i$ . Given a flag  $F_\bullet$  in  $\mathbb{P}^d$  and a sequence  $\alpha : 0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_r \leq d$  of integers, the *Schubert cell*  $X_\alpha F_\bullet \subset Grass_{d-r-1}\mathbb{P}^d$  is the set of all linear projections  $\pi : \mathbb{P}^d \dashrightarrow \mathbb{P}^r$  such that

$$(1.1) \quad \dim \pi(F_{\alpha_i}) = i, \quad \text{and, if } i \neq 0, \quad \dim \pi(F_{\alpha_{i-1}}) = i - 1.$$

The Schubert variety  $X_\alpha F_\bullet$  is the closure of the Schubert cell; it is obtained by replacing the equalities in (1.1) by inequalities ( $\leq$ ).

Replacing  $\mathbb{P}^d$  by the dual projective space  $\hat{\mathbb{P}}^d$  gives an isomorphism  $Grass_{d-r-1}\mathbb{P}^d \simeq Grass_r\hat{\mathbb{P}}^d$ ; a  $(d-r-1)$ -plane  $\Lambda$  in  $\mathbb{P}^d$  corresponds to the  $r$ -plane in  $\hat{\mathbb{P}}^d$  consisting of hyperplanes of  $\mathbb{P}^d$  that contain  $\Lambda$ . Under this isomorphism, the Schubert variety  $X_\alpha F_\bullet$  is mapped isomorphically to the Schubert variety  $X_{\hat{\alpha}} \hat{F}_\bullet$ , where  $\hat{F}_\bullet$  is the flag dual to  $F_\bullet$  ( $\hat{F}_i$  consists of the hyperplanes containing  $F_{d-i}$ ) and  $\hat{\alpha}$  is the sequence  $0 \leq \beta_0 < \beta_1 < \cdots < \beta_{d-r-1} \leq d$  such that

$$(1.2) \quad \{0, 1, \dots, d\} = \{\alpha_0, \alpha_1, \dots, \alpha_r\} \cup \{d - \beta_0, d - \beta_1, \dots, d - \beta_{d-r-1}\}.$$

**1.2. Parameterized rational curves.** Given  $r+1$  coprime linearly independent binary homogeneous forms of the same degree  $d$ , we have a morphism  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^r$  whose image is nondegenerate and of degree  $d$  in that  $\varphi_*([\mathbb{P}^1]) = d[\ell]$ , where  $[\ell]$  is the standard generator in  $H_2(\mathbb{P}^r)$ . Reciprocally, every morphism from  $\mathbb{P}^1$  to  $\mathbb{P}^r$  whose image is nondegenerate and of degree  $d$  is given by  $r+1$  coprime linearly independent binary homogeneous forms of the same degree  $d$ . We consider two such maps to be equivalent when they differ by a projective transformation of  $\mathbb{P}^r$ . In what follows, we will call an equivalence class of such maps a *rational curve of degree  $d$  in  $\mathbb{P}^r$* .

A rational curve  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  is said to be *ramified* at  $s \in \mathbb{P}^1$  if the derivatives  $\varphi(s), \varphi'(s), \dots, \varphi^{(r)}(s) \in \mathbb{A}^{r+1}$  do not span  $\mathbb{P}^r$ . This occurs at the roots of the Wronskian

$$\det \begin{bmatrix} \varphi_1(s) & \cdots & \varphi_{r+1}(s) \\ \varphi'_1(s) & \cdots & \varphi'_{r+1}(s) \\ \vdots & \ddots & \vdots \\ \varphi_1^{(r)}(s) & \cdots & \varphi_{r+1}^{(r)}(s) \end{bmatrix},$$

of  $\varphi$ , a form of degree  $(r+1)(d-r)$ . (Here,  $\varphi_1, \dots, \varphi_{r+1}$  are the components of  $\varphi$ .) The Wronskian is defined by the curve  $\varphi$  only up to multiplication by a scalar; in particular its set of roots and their multiplicities are well-defined in the equivalence class of  $\varphi$ . An equivalence class of such maps containing a real map  $\varphi$  is a *maximally inflected curve* when the Wronskian has only real roots.

An equivalent formulation is provided by linear series on  $\mathbb{P}^1$ . A rational curve  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  whose image is nondegenerate may be factored in the following way

$$(1.3) \quad \mathbb{P}^1 \xrightarrow{\gamma} \mathbb{P}^d \xrightarrow{-\pi} \mathbb{P}^r$$

where  $\gamma$  is the *rational normal curve*, which is given by

$$[s, t] \longmapsto [s^d, s^{d-1}t, s^{d-2}t^2, \dots, st^{d-1}, t^d],$$

and  $\pi$  is a projection whose center  $\Lambda$  is a  $(d-r-1)$ -plane which does not meet the curve  $\gamma$ . The hyperplanes of  $\mathbb{P}^d$  which contain  $\Lambda$  constitute a base point free linear series of dimension  $r$  in  $\mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}(d)) = \mathbb{P}^d$ . (Base point freeness is equivalent to our requirement that the original forms  $\varphi_i$  do not have a common factor.) The center of projection  $\Lambda$  and hence the linear series depends only upon the original map. In this way, the space of rational curves  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  is identified with the (open) subset of the Grassmannian  $Grass_{d-r-1}\mathbb{P}^d$  of  $(d-r-1)$ -planes in  $\mathbb{P}^d$  which do not meet the image of the rational normal curve  $\gamma$ .

Let us define the *ramification sequence* of  $\varphi$  at a point  $s$  in  $\mathbb{P}^1$  to be the vector  $\alpha = \alpha(s)$  whose  $i$ th component is the smallest number  $\alpha_i$  such that the linear span of  $\varphi(s), \varphi'(s), \dots, \varphi^{(\alpha_i)}(s)$  in  $\mathbb{P}^r$  has dimension  $i$ . Equivalently,  $\mathbb{P}^r$  has coordinates such that  $\varphi = [\varphi_0, \varphi_1, \dots, \varphi_r]$  with  $\varphi_i$  vanishing to order  $\alpha_i$  at  $s$ . We will also call this sequence  $\alpha$  the *ramification* of  $\varphi$  at  $s$ . Note that  $\alpha_0 = 0$  and  $d \geq \alpha_i \geq i$  for  $i = 0, \dots, r$ , and the sequence  $\alpha$  is increasing. Call such an integer vector  $\alpha$  a *ramification sequence* for degree  $d$  curves in  $\mathbb{P}^r$  if  $\alpha_r > r$ . If  $s$  is a point of  $\mathbb{P}^1$  with ramification sequence  $\alpha$ , then the Wronskian vanishes to order  $|\alpha| := (\alpha_1 - 1) + (\alpha_2 - 2) + \dots + (\alpha_r - r)$  at  $s$ . When this is zero, so that  $\alpha_r = r$ , we say that  $\varphi$  is *unramified* at  $s$ . Since the Wronskian has degree  $(r+1)(d-r)$ , the following fact holds.

**Proposition 1.1.** *Let  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^r$  be a rational curve of degree  $d$ . Then*

$$\sum_{s \in \mathbb{P}^1} |\alpha(s)| = (r+1)(d-r). \quad \blacktriangleright$$

A collection  $\alpha^1, \dots, \alpha^n$  of ramification sequences for degree  $d$  curves in  $\mathbb{P}^r$  with  $|\alpha^1| + \dots + |\alpha^n| = (r+1)(d-r)$  will be called *ramification data* for degree  $d$  rational curves in  $\mathbb{P}^r$ .

A ramification sequence is expressed in terms of Schubert cycles in  $Grass_{d-r-1}\mathbb{P}^d$  in the following manner. For each  $s \in \mathbb{P}^1$ , let  $F_\bullet(s)$  be the flag of subspaces of  $\mathbb{P}^d$  which osculate the rational normal curve  $\gamma$  at the point  $\gamma(s)$  and  $\pi$  the linear projection defining  $\varphi$  as in (1.3). Then the image  $\pi(F_i(s))$  of the  $i$ -plane  $F_i(s)$  in  $F_\bullet(s)$  is the linear span of  $\varphi(s), \varphi'(s), \dots, \varphi^{(i)}(s)$  in  $\mathbb{P}^r$ .

Comparing the definitions of ramification sequence and of Schubert cell, we deduce that the curve  $\varphi$  has ramification  $\alpha$  at  $s \in \mathbb{P}^1$  when the center of projection lies in the Schubert cell  $X_\alpha^\circ F_\bullet(s)$  of the Grassmannian. The closure of this cell is the Schubert cycle  $X_\alpha F_\bullet(s)$ , which has codimension  $|\alpha|$  in the Grassmannian. Thus, given ramification data  $\alpha^1, \dots, \alpha^n$  and distinct points  $s_1, \dots, s_n \in \mathbb{P}^1$ , a center  $\Lambda$  provides ramification  $\alpha^i$  at point  $s_i$  for each  $i$  if and only if it belongs to the intersection


$$\bigcap_{i=1}^n X_{\alpha^i}^\circ F_\bullet(s_i)$$

On the other hand,

$$\bigcap_{i=1}^n X_{\alpha^i} F_\bullet(s_i) = \bigcap_{i=1}^n X_{\alpha^i} F_\bullet(s_i)$$

To see this, suppose that a point in  $\bigcap_{i=1}^n X_{\alpha^i} F_\bullet(s_i)$  did not lie in a Schubert cell  $X_{\alpha^i}^\circ F_\bullet(s_i)$ . Then the corresponding rational curve would have ramification exceeding  $\alpha^i$  at the point  $s_i$ , which would contradict Proposition 1.1.

The intersection of the above Schubert cycles has dimension at least  $(r+1)(d-r) - \sum_i |\alpha^i| = 0$ . If it had positive dimension, then it would have a non-empty intersection with any hypersurface Schubert variety  $X_\beta F_\bullet(s)$ , where  $|\beta| = 1$  and  $s$  is not among  $\{s_1, \dots, s_n\}$ . This gives a rational curve violating Proposition 1.1. Thus the intersection is zero-dimensional. Let  $N(\alpha^1, \dots, \alpha^n)$  be its degree, which may be computed using the classical Schubert calculus of enumerative geometry [12]. Thus we deduce the following Corollary of Proposition 1.1.

**Corollary 1.2.** *Given  $d > r$ , ramification data  $\alpha^1, \dots, \alpha^n$  for degree  $d$  rational curves in  $\mathbb{P}^r$ , and distinct points  $s_1, \dots, s_n \in \mathbb{P}^1$ , the number of nondegenerate rational curves in  $\mathbb{P}^r$  of degree  $d$  with ramification  $\alpha^i$  at point  $s_i$  for each  $1 \leq i \leq n$  is  $N(\alpha^1, \dots, \alpha^n)$ , counted with multiplicity.* 

Let us consider some special cases. A curve  $\varphi$  has a *cuspidal point* at  $s$  if the center of projection  $\Lambda$  meets the tangent line to the rational normal curve  $\gamma$  at  $\gamma(s)$ . The corresponding Schubert cycle has codimension  $r$ , and so a rational curve of degree  $d$  in  $\mathbb{P}^r$  has at most  $(d-r)(r+1)/r$  cusps. A curve  $\varphi$  is *simply ramified* (or has a *flex*) at  $s$  if  $\varphi(s), \varphi'(s), \dots, \varphi^{(r-1)}(s)$  span a hyperplane in  $\mathbb{P}^r$  which contains  $\varphi^{(r)}(s)$ . This occurs if the center of the projection  $\Lambda$  meets the osculating  $r$ -plane  $F_r(s)$  in a point. In this case the codimension is 1 and so a rational curve with only flexes has  $(r+1)(d-r)$  flexes.

## 2. MAXIMALLY INFLECTED CURVES

2.1. **Existence.** We ask the following question:

**Question 2.1.** Given  $d > r$ , ramification data  $\alpha^1, \dots, \alpha^n$  for degree  $d$  rational curves in  $\mathbb{P}^r$ , and distinct points  $s_1, \dots, s_n \in \mathbb{RP}^1$ , are there any *real* rational curves  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  with ramification  $\alpha^i$  at the point  $s_i$  for each  $i = 1, \dots, n$ ?

Corollary 1.2 guarantees the existence of  $N(\alpha^1, \dots, \alpha^n)$  such *complex* rational curves, counted with multiplicity. The point of this question is, if the ramification occurs at real points in the domain  $\mathbb{P}^1$ , are any of the resulting curves real? Call a real rational curve whose ramifications occurs only at real points a *maximally inflected* curve. The answer to Question 2.1 is unknown in general. There are however many cases for which the answer is yes. Moreover, one, still open, conjecture of Shapiro and Shapiro in the real Schubert calculus would imply a very strong resolution of Question 2.1. We formulate that conjecture in terms of maximally inflected curves.

**Conjecture 2.2.** (Shapiro-Shapiro) Let  $d > r$  be integers and  $\alpha^1, \dots, \alpha^n$  be ramification data for degree  $d$  rational curves in  $\mathbb{P}^r$ . For any choice of distinct real points  $s_1, \dots, s_n \in \mathbb{RP}^1$ , **every** curve  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^r$  of degree  $d$  with ramification  $\alpha^i$  at  $s_i$  for each  $i = 1, \dots, n$  is real.

Eremenko and Gabriellov [9] proved this conjecture in the cases when  $r$  is 1 or  $d-2$ . It is also known to be true for a few sporadic cases of ramification data and some cases of the conjecture imply others. There is also substantial computational evidence in support of Conjecture 2.2 and no known counterexamples. For an account of this conjecture, see [39] or the web page [37].


A ramification sequence is *special* if it is one of the following.

$$(0, 1, \dots, r-1, r+a) \quad \text{or} \quad (0, 1, \dots, r-a, r-a+2, \dots, r+1),$$

for some  $a > 0$ . When  $a = 1$  these coincide and give the sequence of a flex. We can guarantee the existence of maximally inflected curves with these special singularities.

**Theorem 2.3.** *If  $\alpha^1, \dots, \alpha^n$  are ramification data for degree  $d$  rational curves in  $\mathbb{P}^r$  with at most one sequence  $\alpha^i$  not special, then there **exist** distinct points  $s_1, \dots, s_n \in \mathbb{RP}^1$  such that there are exactly  $N(\alpha^1, \dots, \alpha^n)$  real rational curves of degree  $d$  which have ramification  $\alpha^i$  at  $s_i$  for each  $i = 1, \dots, n$ .*

The point of this theorem is that there are the expected number of such curves, and *all* of them are real.

**Proof.** Let  $\alpha^1, \dots, \alpha^n$  be a ramification data for degree  $d$  rational curves in  $\mathbb{P}^r$  with at most one sequence  $\alpha^i$  not special. These are types of Schubert varieties in the Grassmannian of  $(d-r-1)$ -planes in  $\mathbb{P}^d$  with at most one not special. By Theorem 1 of [38], there exist points  $s_1, \dots, s_n \in \mathbb{RP}^1$  so that the Schubert varieties  $X_{\alpha^i} F_*(s_i)$  defined by flags osculating the rational normal curve at points  $s_i$  intersect transversally with all points of intersection real. Every  $(d-r-1)$ -plane  $\Lambda$  in such an intersection is a center of projection giving a maximally inflected real rational curve  $\varphi$  with ramification sequence  $\alpha^i$  at  $s_i$  for each  $i = 1, \dots, n$ . 

The proof in [38] gives points of ramification that are clustered together. More precisely, suppose that the sequences  $\alpha^2, \dots, \alpha^n$  are special and only (possibly)  $\alpha^1$  is not special. By

$$\forall s_2 \ll s_3 \ll \dots \ll s_n$$

we mean

$$\forall s_2 > 0 \exists \mathcal{N}_3 > 0, \text{ such that } \forall s_3 > \mathcal{N}_3 \exists \mathcal{N}_4 > 0, \dots \\ \forall s_{n-1} > \mathcal{N}_{n-1} \exists \mathcal{N}_n > 0, \text{ such that } \forall s_n > \mathcal{N}_n.$$

Then the choice of points  $s_i$  giving all curves real in Theorem 2.3 is

$$s_1 = \infty \quad \text{and} \quad \forall s_2 \ll s_3 \ll \dots \ll s_n.$$

In short, Theorem 2.3 only guarantees the existence of many maximally inflected curves when almost all of the ramification is special, and the points of ramification are clustered together in this way. A modification of the proof in [38] (along the lines of the Pieri homotopy algorithm in [18]) shows that there can be two ramification indices that are not special, and also two ‘clusters’ of ramification points.

Eremenko and Gabrielov [8, Corollary 4] prove the following.

**Proposition 2.4** (Eremenko-Gabrielov). *Suppose  $0 < r < d$  and  $d$  is even. Let  $m := \max\{r+1, d-r\}$  and  $p := \min\{r+1, d-r\}$ . Set  $M := (r+1)(d-r)$ . Then for any distinct points  $s_1, \dots, s_M \in \mathbb{RP}^1$ , there exist at least*

$$(2.1) \quad \frac{1!2! \cdots (p-1)!(m-1)!(m-2)! \cdots (m-p+1)! \left(\frac{pm}{2}\right)!}{(m-p+2)!(m-p+4)! \cdots (m+p-2)! \left(\frac{m-p+1}{2}\right)! \left(\frac{m-p+3}{2}\right)! \cdots \left(\frac{m+p-1}{2}\right)!}$$

*maximally inflected curves of degree  $d$  in  $\mathbb{P}^r$  with flexes at the points  $s_1, \dots, s_M$ .*

The constructions of Section 4.1 show that there exist maximally inflected plane curves of any degree  $d$  with any number up to  $d-2$  cusps for some choices of ramification points not clustered together.

**Remark 2.5.** It is worthwhile to compare this number (2.1) of Eremenko-Gabrielov to the number of (complex) curves with the same flexes, as computed by Schubert [29]:

$$(2.2) \quad \frac{1! 2! \cdots (p-2)! (p-1)! \cdot (mp)!}{(m)! (m+1)! \cdots (m+p-1)!}.$$

The ratio of (2.2) to (2.1) is

$$r(m, p) := \frac{\left(\frac{m-p+1}{2}\right)! \left(\frac{m-p+3}{2}\right)! \cdots \left(\frac{m+p-1}{2}\right)! \cdot (mp)!}{(m-p+1)! (m-p+3)! \cdots (m+p-1)! \cdot \left(\frac{pm}{2}\right)!}.$$

According to Stirling's formula,  $\log r(m, p)$  grows as

$$mp \log \frac{mp}{2} - (m-p+1) \log \frac{m-p+1}{2} - \cdots - (m+p-1) \log \frac{m+p-1}{2}.$$

Fixing  $p$  and letting  $m$  grow (that is, fixing the target  $\mathbb{P}^r$  and letting the degree  $d$  grow), we obtain the asymptotic value of  $\frac{1}{2}mp \log p$  for  $\log r(m, p)$ . Similar arguments show that, asymptotically, the logarithm of Schubert's number grows like  $mp \log p$ . Thus we see that the number of real solutions guaranteed by Eremenko-Gabrielov is approximately the square root of the total number of solutions, in this asymptotic limit.

This asymptotic result is reminiscent of (but different than) results of E. Kostlan [23] and M. Shub and S. Smale [32] concerning the expected number of real solutions to a system of polynomial equations. The set of real polynomial systems on  $\mathbb{R}\mathbb{P}^n$

$$(2.3) \quad f_1(x_0, x_1, \dots, x_n) = f_2(x_0, x_1, \dots, x_n) = \cdots = f_n(x_0, x_1, \dots, x_n) = 0,$$

where  $f_i$  is homogeneous of degree  $d_i$  is parameterized by  $\mathbb{R}\mathbb{P}^{d_1} \times \mathbb{R}\mathbb{P}^{d_2} \times \cdots \times \mathbb{R}\mathbb{P}^{d_n}$ . Integrating the number of real roots of a system (2.3) against the Fubini-Study probability measure on this space of systems gives the expected number of real roots

$$(d_1 \cdot d_2 \cdots d_n)^{\frac{1}{2}},$$

the square root of the expected number  $d_1 \cdot d_2 \cdots d_n$  of complex roots.

**2.2. Deformations.** Deforming a maximally inflected curve  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^r$  means deforming the positions of its ramifications in  $\mathbb{R}\mathbb{P}^1$ . A set  $S := \{s_1(t), \dots, s_n(t)\}$  of continuous functions  $s_i: [0, 1] \rightarrow \mathbb{R}\mathbb{P}^1$  where, for each  $t$ , the points  $s_1(t), \dots, s_n(t)$  are distinct is an *isotopy* between  $\{s_1(0), \dots, s_n(0)\}$  and  $\{s_1(1), \dots, s_n(1)\}$ . Given such an isotopy  $S$ , suppose  $\varphi$  has ramification  $\alpha^i$  at  $s_i(0)$ , for  $i = 1, \dots, n$ . A continuous family  $\varphi_t$  for  $t \in [0, 1]$  of maximally inflected curves with  $\varphi_0 = \varphi$  and where  $\varphi_t$  has ramification  $\alpha^i$  at  $s_i(t)$ , for  $i = 1, \dots, n$ , is a *deformation of  $\varphi$  along  $S$*  and  $\varphi_1$  is a deformation of  $\varphi$ . A maximally inflected curve  $\varphi$  is said to admit *arbitrary deformations* if  $\varphi$  has a deformation along any isotopy  $S$  deforming the ramification points of  $\varphi$ . Since reparameterization by a projective transformation of  $\mathbb{R}\mathbb{P}^1$  does not change the image of  $\varphi$ , a basic question in the topology of maximally inflected curves is to classify them up to deformation and reparameterization.

Experimental evidence and geometric intuition suggest that maximally inflected curves admit arbitrary deformations, in a strong sense that we make precise in Theorem 2.7(2) below. First we state a non-degeneracy conjecture.

**Conjecture 2.6.** (Conjecture 5 of [38]) Let  $d > r$  and  $\alpha^1, \dots, \alpha^n$  be ramification data for degree  $d$  rational curves in  $\mathbb{P}^r$ . If  $s_1, \dots, s_n \in \mathbb{RP}^1$  are distinct, then there are exactly  $N(\alpha^1, \dots, \alpha^n)$  **complex** rational curves of degree  $d$  in  $\mathbb{P}^r$  with ramification sequence  $\alpha^i$  at  $s_i$  for each  $i = 1, \dots, n$ .

That is, when the points  $s_i$  are real, there should be no multiplicities in Corollary 1.2.


**Theorem 2.7.**

- (1) *Suppose Conjecture 2.6 holds in all cases when the ramification consists only of flexes. Then Conjecture 2.2 holds for all ramification data.*
- (2) *If Conjecture 2.6 holds for all ramification data, then given  $s_1, \dots, s_n \in \mathbb{RP}^1$ , each of the  $N = N(\alpha^1, \dots, \alpha^n)$  maximally inflected curves with ramification sequence  $\alpha_i$  at  $s_i$  admit arbitrary deformations, and the  $N$  deformations along a given isotopy are distinct at each point  $t \in [0, 1]$ .*

**Proof.** Statement 1 is just Theorem 6 of [38], adapted to maximally inflected curves.

For the second statement, let  $\{s_1(t), \dots, s_n(t)\}$  be continuous functions  $s_i: [0, 1] \rightarrow \mathbb{RP}^1$  with the points  $s_1(t), \dots, s_n(t)$  distinct for each  $t$ . For each  $t \in [0, 1]$ , consider the intersection of Schubert varieties

$$\bigcap_{i=1}^n X_{\alpha^i} F_*(s_i(t)) \subset \text{Grass}_{d-r-1} \mathbb{P}^d \times [0, 1].$$

By Conjecture 2.6, this consists of exactly  $N := N(\alpha^1, \dots, \alpha^n)$  points for each  $t$ , and by the first statement, they are all real. Since  $N$  is the degree of such an intersection, it must be transverse, and so the totality of these intersections define  $N$  continuous and non-intersecting arcs in the real points of  $\text{Grass}_{d-r-1} \mathbb{P}^d \times [0, 1]$ . Each arc is a deformation of the corresponding curve at  $t = 0$ , which proves the second statement. 

We remark that Conjecture 2.6 is not true if we remove the restriction that the points  $s_i$  are real. For example, there is a unique map  $\mathbb{P}^1 \rightarrow \mathbb{P}^1$  of degree 3 with simple critical points (simple ramification)  $0, 1, \omega$ , and  $\omega^2$ , where  $\omega$  is a primitive third root of unity. Given 4 critical points in general position, there will be two such maps, which happen to coincide for this particular choice. If however, all critical points are real, then there will always be 2 such maps. (Details are found in [6, Section 9].)

Conjecture 2.6 is known to hold whenever it has been tested. This includes when  $r = 1$  or  $d - 2$  and the ramification consists only of flexes [9] and for some other ramification data. The case  $r = 2$  and  $d = 4$  of plane quartics with arbitrary ramification was shown earlier, by direct computation [39, Theorem 2.3].

**2.3. Constructions using duality.** We give two elementary constructions of new maximally inflected curves from old ones, each invoking a different notion of duality for these curves. The first relies on Grassmann duality—the Grassmannian of  $(d - r - 1)$ -planes in  $\mathbb{P}^d$  is isomorphic to the Grassmannian of  $r$ -planes in  $\mathbb{P}^d$ . The second construction relies on projective duality and has new implications for the Shapiro conjecture.

Let  $F_*(s)$  be the flag of subspaces in  $\mathbb{P}^d$  osculating the rational normal curve  $\gamma(s) = F_0(s)$ . Then its dual flag  $\hat{F}_*(s)$  is the flag of subspaces osculating the rational normal curve  $F_{d-1}(s)$  in the dual projective space. In particular,  $\hat{F}_i(s)$  is dual to  $F_{d-i}(s)$ . Consider a possible ramification sequence  $\alpha = (0, \alpha_1, \dots, \alpha_r)$  for degree  $d$  curves in  $\mathbb{P}^r$ , with the **additional restriction** that  $\alpha_r < d$ . Recall from Section 1.1 that a  $(d - r - 1)$ -plane  $\Lambda$



lies in the Schubert variety  $X_\alpha F_\bullet(s)$  if and only if the dual  $r$ -plane lies in the Schubert variety  $X_{\hat{\alpha}} \hat{F}_\bullet(s)$ . Here, the sequence  $\hat{\alpha}$  is defined from  $\alpha$  by (1.2). Identifying  $\mathbb{P}^d$  with its dual space, this gives a bijection between the two algebraic sets

$$(2.4) \quad \left\{ \begin{array}{l} \text{Curves } \varphi \text{ of degree } d \text{ in} \\ \mathbb{P}^r \text{ with ramification } \alpha^i \\ \text{at } s_i \text{ for } i = 1, \dots, n. \end{array} \right\} \iff \left\{ \begin{array}{l} \text{Curves } \varphi \text{ of degree } d \text{ in} \\ \mathbb{P}^{d-r-1} \text{ with ramification} \\ \hat{\alpha}^i \text{ at } s_i \text{ for } i = 1, \dots, n. \end{array} \right\}$$

For a maximally inflected curve in  $\mathbb{P}^r$ , this *Grassmann duality* gives a maximally inflected curve of the same degree in a possibly different projective space ramified at the same points, but with different ramification sequences.

Now let us give another construction involving the usual projective duality. Given a curve  $C$  in  $\mathbb{P}^r$ , its dual curve  $\check{C}$  is the curve in the dual projective space  $\check{\mathbb{P}}^r$  which is the closure of the set of hyperplanes osculating  $C$  at general points. If  $C$  is the image of a maximally inflected curve  $\varphi$ , then  $\check{C}$  is also rational with a parameterization  $\check{\varphi}$  induced from  $\varphi$ . The ramification points of  $\check{\varphi}$  coincide with those of  $\varphi$ , but the ramification indices and degree of  $\check{\varphi}$  will be different. We compute this degree and the transformation of ramification indices and thus show that  $\check{C}$  is a maximally inflected curve.

The osculating hyperplane at a general point  $\varphi(s)$  of  $C$  is determined by the 1-form  $\psi(s)$  whose  $i$ th coordinate is the determinant

$$(2.5) \quad \left( \varphi_b^{(a)}(s) \right)_{\substack{a=0,1,\dots,r-1 \\ b=0,1,\dots,\widehat{r-i},\dots,r}},$$

where  $\varphi = (\varphi_1, \dots, \varphi_r)$  and the derivatives are taken with respect to some local coordinate of  $\mathbb{P}^1$  at  $s$ , and  $\widehat{r-i}$  indicates that the index  $r-i$  is omitted from the list. We may assume for simplicity that in this local coordinate  $\varphi_i$  has degree  $d-i$ . Thus the 1-form  $\psi$  has degree  $r(d-r+1)$  and its coordinates define a linear series of dimension  $r$  and degree  $r(d-r+1)$  on  $\mathbb{P}^1$ . In general, this linear series will have base points and so the degree of the resulting map will be less than  $r(d-r+1)$ .

Let us determine the base point divisor and the ramification of the map determined by  $\psi$ . Suppose that  $\alpha$  is the ramification sequence of  $\varphi$  at a point  $s$  of  $\mathbb{P}^1$ . We may assume that  $\varphi_i$  vanishes to order  $\alpha_i$  at  $s$ , and a calculation shows that the determinant (2.5) vanishes to order  $|\alpha| + r - \alpha_{r-i}$  at  $s$ . Thus  $s$  has multiplicity  $|\alpha| + r - \alpha_r$  in the base point divisor. Removing this base point divisor from the map  $\psi$  gives a map  $\check{\varphi}$  whose  $i$ th coordinate vanishes to order  $\alpha_r - \alpha_{r-i}$  at  $s$ , and so the ramification sequence of the dual curve at a point  $s$  where  $\alpha = \alpha(s)$  is

$$(2.6) \quad \check{\alpha} := (0, \alpha_r - \alpha_{r-1}, \dots, \alpha_r - \alpha_1, \alpha_r).$$


Thus the map determined by  $\psi$ , and hence the dual curve, has degree

$$(2.7) \quad r(d-r+1) - \sum_{s \in \mathbb{P}^1} (|\alpha(s)| + r - \alpha(s)_r),$$

where  $\alpha(s)$  is the ramification sequence at  $s$ , which is typically  $(0, 1, \dots, r)$ . The following theorem is immediate.

**Theorem 2.8.** *Let  $d > r$  and suppose that  $\alpha^1, \dots, \alpha^n$  is ramification data for rational curves of degree  $d$  in  $\mathbb{P}^r$ . Then  $\check{\alpha}^1, \dots, \check{\alpha}^n$  is ramification data for rational curves in  $\mathbb{P}^r$  of degree*

$$m := r(d-r+1) - \sum_{s \in \mathbb{P}^1} (|\alpha(s)| + r - \alpha(s)_r).$$

- (1) For any choice of distinct points  $s_1, \dots, s_n \in \mathbb{RP}^1$ , there is a one to one correspondence between maximally inflected curves of degree  $d$  with ramification  $\alpha^i$  at  $s_i$  for  $i = 1, \dots, n$  and maximally inflected curves of degree  $m$  with ramification  $\tilde{\alpha}^i$  at  $s_i$  for  $i = 1, \dots, n$ .
- (2) Conjecture 2.2 holds for  $r, d$ , and the sequences  $\alpha^1, \dots, \alpha^n$  if and only if it holds for the integers  $r, m$ , and the sequences  $\tilde{\alpha}^1, \dots, \tilde{\alpha}^n$ . 

We compute the degree  $m$  of the dual curve when  $r = 2$ . Suppose the original curve has ramification indices  $\alpha^1, \dots, \alpha^n$ . The formula (2.7) becomes

$$(2.8) \quad m = 2(d - 1) - \sum_i (|\alpha^i| + 2 - \alpha_2^i) = 4 - d + \sum_i (\alpha_2^i - 2).$$

### 3. MAXIMALLY INFLECTED PLANE CURVES

For plane curves, we have  $r = 2$ . Suppose, in the beginning, for sake of discussion that we have a rational plane curve whose only ramifications are cusps and flexes, and whose only other singularities are ordinary double points. Such a curve of degree  $d$  with  $\kappa$  cusps has  $\iota = 3(d - 2) - 2\kappa$  flexes. This Plücker formula follows, for example, from Proposition 1.1 since a cusp has ramification sequence  $(0, 2, 3)$  and a flex  $(0, 1, 3)$ , and these have weights 2 and 1 respectively. Since the curve is of genus zero, it must have  $g_\kappa := \binom{d-1}{2} - \kappa$  double points.

Suppose now that the curve is real. Each visible (in the real part of  $\mathbb{P}^2$ ) node is either a *real node*—a real ordinary double point with real tangents, or a *solitary point*—a real ordinary double point with complex conjugate tangents. All other nodes are *complex nodes*; they occur in complex conjugate pairs. Let  $\eta$  be the number of real nodes,  $\delta$  the number of solitary points, and  $c$  the number of complex nodes.

Up to projective transformation and reparametrization, there are only three real rational plane cubic curves. They are represented by the equations  $y^2 = x^3 + x^2$ ,  $y^2 = x^3 - x^2$ , and  $y^2 = x^3$ , and they have the shapes shown in Figure 3.1. All three have a real flex at

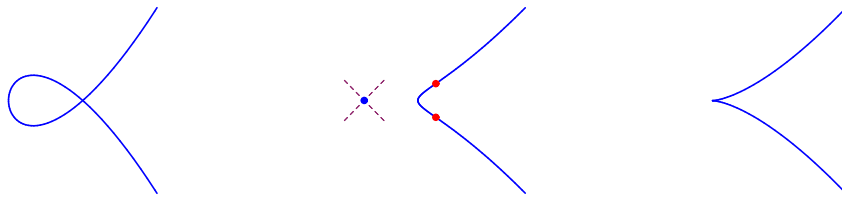


FIGURE 3.1. Real rational cubics

infinity and are singular at the origin. The first has a real node and no other real flexes, the second has a solitary point and two real flexes at  $(\frac{4}{3}, \pm \frac{4}{3\sqrt{3}})$  (we indicate these with dots and the complex conjugate tangents at the solitary point with dashed lines), and the third has a real cusp. The last two are maximally inflected, while the first is not. In general, as is shown below, the number of real nodes is restricted for maximally inflected curves.

In what follows we consider maximally inflected curves with arbitrary ramifications and define a *solitary point* of a real rational curve  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^2$  to be a pair of distinct complex conjugate points  $s, \bar{s} \in \mathbb{P}^1$  with  $\varphi(s) = \varphi(\bar{s})$ , which necessarily represents a point in  $\mathbb{RP}^2$ .

Let  $\delta$  be the number of such solitary points. We similarly define solitary bitangents to be solitary points of the dual curve, and let  $\tau$  be their number.

**Theorem 3.1.** *Let  $\varphi$  be a maximally inflected plane curve of degree  $d$  with ramification  $\alpha^1, \dots, \alpha^n$ . Then the numbers  $\tau$  of solitary bitangents and  $\delta$  of solitary points satisfy*

$$\delta = \tau + d - 2 - \sum_i (\alpha_1^i - 1).$$

**Proof.** Let  $C$  be the real points of image of the curve  $\varphi$  and  $\check{C}$  be its dual curve, the real points of the image of  $\check{\varphi}$ . The generalized Klein formula due to Schuh [30] (see also [42]) gives the relation

$$(3.1) \quad m + \sum_{z \in C} (\mu_z - r_z) = d + \sum_{z \in \check{C}} (\mu_z - r_z),$$

where  $\mu_z$  is the multiplicity of a point  $z$  in a curve and  $r_z$  is the number of real branches of the curve at  $z$ .

We first evaluate the sum

$$(3.2) \quad \sum_{z \in C} (\mu_z - r_z).$$

The multiplicity  $\mu_z$  at a point  $z \in C$  is the local intersection multiplicity (at  $z$ ) of the curve  $C$  with a general linear form  $f$  vanishing at  $z$ . This is the sum over all  $s \in \varphi^{-1}(z)$  of the order of vanishing at  $s$  of the pullback  $\varphi^*(f)$ . This order of vanishing is  $\alpha(s)_1$ , where  $\alpha(s)$  is the ramification sequence of  $\varphi$  at  $s$ . There are two cases to consider: Either  $s$  is real ( $s \in \mathbb{RP}^1$ ) or it is not.

In the first case, the image of a neighborhood of  $s$  in  $\mathbb{RP}^1$  is a branch of  $C$  at  $z$ , so the contribution of  $s$  to the sum (3.2) is  $\alpha(s)_1 - 1$ . Since  $\alpha(s)_1 - 1$  vanishes except at some points of ramification, the contribution of points in  $\mathbb{RP}^1$  to (3.2) is the sum

$$\sum_{i=1}^n (\alpha_1^i - 1).$$

In the second case, the complex conjugate  $\bar{s}$  of  $s$  is also in  $\varphi^{-1}(z)$ . Since  $\varphi$  is maximally inflected, it is unramified at these points, so  $\alpha(s)_1 = \alpha(\bar{s})_1 = 1$ . Thus each solitary point contributes 2 to the sum (3.2). Combining these observations gives


$$\sum_{z \in C} (\mu_z - r_z) = 2\delta + \sum_{i=1}^n (\alpha_1^i - 1).$$

If  $s \in \mathbb{P}^1$  and  $\varphi$  has ramification  $(0, a, b)$  at  $s$ , then  $\check{\varphi}$  has ramification  $(0, b-a, b)$  at  $s$ , by (2.6). Thus

$$\sum_{z \in \check{C}} (\mu_z - r_z) = 2\tau + \sum_{i=1}^n (\alpha_2^i - \alpha_1^i - 1).$$

Substituting these expressions and the formula (2.8) for  $m$  into Schuh's formula (3.1), we obtain

$$4 - d + \sum (\alpha_2^i - 2) + 2\delta + \sum (\alpha_1^i - 1) = d + 2\tau + \sum (\alpha_2^i - \alpha_1^i - 1)$$

or  $2\delta = 2d - 4 + 2\tau - 2 \sum (\alpha_1^i - 1)$ , which completes the proof. 


**Corollary 3.2.** *Let  $\varphi$  be a maximally inflected plane curve of degree  $d$ . Let  $\delta, \eta$ , and  $c$  be the numbers of solitary points, real nodes, and complex nodes (respectively). Then,*

$$\begin{aligned} d-2-\sum(\alpha_1^i-1) &\leq \delta \leq \binom{d-1}{2}-\frac{1}{2}\sum(\alpha_1^i-1)(\alpha_2^i-1) \\ 0 &\leq \eta+2c \leq \binom{d-2}{2}-\frac{1}{2}\sum(\alpha_1^i-1)(\alpha_2^i-1)+\sum(\alpha_1^i-1). \end{aligned}$$

**Proof.** Since  $\tau \geq 0$ , the lower bound for  $\delta$  follows from the formula of Theorem 3.1. The upper bound is, in fact, an upper bound for the number of *virtual double points* which can appear, outside a given ramification, on a curve of degree  $d$ . Namely, the total number of virtual double points (including the virtual double points accounted for by the fixed ramification points) is equal to the genus of a nonsingular curve of degree  $d$  and the number of virtual double points contained in a ramification point equals  $\frac{1}{2}(\mu + r - 1)$ , where  $\mu$  is the Milnor number and  $r$  is the number of local branches (this formula can be found in [2]; for a more modern treatment and generalizations see [24]). Now, it remains to notice that  $\mu \geq (\alpha_1-1)(\alpha_2-1)$  at a ramification point of type  $(0, \alpha_1, \alpha_2)$ .

Since a maximally inflected curve has genus 0, the genus formula gives

$$(3.3) \quad \delta + \eta + 2c \leq g_\alpha,$$

where  $g_\alpha$  is the genus of a generic curve with singularities prescribed by the ramification data. The upper bound for  $\eta + 2c$  now follows from the lower bound on  $\delta$  and the upper bound on the genus  $g_\alpha \leq \binom{d-1}{2} - \frac{1}{2}\sum(\alpha_1^i-1)(\alpha_2^i-1)$  is given by the above bound on the number of virtual double points. 

**Corollary 3.3.** *Let  $\varphi$  be a maximally inflected plane curve of degree  $d$  with  $\kappa$  (real) cusps and  $\iota = 3(d-2) - 2\kappa$  (real) flexes whose remaining singularities are nodes. Let  $\delta, \eta$ , and  $c$  be the numbers of solitary points, real nodes, and complex nodes (respectively) of  $\varphi$ , which satisfy  $\delta + \eta + 2c = \binom{d-1}{2} - \kappa =: g_\kappa$ . If  $\kappa \leq d-2$ , then these additionally satisfy*

$$(3.4) \quad \begin{aligned} d-2-\kappa &\leq \delta \leq g_\kappa \\ 0 &\leq \eta+2c \leq \binom{d-2}{2} \end{aligned}$$

If  $d-2 \leq \kappa$  (which is at most  $3(d-2)/2$ ), then  $\delta, \eta$ , and  $c$  satisfy

$$(3.5) \quad \begin{aligned} 0 &\leq \delta \leq \binom{2d-4-\kappa}{2} \\ g_\kappa - \binom{2d-4-\kappa}{2} &\leq \eta+2c \leq g_\kappa \end{aligned}$$

This may be deduced from the Klein [21] and Plücker [26] formulas alone.

**Proof.** Since the ramification sequence of a flex is  $(0, 1, 3)$  and that of a cusp is  $(0, 2, 3)$ , we have

$$\sum(\alpha_1^i-1) = \frac{1}{2}\sum(\alpha_1^i-1)(\alpha_2^i-1) = \kappa,$$

and so (3.4) is just a special case of Corollary 3.2.


Now suppose that  $d-2 \leq \kappa$  and consider the dual curve to  $\varphi$ , which has degree  $m := 2(d-1) - \kappa$  by (2.8),  $\kappa$  flexes and  $\iota = 3(d-2) - 2\kappa$  cusps. The total number of

double points of this curve (bitangents of  $\varphi$ ) is given by the genus formula

$$\binom{m-1}{2} - \iota = \binom{2d-3-\kappa}{2} - (3(d-2) - \kappa).$$

This expression is an upper bound for the number  $\tau$  of solitary bitangents of  $\varphi$ . This gives the upper bound for  $\delta$

$$\begin{aligned} \delta &\leq \binom{2d-3-\kappa}{2} - (3(d-2) - 2\kappa) + d - 2 - \kappa \\ &= \binom{2d-3-\kappa}{2} - (2d-4-\kappa) = \binom{2d-4-\kappa}{2}. \end{aligned}$$

Combining the bounds for  $\delta$  with the genus formula  $\delta + \eta + 2c = g_\kappa$  gives the bounds for  $\eta + 2c$ . 

**Remark 3.4.** A fundamental question about the statement of Corollary 3.3 is whether its hypotheses are satisfied by any maximally inflected curve with  $\kappa$  cusps and  $3(d-2) - 2\kappa$  flexes. That is, is there a curve with  $\kappa$  cusps and whose other singularities are only ordinary double points not occurring at points of ramification? (If there is one such curve, then the general curve has this property.) As the construction in Section 4.1 shows, this is true when  $\kappa \leq d - 2$ .

While we believe that a generic maximally inflected curve has only ordinary double points not occurring at ramification points, we do not have a proof. A difficulty is that there are few constructions of maximally inflected curves.

Corollary 3.3 raises our first question concerning the classification of maximally inflected curves by their topological invariants.

**Question 3.5.** Let  $d, \iota, \kappa$  be positive integers with  $\iota + 2\kappa = 3(d-2)$ .

- (1) Which numbers  $\delta$  in the range allowed by Corollary 3.3 occur as the number of solitary points in a maximally inflected plane curve of degree  $d$  with  $\iota$  flexes and  $\kappa$  cusps, and whose other singularities are all ordinary double points?
- (2) Given a number  $\delta$  of solitary points which occurs,  $\eta + 2c = g_\kappa - \delta$ . Which numbers  $\eta$  in the range  $[0, g_\kappa - \delta]$  with the same parity as  $g_\kappa - \delta$  occur as the number of real nodes of such a maximally inflected curve?

**Remark 3.6.** For example, when  $d = 5$ ,  $\kappa = 4$ , and  $\iota = 1$ , we are in the case of  $\kappa \geq d - 2$  in Corollary 3.3. The bounds give  $\delta = 0$  or  $1$ . Question 3.5(1) asks: do both values of  $\delta$  occur? If  $\delta = 1$ , then  $\eta + 2c = 1$ , so a curve with  $\delta = 1$  has a single real node. If  $\delta = 0$ , then  $\eta + 2c = 2$ , so there are 2 possibilities for  $\eta$  of 0 or 2. Question 3.5(2) asks: do both values of  $\eta$  occur? In fact, both values for  $\delta$  occur, but when  $\delta = 0$ , only the value of 2 for  $\eta$  occurs.

The first part of the latter statement follows from the results of Section 5, as explained in Remark 3.8. Two such curves are displayed in Figure 6.1. The impossibility of the values  $\delta = \eta = 0$  for a rational quintic plane curve with four cusps and one flex is easily explained using V. Rokhlin's theory of complex orientations [28] as extended by N. Mishachev [25] and V. Zvonilov [46]. Let  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  be a real rational plane quintic with four cusps and no solitary points or nodes. We first perturb it to a new real rational quintic with 4 real nodes instead of the cusps, and then smooth each real node in such a way to obtain an

oval, and get in this way a dividing real quintic  $Y$  of genus 4. (This smoothing of a cusp is illustrated in Figure 3.2.) Since the Rokhlin complex orientation formula extends to

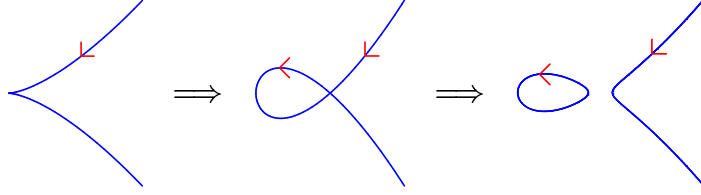


FIGURE 3.2. Smoothing a cusp and complex orientation

“flexible” curves, which are only almost holomorphic near the real plane, we do not need the existence of such an algebraic deformation and may instead just glue a proper local model in place of the cusps.

Choosing one component of  $Y \setminus \text{Re}Y$  gives a complex orientation to the ovals and the odd branch of the curve  $Y$ . This complex orientation must satisfy

$$25 = 1 + 4\mu + 4 + 4(R_+ - R_-),$$

where  $\mu = 0$  or  $2$  counts the intersection of the two components of  $Y \setminus \text{Re}Y$  at the complex nodes and  $R_{\pm}$  counts the relative orientation of the ovals to the odd branch. As we may see from Figure 3.2, the ovals are all coherently oriented with respect to the odd branch, so we have  $R_+ = 4$  and  $R_- = 0$ , which gives the contradiction.

More generally, we may ask the analog of Question 3.5 for maximally inflected curves with arbitrary ramification. We leave the formulation of this to the reader.

**Remark 3.7.** We indicate another approach to show the impossibility of the values  $\delta = \eta = 0$  for a rational quintic plane curve with four cusps and one flex. Consider the dual curve. It is a rational quartic with 4 flexes and 1 cusp. By formula (12),  $\tau \geq 1$  and, thus, the dual curve has at least one solitary point. Trace a straight line through such a solitary point and the cusp. By Bézout’s theorem, this line contains no other points of the curve. Hence, choosing a nearby line as the line at infinity, we may assume the real part of the quartic lies in an affine part of  $\mathbb{RP}^2$ .

According to the Fabricius-Bjerre formula [10],  $\iota + 2\eta + 2\kappa = 2(t_+ - t_-)$ , where  $\iota$  is the number of flexes,  $\eta$  the number of nodes,  $\kappa$  is the number of cusps, and  $t_+$  (respectively,  $t_-$ ) is the number of one-sided (respectively, two-sided) double tangents. Here, a line going through a cusp and tangent at another point is counted as a double tangent as well. Because this quartic curve is connected, any double tangent has both germs of the curve at the tangencies on the same side, that is, it is a one-sided double tangent. The projection from a cusp is 2-sheeted and, thus the number of such tangents through the cusp is at most 2. All this together implies that there is at least one ordinary (i.e., not going through a cusp) double tangent. Hence, the quintic has a real node.

**Remark 3.8.** Yet another proof of this impossibility of  $\eta = 0$  for a maximally inflected quintic with 4 cusps and one flex uses results of Section 5. As we have seen, the dual of such a curve is a maximally inflected rational quartic in  $\mathbb{RP}^2$  with 4 flexes and a single cusp. By Theorem 5.6, the possible topological types of such curves and of arrangements of their bitangents with respect to the curve are exhibited by the curves in the second column of Table 5.2 (the bitangents are not drawn there, but may be inferred). One curve

has one bitangent, while the other (nodal) curve has two bitangents. Thus our original quintic must have one or two real nodes, so that  $\eta = 0$  is impossible.

#### 4. TWO CONSTRUCTIONS OF MAXIMALLY INFLECTED CURVES

In the study of real plane curves, examples are typically constructed either by deforming a reducible curve (for example, in the constructions of Harnack [15] and Hilbert [17]) or by Viro's patchworking method deforming a curve in a reducible toric variety [40, 41] (see also [27]), sometimes combined with Cremona transformations. These methods were used initially to build smooth curves. Gudkov and, then, Shustin extended these methods to use and to obtain singular curves [33, 36]. We use such extensions to construct maximally inflected plane curves of degree  $d$  with no complex nodes and the extreme numbers of 0 and  $\binom{d-2}{2}$  real nodes.

Gudkov and Shustin have established a number of theorems extending the classical result of Brusotti [3] which state that given a singular curve satisfying a numerical condition and local models for certain deformations of the singular points of the curve, there exists a deformation of the singular curve where each singularity is deformed according to the corresponding local model. It is this approach that we use in the proof of the following theorem. (We give the details in our proof, as the results in the literature are vastly more general than we need and use subtle hypotheses.)

##### **Theorem 4.1.**

- (1) *For any  $d$  and  $\kappa$  with  $0 \leq \kappa \leq d-2$ , there exists a maximally inflected plane curve with  $\kappa$  cusps,  $3(d-2)-2\kappa$  flexes, and  $\binom{d-2}{2}$  real nodes (hence  $d-2-\kappa$  solitary points).*
- (2) *For any  $d$  there exists a maximally inflected plane curve of degree  $d$  with  $3(d-2)$  flexes and  $\binom{d-1}{2}$  solitary points. (Hence without real or imaginary nodes.)*

We prove the first statement in Section 4.1 and the second in Section 4.2. These realize the maximum and minimum possible numbers of real nodes  $\eta$  and solitary points  $\delta$  allowed by Corollary 3.3 for degree  $d$  curves with  $3(d-2)$  flexes. In Section 6 we discuss some implications of the constructions in Section 4.1.

**Remark 4.2.** Theorem 4.1 asserts the existence of maximally inflected curves with the given ramification, for some choices of ramification. Such an existence is something new when  $d$  is odd. Eremenko and Gabrielov's result (Proposition 2.4) does not guarantee the existence of any maximally inflected curves of odd degree with given ramification. From the constructions below, we can see that the ramification points are not 'clustered together', as in the discussion following Theorem 2.3, so Theorem 2.3 also does not apply.

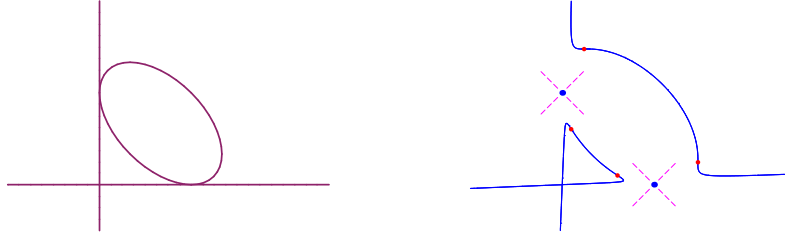
It follows from the relation between maximally inflected curves and the real Schubert calculus that these classical constructions of curves from Theorem 4.1 imply the existence of real solutions to some problems in the Schubert calculus. When  $d$  is odd and for the ramifications of Theorem 4.1, this result is new and gives further evidence in support of the conjecture of Shapiro and Shapiro.

Let us notice also that whatever is the value of  $d$ , even or odd, the proof of the Theorem gives not just existence but some explicit maximally inflected curves with well controlled topology.

4.1. **Deformations of Singular Curves.** For sufficiently small  $\epsilon > 0$  the equation

$$12xy((x+y-1)^2 - xy) - \epsilon(x+y)^2(x+y-1)^2 = 0$$

defines a rational quartic curve  $C(\epsilon)$  with 6 flexes, one real node, and 2 solitary points. The curve  $C(0)$  is the union of the coordinate axes and the ellipse shown on the left below. The curve  $C(\frac{1}{20})$  is displayed on the right below.



The quartic curve on the right has 2 solitary points at  $(1, 0)$  and  $(0, 1)$  and a node at the origin. By the genus formula the curve is rational. It also has 6 flexes. Four are indicated by circles, and there is one more along each of the branches close to the coordinate axes, as the oriented geodesic curvature (which is given by the Wronskian  $\det(\phi, \phi', \phi'')$  with respect to the local orientation defined by  $\phi$ , where  $\phi$  is the parametrization of the curve) of the segment of the curve along such a branch takes values of opposite sign at the extremal points (close to the initial tangency points) of these branches. (The local orientation changes with respect to an affine one when the branch crosses infinity.) Thus any parameterization of this curve is a maximally inflected quartic with 6 flexes, 1 node, and 2 solitary points.

**Proof of Theorem 4.1(1).** Fix an integer  $d > 2$  throughout and let  $P_0$  be the union of a nonsingular conic and any  $d-2$  distinct lines tangent to the conic. Then  $P_0$  is a reducible curve of degree  $d$ . Each pair of tangents meet and no three meet in a point as the dual curve to the conic is another nonsingular conic. Thus  $P_0$  has  $\binom{d-2}{2}$  real nodes and  $d-2$  other singularities at the points of tangency. We deform those tangency singularities while preserving the nodes.

In a neighborhood  $V$  of each point of tangency,  $P_0$  is isomorphic to the reducible curve  $I_0$  given by the equation

$$y(y - x^2) = 0,$$

in some neighborhood  $U$  of the origin. For each  $t \in \mathbb{R}$ , let  $I_t$  be the deformation of  $I_0$  defined in  $U$  by

$$(4.1) \quad y(y - x^2) + tx^2 = 0.$$

For  $t$  positive but sufficiently small,  $I_t$  has a solitary point at the origin and 2 flexes near the parabola  $y = x^2$ , one flex along each branch and within  $U$ . Moreover,  $I_t$  lies above the  $x$ -axis. Counting the Whitney index by means of the Gauss map shows the existence of two such flexes.

To construct curves with cusps we replace the local deformation model  $I_t$  (4.1) of the tangent points by the local model  $K_t$  defined by

$$(4.2) \quad y(y - x^2) + tx^3 = 0.$$

For  $t > 0$ ,  $K_t$  has a cusp at the origin and one flex near the origin.



Suppose that we can deform each tangency singularity (a tacnode) according to the local model  $I_t$  while preserving the nodes so that we get a deformation  $P_t$  of the curve  $P_0$  for  $t \in (0, \epsilon)$  such that (1)  $P_t$  has degree  $d$ , (2)  $P_t$  has a node in a neighborhood of each node of  $P_0$ , and (3) in a neighborhood  $V$  of each point of tangency of  $P_0$ ,  $P_t$  is isomorphic to  $I_t$ , in the neighborhood  $U$  of the origin. For  $0 < t < \epsilon$ , the curve  $P_t$  has  $\binom{d-2}{2}$  nodes and  $d-2$  solitary points, by the construction. Since it has degree  $d$ , it is rational. Each solitary point contributes 2 flexes, accounting for  $2(d-2)$  flexes. Furthermore, there is an additional flex along each asymptote (the original tangent lines) as the concavity of  $P_t$  changes while passing through infinity. Thus any parameterization of the curve  $P_t$  gives a maximally inflected curve of degree  $d$  with  $3(d-2)$  flexes,  $\binom{d-2}{2}$  nodes, and  $d-2$  solitary points.

If we replace some local deformation models  $I_t$  by  $K_t$ , every tangency point where we use the local perturbation  $K_t$  gives us a cusp, no solitary point, and only one flex (the flex along the tangent line does not appear, as the concavity of  $K_t$  does not change along that line). Doing this for  $\kappa$  of the  $d-2$  tangency points of  $P_0$  gives a maximally inflected curve with  $\kappa$  cusps,  $3(d-2) - 2\kappa$  flexes,  $d-2-\kappa$  solitary points, and  $\binom{d-2}{2}$  real nodes, proving statement (1) (the curve is rational because  $g = 0$ ).

This simultaneous deformation of the tacnodes according to arbitrary independent local models while preserving the nodes follows from the transversality of the equisingularity strata of the singularities. The equisingularity stratum of a node is smooth and its tangent space is given by polynomials vanishing at the node. The equisingularity stratum of a tacnode is also smooth and its tangent space is given by polynomials vanishing at the tacnode that satisfy two additional conditions: their first derivative at a tangent direction of the branches is zero, and the second derivatives along both branches (taken in the same direction) are equal.

So, to check the transversality it is sufficient merely to count the dimension of the intersection of the tangent spaces. In our case, the intersection is contained in the linear span of polynomials  $L'_0 Q_0 L_1 \cdots L_{d-3} + L'_1 Q_0 L_0 \cdots L_{d-3} + \cdots + L_0 L_1 \cdots L_{d-3} Q'_0$ , where  $Q_0$  is our initial nonsingular conic,  $L_0, \dots, L_{d-3}$  our tangent lines, and  $L'_0, L'_1, \dots, Q'_0$  are equations for arbitrary lines and a conic. So, the condition on the second derivatives takes the form of a triangular linear system and, hence, the subspace we are looking for is of dimension at most  $2(d-2) + 6 - (d-2) = d+4$ , which implies transversality. ▶

Figure 4.1 shows these curves for  $d = 5$  and  $\kappa = 0, 1, 2, 3$ .

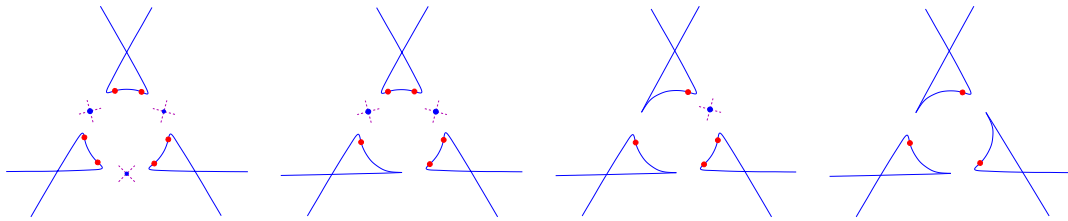


FIGURE 4.1. Maximally inflected degree 5 curves with 3 real nodes

**Remark 4.3.** For  $t > 0$ , the cusp in the curve  $K_t$  (4.2) is on the branch to the left of the origin. Had we instead used the perturbation  $K'_t$  given by

$$y(y - x^2) - tx^3 = 0,$$

then the cusp is now on the right branch for  $t > 0$ . In Section 6.1, we use this to study Question 5.7 concerning possible necklaces for a given set of ramifications and numbers of solitary points.

**4.2. Patchworking of Singular Curves.** Viro's method for constructing real plane curves with prescribed topology [40, 41] (see also [27]) begins with a subdivision of the simplex

$$\Delta_d := \{(i, j) \in (\mathbb{Z}_+)^2 \mid i + j \leq d\}$$

defined by a piecewise linear convex *lifting function*. Reflecting this subdivision in the coordinate axes and in the origin gives a subdivision of the region

$$\diamond_d := \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leq d\}.$$

Gluing opposite edges of  $\diamond_d$  gives a PL-space homeomorphic to  $\mathbb{RP}^2$ .

The other ingredient is, for each facet  $F$  of the subdivision, a real polynomial  $f_F$  whose Newton polytope is  $F$  and such that  $f_F = 0$  defines a smooth curve in the torus  $(\mathbb{C}^\times)^2$ . These polynomials additionally satisfy a compatibility condition: For an edge  $e$  common to two facets  $F$  and  $G$ , we have  $f_F|_e = f_G|_e$ , where the latter expressions denote the truncations of the polynomials  $f_F$  and  $f_G$  to the edge  $e$  (i.e., those monomials whose exponent vector is in  $e$ ). Furthermore, this common truncation has no multiple factors (except  $x$  and  $y$ ).

The real points of the real curve  $f_F = 0$  lie naturally in the four copies of the facet  $F$  in  $\diamond_d$  (with the boundary points representing the asymptotic behavior of solutions, or, equivalently, the zeros of the restriction  $f_F$  to the corresponding edge). The pair consisting of these facets and curves will be called the *Newton portrait* of  $f_F = 0$ . By the compatibility condition, the Newton portraits of the facet curves glue naturally together, giving a topological curve  $\Gamma$  in  $\mathbb{RP}^2$ . Viro's theorem asserts that there exists a curve  $C$  of degree  $d$  in  $\mathbb{RP}^2$  such that the pair  $(\mathbb{RP}^2, C)$  is homeomorphic to the pair  $(\mathbb{RP}^2, \Gamma)$ . The homeomorphism preserves each coordinate axis and each quadrant. The complex points of  $C$  are smooth, and  $\Gamma$  and  $C$  meet each coordinate axis in the same number of points.

Shustin [33] (see also [34, 35]) showed how to modify this construction when the facet curves ( $f_F = 0$ ) have singularities in  $(\mathbb{C}^\times)^2$ . If a numerical criterion is satisfied (such a criterion is given in Theorem 1.7 of [34]), there exists a curve of degree  $d$  in  $\mathbb{RP}^2$  whose singularities are the disjoint union of the singularities of the facet curves, and whose topology is glued from that of the facet curves as before. (See Theorem 1.8 of [34], originally proven in [33].) In particular, Shustin shows the following.

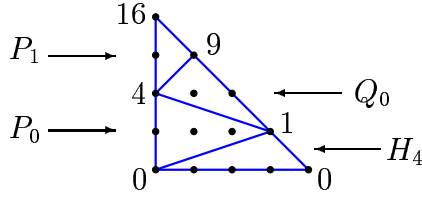
**Proposition 4.4** (See [34] and [33]). *If the singularities of the facet curves are only nodal, then there exists a curve of degree  $d$  in  $\mathbb{RP}^2$  whose only (complex) singularities are the disjoint union of the singularities of the facet curves, and whose topology is given by gluing the facet curves as in the Viro construction.*

We use this to prove Statement (2) of Theorem 4.1. (Another approach is indicated in a remark in the end of Section.)

**Proof of Theorem 4.1(2).** Consider the subdivision of  $\Delta_d$  given by the piecewise linear convex lifting function  $w$  which we define for some the vertices of  $\Delta_d$ . Set  $w(0, 0) = w(d, 0) = 0$ , and

$$\begin{aligned} w(0, 2i+2) &= (2i+2)^2 \\ w(d-1-2i, 2i+1) &= (2i+1)^2 \end{aligned} \quad \text{for } i = 0, \dots, \left\lfloor \frac{d-1}{2} \right\rfloor.$$

Here is the resulting subdivision of  $\Delta_4$  and the values of the lifting function.



This regular subdivision of  $\Delta_d$  has three types of triangles

- (i) The triangle  $H_d$  with vertices  $\{(0, 0), (d, 0), (d-1, 1)\}$ .
- (ii) The triangle  $P_i$  with vertices  $\{(0, 2i), (0, 2i + 2), (d-1-2i, 2i + 1)\}$  for each  $i$  from 0 to  $\lfloor \frac{d-2}{2} \rfloor$ .
- (iii) The triangle  $Q_i$  with vertices  $\{(0, 2i + 2), (d-1-2i, 2i + 1), (d-3-2i, 2i + 3)\}$  for each  $i$  from 0 to  $\lfloor \frac{d-3}{2} \rfloor$ .

For each facet, we give polynomials  $f_F$  that define curves with only solitary points. These will not necessarily satisfy the compatibility condition, but rather a weaker one: A common edge  $e$  between two facets  $F$  and  $G$  contains only two lattice points, and after possibly multiplying one of the facet polynomials by  $-1$ , the signs of the monomials from the two facet polynomials agree. This weak compatibility implies that there are monomial transformations with positive coefficients of the facet polynomials which do satisfy the compatibility criteria after adjusting the sign of one of the two polynomials. Since these monomial transformations do not change the geometry (number of solitary points, topology of the glued curve  $\Gamma$ ), and the dual graph to the triangulation is a chain, it will suffice to construct polynomials satisfying the weaker criteria and giving the desired topology.

We describe the monomial transformations. Consider first a common edge  $e$  between adjacent facets  $F$  and  $G$  of the triangulation. Since  $e$  has no interior lattice points, the restrictions of the facet polynomials to  $e$  will be binomials of the form

$$f_F|_e = Ax^a y^b + Bx^c y^{b+1} \quad f_G|_e = Cx^a y^b + Dx^c y^{b+1}.$$

(Not only do the exponents of  $y$  differ by 1, but one of the exponents  $a$  or  $c$  is zero.) For  $z \neq 0$ , let  $\text{sgn}(z) := z/|z|$ . We compute

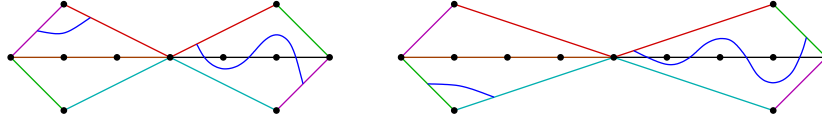
$$\begin{aligned} \text{sgn}\left(\frac{A}{C}\right) \left|\frac{C}{A}\right|^{\frac{-c}{a-c}} \left|\frac{D}{B}\right|^{-b} f_F \left(\left|\frac{C}{A}\right|^{\frac{1}{a-c}} x, \left|\frac{D}{B}\right| y\right) \\ &= \text{sgn}\left(\frac{A}{C}\right) \left|\frac{C}{A}\right|^{\frac{-c}{a-c}} \left|\frac{D}{B}\right|^{-b} \left(A \left|\frac{C}{A}\right|^{\frac{a}{a-c}} \left|\frac{D}{B}\right|^b x^a y^b + B \left|\frac{C}{A}\right|^{\frac{c}{a-c}} \left|\frac{D}{B}\right|^{b+1} x^c y^{b+1}\right) \\ &= \text{sgn}\left(\frac{A}{C}\right) \left(A \left|\frac{C}{A}\right| x^a y^b + B \left|\frac{D}{B}\right| x^c y^{b+1}\right) \\ &= Cx^a y^b + Dx^c y^{b+1}, \end{aligned}$$

as the weak compatibility criteria ensures that  $\text{sgn}\left(\frac{A}{C}\right) = \text{sgn}\left(\frac{D}{B}\right)$ .

Since the dual graph of the triangulation is a chain, we encounter no obstructions when transforming the facet polynomials so that they satisfy the compatibility condition. More precisely, given the facet polynomial for  $H_d$ , we transform the facet polynomial for  $P_0$  as above, then the facet polynomial for  $Q_0$ , then for  $P_1$ , and *et cetera*.

We now give the facet polynomials. The reader is invited to check that the weak compatibility conditions are satisfied. The facet  $H_d$  is the Newton polytope of the polynomial

$h_d := x^{d-1}y - (1-x)(2-x)\cdots(d-x)$ . Here are the Newton portraits of  $h_3$  and  $h_4$ .



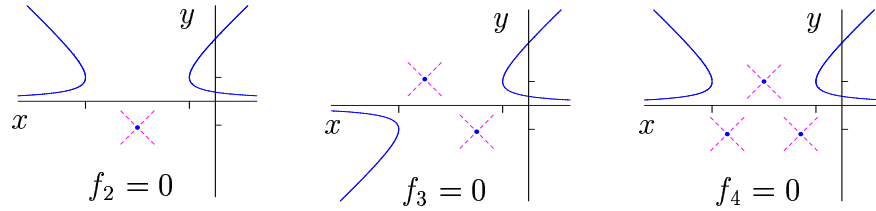
The remaining facet polynomials are based on an idea of Shustin [34, p. 849]. Recall that the Chebyshev polynomials  $\mathfrak{U}_k(x)$  (defined recursively by

$$\mathfrak{U}_0 := 1, \quad \mathfrak{U}_1 := x, \quad \text{and for } k > 1, \quad \mathfrak{U}_k(x) := 2x\mathfrak{U}_{k-1}(x) - \mathfrak{U}_{k-2}(x).$$

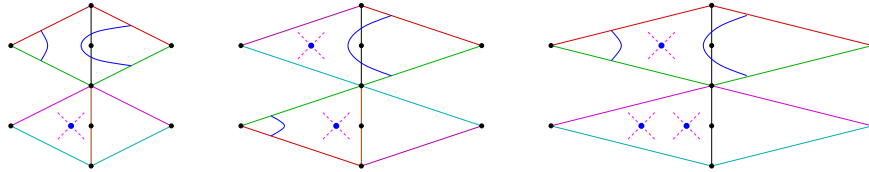
have the property that  $\mathfrak{U}_k(x)$  has exactly  $k$  roots in the interval  $(-1, 1)$  and  $k-1$  local extrema in this interval with values  $\pm 1$ , and for  $|x| > 1$ ,  $|\mathfrak{U}_k(x)| > 1$ . Lastly, the leading term of  $\mathfrak{U}_k(x)$  is  $2^{k-1}x^k$ . Then the polynomial  $f_k(x) := y^2 - 2y\mathfrak{U}_k(x+2) + 1$  has Newton polytope the triangle

$$\text{Conv}\{(0, 0), (1, k), (0, 2)\},$$

which is a translate of the polytope  $P_i$  by  $(0, -2i)$  when  $d-1-2i = k$ . The curve  $f_k = 0$  in  $\mathbb{R}^2$  has 2 connected components and  $k-1$  solitary points. We display these curves for  $k = 2, 3$ , and 4, scaling the  $y$ -axis by the transformation  $y \mapsto \text{sign}(y)|y|^{1/k}$ .



Here are the Newton portraits of these curves. We omit the interior lattice points in the triangles.

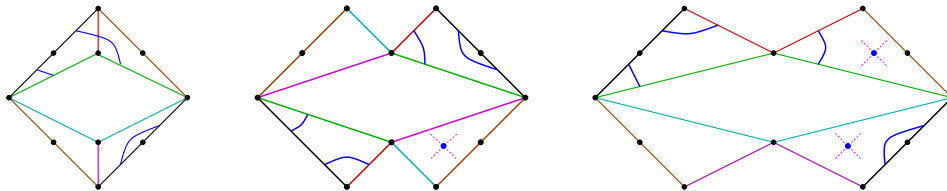


Let  $y^{2i}f_{d-1-2i}$  be the facet polynomial for the facet  $P_i$ .

Finally, set

$$g_k(x, y) := f_k\left(-\frac{1}{x}, (-1)^k \frac{y}{x}\right) = \frac{y^2}{x^2} - (-1)^k 2 \frac{y}{x} \mathfrak{U}_k\left(2 - \frac{1}{x}\right) + 1.$$

The Newton polytope of  $g_k$  has vertices  $\{(0, 0), (-2, 2), (-k-1, 1)\}$ . We display the Newton portraits of  $g_1$ ,  $g_2$ , and  $g_3$ . (For this, we first translate their Newton polytopes by  $(k+1, 0)$ , placing it into the positive quadrant.)



Let  $x^{d-1-2i}y^{1+2i}g_{d-2-2i}$  be the facet polynomial of the facet  $Q_i$ .

The curve  $C_d$  constructed from these data by Proposition 4.4 has  $3(d-2)$  flexes and  $\binom{d-1}{2}$  solitary points as claimed. First, since the facet curves  $f_k$  and  $g_k$  each have  $k-1$  solitary points,  $C_d$  has  $(d-2)+(d-3)+\cdots+1+0 = \binom{d-1}{2}$  solitary points, and so is rational (in fact, the solitary points correspond to internal integer points of the Newton polygon). From the Newton portrait of  $h_d$ , we see that  $C_d$  meets the  $x$ -axis in  $d$  points. Each facet  $P_k$  contributes 2 points of intersection of  $C_d$  with the  $y$ -axis. When  $d$  is even, this gives  $d$  points of intersection, and when  $d$  is odd,  $d-1$  points of intersection. When  $d$  is odd, the last facet  $Q_1$  (corresponding to  $g_1$ ) contributes an additional point of intersection with the  $y$ -axis. Finally, each facet  $Q_k$  contributes 2 points of intersection of  $C_d$  with the  $z$ -axis, giving  $d-2$  points of intersection when  $d$  is even and  $d-1$  points of intersection when  $d$  is odd. The facet  $H_d$  contributes an additional point, and when  $d$  is even, the facet  $P_{\frac{d-2}{2}}$  (corresponding to  $f_2$ ) contributes one point.

As a result, the curve  $C_d$  has three separate segments, each intersecting a different coordinate axis in  $d$  consecutive points going in the same order on the curve and on the axis. Thus, by counting the Whitney indices by means of the Gauss map we find at least  $d-2$  flexes on each segment. Hence, the curve constructed has  $3(d-2)$  flexes.  $\blacktriangleright$

Figure 4.2 shows the curves  $C_4$  and  $C_5$ . The curve  $C_4$  has the topological type of the quartic in Table 5.2 with six flexes and no real nodes.

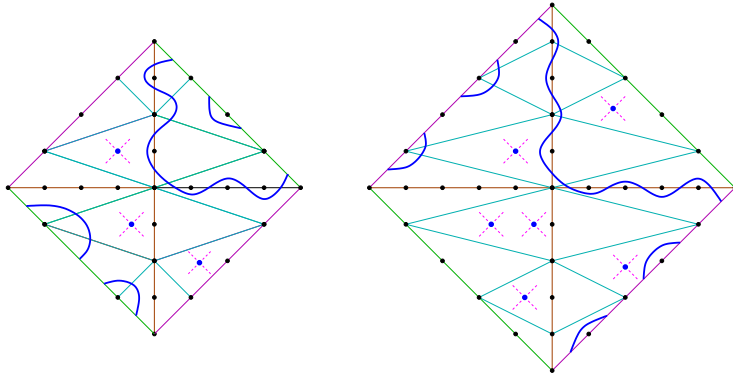


FIGURE 4.2. The curves  $C_4$  and  $C_5$ .

**Remark 4.5.** Another patchworking is via gluing parameterizations of the facet curves  $f$  and  $g$  (which are rational). Since the dual graph of the triangulation is a chain, it is sufficient to have a gluing construction for a pair of rational plane curves intersecting transversally. For that purpose, one can pick parameterizations  $F$  and  $G$  such that  $F(0) = G(0)$  and consider, for generic  $\lambda, \mu$  and sufficiently small  $\epsilon > 0$  the rational curves  $H_\epsilon$  given by  $\lambda F(u) + \mu G(v)$ ,  $uv = \epsilon$ . The flexes and the nodes are preserved under this patchworking construction, since they are stable under small deformations.

## 5. MAXIMALLY INFLECTED PLANE CURVES OF DEGREES THREE AND FOUR

**5.1. Cubic Plane Curves.** In Figure 3.1 we saw that a plane cubic with 3 real flexes has a solitary point and no nodes and a plane cubic with one real cusp has no nodes. These are the only possible maximally inflected cubics and they exhaust all the possibilities allowed by Corollary 3.3.

**5.2. Quartic Plane Curves.** Consider now quartics whose only ramifications are flexes and cusps. The upper bound for  $\eta + 2c$  allowed by Corollary 3.3 for quartics is 1, so maximally inflected quartics have no complex nodes and either 1 or 0 real nodes. Table 5.1 summarizes the possible numbers  $\eta$  of real nodes and  $\delta$  of solitary points. Clearly, for

$\kappa$	0	1	2	3			
$\iota$	6	4	2	0			
$\eta$	0	1	0	1	0	1	0
$\delta$	3	2	2	1	1	0	0

TABLE 5.1. Topological invariants of quartics allowed by Corollary 3.3.

quartics, these numbers determine the real part of the image up to homeomorphism. In fact, for quartics, they determine it up to isotopy in  $\mathbb{RP}^2$  (see Theorem 5.2, Remark 5.4, and Theorem 5.6).

**Theorem 5.1.** *For every quadruple  $(\kappa, \iota, \eta, \delta)$  given in Table 5.1, there is a (real) plane quartic with  $\kappa$  cusps,  $\iota$  inflection points,  $\eta$  nodes, and  $\delta$  solitary points.*

**Proof.** This may be deduced the classification of real rational quartics given by D.A. Gudkov, et. al. [14] (see also F. Klein [22], H.G. Zeuthen [45], and C.T.C. Wall [43]). However, we will instead exhibit examples of curves with each possible ramification and singularity. In Table 5.2, we display a maximally inflected plane quartic curve for each quadruple  $(\kappa, \iota, \eta, \delta)$  of Table 5.1. These were generated using a computer calculation, by

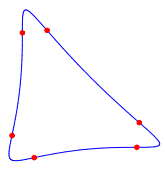
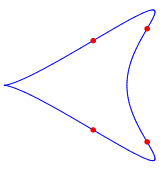
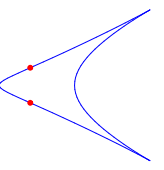
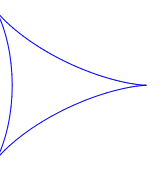
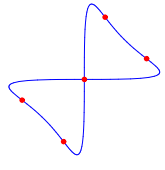
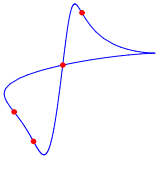
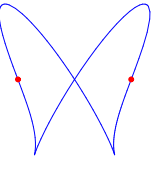
		$\kappa, \iota$			
		0,6	1,4	2,2	3,0
$\eta$	0				
	1				not allowed

TABLE 5.2. Quartics realizing all topological types allowed by Corollary 3.2.

first solving for the centers of projection (as described, for example in [39, Section 2] or in [37, Section 2]), and then drawing the resulting parameterized curve using MAPLE. (This method is used to draw most of the curves we display.) The positions of the flexes are marked with dots and the curve with 6 flexes and one node has 2 flexes at its node.



Recall from Section 2.2 that Conjecture 2.6 holds for plane quartics and so we have the strong information of Theorem 2.7 about deformations of plane quartics. We explore some

consequences of that fact. There is a single isotopy class of sextuples of distinct points on  $\mathbb{RP}^1$ . Thus, given any sextuple  $S = \{s_1, \dots, s_6\}$  of distinct (non ordered) points, there is an isotopy from the sextuple  $\{-3, -1, 0, 1, 3, \infty\}$  to  $S$ .

The Schubert calculus gives 5 rational quartics with 6 given points of inflection, thus each of the 5 maximally inflected quartics with flexes at  $S$  are deformations of one of the 5 maximally inflected curves with flexes at  $\{-3, -1, 0, 1, 3, \infty\}$ , which we display in Figure 5.1. (Each nodal curve has 2 flexes at its node.) We indicate the differences in the

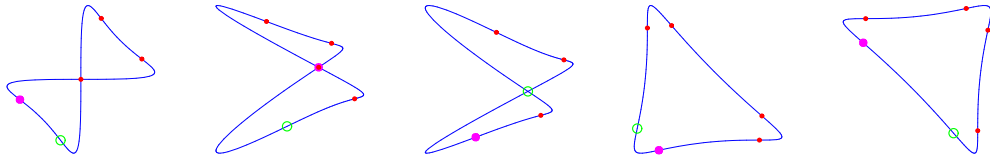


FIGURE 5.1. The 5 curves with flexes at  $\{-3, -1, 0, 1, 3, \infty\}$

parameterizations of these curves, labeling the flex at  $-3$  by the larger dot and the flex at  $-1$  by the circle.

**Theorem 5.2.** *For each sextuple  $S = \{s_1, \dots, s_6\}$  of distinct points in  $\mathbb{RP}^1$ , there are exactly five maximally inflected quartics with flexes at  $S$ . Of these five, two have three solitary points and no real nodes, while three have two solitary points and one real node. Moreover, the possible arrangements of solitary points and bitangents are as indicated in Figure 5.2 for each of these types of curves.*

**Proof.** Since the five quartics in Figure 5.1 show that the statement is valid for the choice of flexes at  $\{-3, -1, 0, 1, 3, \infty\}$ , it suffices to show that the number of solitary points does not change under a deformation of such a quartic curve.

Any deformation of a quartic with one real node must have one real node; in passing to a curve without a real node, the deformation would include a curve with some other singularity, which would necessarily be a ramification point that is not a flex. Since the curve already contains six flexes, this would violate Proposition 1.1. Thus every deformation of the first three curves has a single real node.

To see that every deformation of any of the five curves has only ordinary double points, consider the arrangements of bitangents and solitary points on an example. Figure 5.2 shows the second and fourth curves with their bitangents and solitary points. Here, each

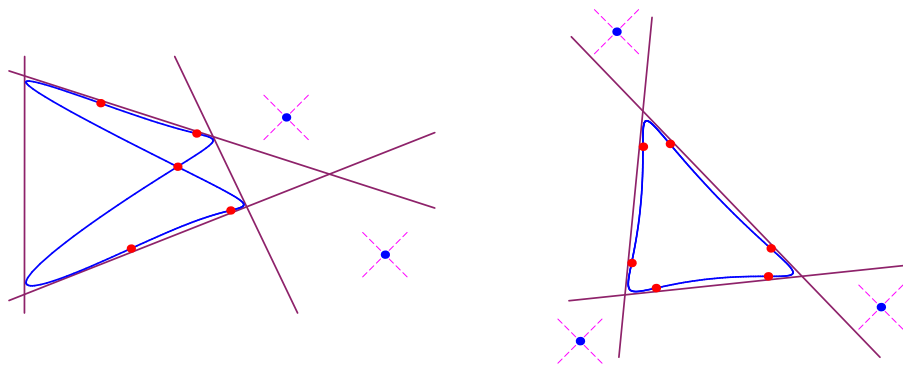



FIGURE 5.2. Bitangents and solitary points.

solitary point is separated from other real, solitary or not, points of the curve by some number of real double tangents. The same phenomena takes place for each of the five maximally inflected curves.

Since the curves have degree 4, the bitangents cannot meet the curves in additional points, by Bézout's theorem. Similarly, a bitangent cannot be tangent to a flex. Thus in a deformation, the flexes are confined to the arcs of the curve between two points of tangency to bitangents, and the number of bitangents does not change. Similarly, the solitary points cannot meet another point of the curve, to do so, they would first have to meet a bitangent, which is not possible, by Bézout's theorem. This completes the proof of the theorem. 

**Remark 5.3.** For reference, in Figure 5.3, we give the solitary points and bitangents, as well as parameterizations  $[\varphi_1(s, t), \varphi_2(s, t), \varphi_3(s, t)]$  for the two curves in Figure 5.2. The decimals are numerical approximations.

NODAL CURVE	ANODAL CURVE
Parameterization	Parameterization
$\varphi_1(s, t) = s^4 - 6t^2s^2 + 9t^4$	$\varphi_1(s, t) = ts^3 - \sqrt{6}t^2s^2 - \frac{3\sqrt{6}}{2}t^4$
$\varphi_2(s, t) = \frac{3}{4}s^4 + ts^3 - \frac{3}{2}t^2s^2 + \frac{9}{4}t^4$	$\varphi_2(s, t) = t^3s + \frac{\sqrt{6}}{2}t^4$
$\varphi_3(s, t) = s^4 + ts^3 - 3t^2s^2 - 2t^3s + \frac{15}{2}t^4$	$\varphi_3(s, t) = s^4 + 6t^2s^2 + 9t^4$
Solitary Points	Solitary Points
(1.514769, 0.854076)	$(\frac{7}{16} - \frac{\sqrt{6}}{32}, -\frac{7}{48} + \frac{13\sqrt{6}}{288})$
(2.088892, 0.040735)	$(-\frac{7}{16} - \frac{\sqrt{6}}{32}, \frac{7}{48} + \frac{13\sqrt{6}}{288})$
Bitangents	Bitangents
$x = 0$	$(-\frac{\sqrt{6}}{4}, -\frac{\sqrt{6}}{36})$
$y = 3 - \frac{25}{12}x$	Bitangents
$y = 1.07221014 - 0.31889744x$	$y = 0.03402069 - \frac{1}{3}x$
$y = -0.17859312 + 0.39336553x$	$y = 1.4984552 + 3.2996598x$
	$y = -0.0015448 + .0336735x$

FIGURE 5.3. Parameterizations, solitary points, and bitangents for the curves of Figure 3

Interestingly, the two solitary points of the nodal curve lie on the line  $y = 3 - \frac{17}{12}x$ , and this line also meets the two points of intersection of the bitangents to the right of and above the quartic. Similarly, the node lies at the intersection of the pairs of lines through the other points of intersection of the bitangents. The other two nodal quartics with flexes at  $\{-3, -1, 0, 1, 3, \infty\}$  shown in Figure 5.1 also have this property. That is clear for the second asymmetric nodal quartic whose image in  $\mathbb{RP}^2$  is isomorphic to this quartic. The symmetric nodal quartic has its solitary points at  $[1, 0, 0]$  and  $[0, 1, 0]$ , and the corresponding pairs of bitangents are parallel, so all four points lie on the line at infinity. The statement about its node is also clear from symmetry.

By the formula of Theorem 3.1 for such quartics,  $\delta = \tau + 2$ , where  $\delta, \tau$  count the solitary points and solitary bitangents. Thus the nodal curve has no solitary bitangents,



while the other curve has a single solitary bitangent, which is the line at infinity. To see this, note that  $\varphi_3$  factors as  $(s^2 + 3t^2)^2$ , and so the line at infinity is a bitangent, with tangencies the points of the curve where  $[s, t] = [\pm\sqrt{-3}, 1]$ . Evaluating, we see these are at  $[-1 \pm 2\sqrt{-2}, 1, 0]$ .

**Remark 5.4.** Theorem 5.2 shows that the isotopy type of the embedding of a quartic with 6 flexes into  $\mathbb{RP}^2$  does not change under an isotopy of the positions of the flexes in  $\mathbb{RP}^1$ . In fact something stronger is true. The space of all possible positions of flexes modulo reparameterizations preserving orientation (i.e., the quotient of  $(\mathbb{RP}^1)^6$  minus all the diagonals by  $SL(2, \mathbb{R})$ ) is contractible (to see this, consider fixing the position of one flex), and because of the confinement property of flexes and the tangent points of the bitangents (see the proof of Theorem 5.2), there is no way to deform one of the five quartics in Figure 5.1 into any other.

However, one may allow two flexes to come together, for example, the flexes in Figure 5.1 represented by the thickened dot and the open circle. When the positions of these two flexes collide, the second, third, and fourth curve will develop a cusp, while the first and fifth will develop a *planar point* with ramification sequence  $(0,1,4)$ . Suppose that we fix five of the ramification points, and let the sixth move along  $\mathbb{RP}^1 \simeq S^1$ . At every position of the sixth point, we get a maximally inflected curve which has 6 flexes, except when the 6th point collides with one of the fixed points, and then we get a curve with either a cusp or a planar point<sup>†</sup>.

We have used symbolic methods to calculate what happens when the sixth point moves. The number of nodes is always preserved, and when the sixth point returns to its original position, we get a curve different than the original one. In fact this monodromy action cyclically permutes the three nodal quartics in Figure 5.1 and interchanges the two quartics without nodes. This can also be inferred by visualizing the effect of moving the the flexes labeled by the open circles in Figure 5.1.

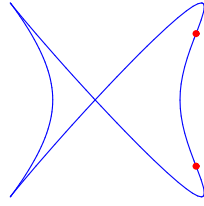
**Remark 5.5.** The proof we gave of Theorem 5.2 is based on the following property going back to Zeuthen [45]: each of the components of the complement in  $\mathbb{RP}^2$  of the double and solitary tangents contains at most 1 connected compact component of the real locus. It is valid for any quartic, and a similar proof shows that every maximally inflected rational quartic has only ordinary double points, apart from the singularities at the ramification points, and thus we obtain the analog of Theorem 5.2 for all maximally inflected rational quartics, which we state below after we introduce a further, necessary notion.

Underlying a deformation of a maximally inflected curve is the isotopy type of its ramification. Consider the curves with 2 flexes and 2 cusps in Table 5.2. In the curve with no real nodes,  $\eta = 0$ , the cusps ( $\kappa$ ) and flexes ( $\iota$ ) occur along  $\mathbb{RP}^1$  in the order  $\kappa\kappa\iota\iota$ , while for the curve with a single real node, the order is  $\kappa\iota\kappa\iota$ . There is another curve with

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<sup>†</sup>Motion pictures of families of curves with such moving ramification points may be found at [www.math.umass.edu/~sottile/pages/inflected](http://www.math.umass.edu/~sottile/pages/inflected).

2 cusps, 2 flexes, and one node.



Here, the ramification occurs in the order  $\kappa\kappa\iota\iota$ . Thus quartics with the ramification  $\kappa\kappa\iota\iota$  may have both possibilities of zero or one real node. Interestingly, all quartics with order  $\kappa\iota\kappa\iota$  have one real node, which we explain below.

Questions of classifying maximally inflected plane rational curves can thus be refined to take into account the isotopy type of the ramification in  $\mathbb{RP}^1$ . The different isotopy types of the placement of ramification data  $\alpha^1, \dots, \alpha^n$  on  $\mathbb{RP}^1$  are encoded by combinatorial objects called necklaces:  $n$  ‘beads’ with ‘colors’  $\alpha^1, \dots, \alpha^n$  are strung along  $S^1 = \mathbb{RP}^1$  to make a necklace. Given a maximally inflected curve  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^2$ , we may reverse the parameterization of  $\mathbb{RP}^1$  to obtain another maximally inflected curve whose necklace is the mirror image of the necklace of the original curve. Thus we identify two necklaces that differ only by the orientation of  $\mathbb{RP}^1$ . For example, Figure 5.4 displays the two necklaces with 2 beads each of color  $\kappa$  and  $\iota$ . To a maximally inflected curve with ramification

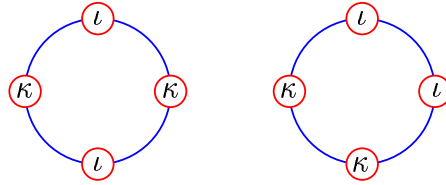


FIGURE 5.4. Necklaces for  $\kappa = \iota = 2$ .

$\alpha^1, \dots, \alpha^n$ , we associate a necklace with beads of colors  $\alpha^1, \dots, \alpha^n$  where the bead with color  $\alpha^i$  is placed at the corresponding point of ramification on  $S^1 = \mathbb{RP}^1$ .

To state the promised generalization of Theorem 5.2 let us denote by  $C(\Omega)$  the space of maximally inflected rational quartics with a given necklace  $\Omega$  and by  $P(\Omega)$  the space of the placements of the necklace in  $\mathbb{RP}^1$ . This generalization may be proven in a manner similar to the proof of Theorem 5.2 (cf. Remark 5.5), and so we omit its proof.

**Theorem 5.6.** *Maximally inflected rational quartic curves in  $\mathbb{RP}^2$  admit arbitrary deformations, and the isotopy type of the image (together with the bitangents and solitary points) in  $\mathbb{RP}^2$  is a deformation invariant. Moreover, for any necklace  $\Omega$  the canonical projection  $C(\Omega) \rightarrow P(\Omega)$  is a trivial covering.*

*In particular, the only isotopy types of the image are those indicated in Table 2 and for any given necklace the number of maximally inflected quartics having a given isotopy type of the image in  $\mathbb{RP}^2$  does not depend on the placement of the necklace.*

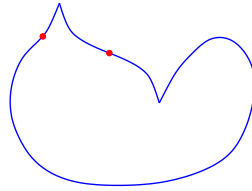
(Note that there are rational real quartics with one real node and nested loops, but such a quartic cannot have solitary points, and, as it follows from 3.3, it cannot be maximally inflected.)

We refine Question 3.5.

**Question 5.7.** Given a necklace  $\Omega$  with beads of color  $\alpha^1, \dots, \alpha^n$ , which are ramification data for degree  $d$  plane curves, what are the possibilities for the numbers  $\delta, \eta, c$  of solitary points, real nodes, and complex nodes of a maximally inflected curve of degree  $d$  whose associated necklace is  $\Omega$ ? Are any of these (or their positions) a deformation invariant?

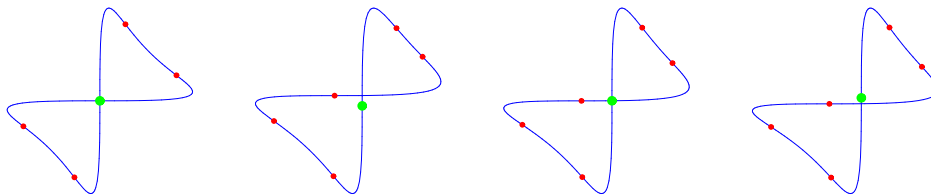
A weaker problem is to give bounds better than those of Corollary 3.3 for these possibilities. For example,  $\eta = 0$  is not possible for the necklace on the left in Figure 5.4. Indeed, the two curves for this necklace with ramification points at  $\{-1, 0, 1, \infty\}$  are obtained from each other by reversing the parameterization, and so their images in  $\mathbb{RP}^2$  are equal. In fact, this common image, which has a single node, is displayed the second row of the third column of Table 5.2. Invoking Theorem 5.6 completes the proof.

We give a more direct proof that  $\eta = 0$  is not possible. To see this, suppose such a curve has no real nodes. Then the real locus is a topologically embedded two-sided circle. The inflection points are the points where the concavity (which can be represented by an oriented curvature covector taking zero on the tangent direction) of the curves changes. Note that the concavity does not change at a cusp, and because of a flex between the cusps, one cusp is pointed outward and another inward, as indicated below.

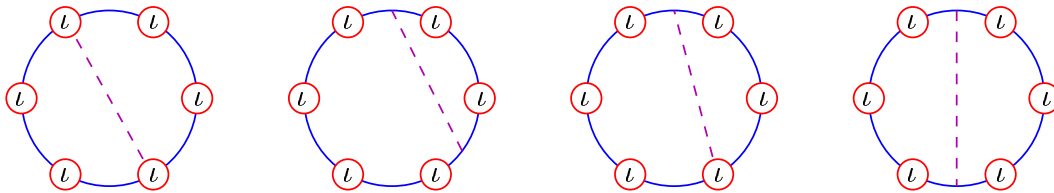


Consider now the line through the cusps. It must meet the curve in at least 1 additional point, or have a local intersection number of 3 with a cusp, and thus it has intersection number at least 5 with the quartic. This contradicts Bézout's theorem, and proves the impossibility of such a curve.

We now consider the possible positions of the nodes in a maximally inflected curve. There are 4 possibilities for the positions of the node with respect to the flexes in a maximally inflected quartic with 6 flexes and one real node. We display all 4:



The position of the node may be represented in the associated necklace by drawing a chord joining the two points whose images coincide:



These are the only 4 possibilities: Since a bitangent to a quartic can neither be tangent at a flex nor meet another point of the curve, the flexes and nodes of such a quartic are constrained to lie on the arcs of the curve in between the contacts of its bitangents. Since,

by Theorem 5.2, every maximally inflected quartic with a single node can be deformed into one of those shown in Figure 5.1, this constraint rules out the other possibilities for the chord in a necklace of such a quartic.

We further refine Questions 3.5 and 5.7.

**Question 5.8.** Which necklaces  $\Omega$  with beads of color  $\alpha^1, \dots, \alpha^n$  and  $\eta$  chords occur as maximally inflected curves? Given such a chord diagram, what are the possible numbers of solitary points (and hence complex nodes)?

Ignoring the beads, we obtain a circle with  $\eta$  chords, and the same questions may be asked of these diagrams. Such pure chord diagrams encode, together with the number of solitary points, the topology of the image as an abstract, not embedded in  $\mathbb{RP}^2$ , topological space, and therefore the classification of chord diagrams is a necessary part of any topological classification.

We make one final observation concerning the orientation of the cusp of a maximally inflected quartic with four flexes and one cusp. The examples in Table 5.2 both have the cusp pointing into the unbounded region in the complement of the curve. This is necessarily the case. If a quartic had a cusp pointing into a bounded region, then it could not have real nodes or solitary points, as the line joining a cusp with such a real double point would meet the curve in at least two additional points, and thus contradicts Bézout's theorem. But Corollary 3.3 requires such a maximally inflected curve to have at least one solitary point. While we are presently unable to formulate a general question concerning restrictions on the disposition of cusps (and other ramification), it is likely there are further topological restrictions.

Further pictures of maximally inflected quartics with more general ramification and many quintics may be found on the web<sup>†</sup>.

## 6. MAXIMALLY INFLECTED PLANE QUINTICS

Unless explicitly stated otherwise, all quintics in this section are assumed to have cusps and flexes as their only ramification points and to have ordinary double points as their only other singularities. In Section 6.1, we additionally assume that all quintics have three real nodes.

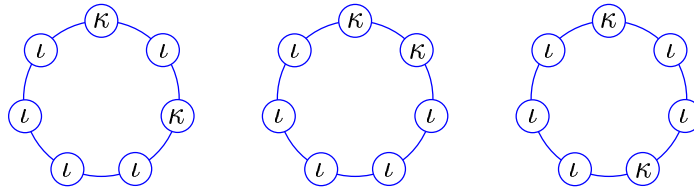
**6.1. Quintics with cusps and flexes having three nodes.** The construction of Theorem 4.1 for quintics gives maximally inflected quintics whose number  $\kappa$  of cusps is 0, 1, 2, or 3. We obtain a quintic with the maximal number of 4 cusps by taking the dual of a maximally inflected quartic with one cusp and 4 flexes, whose existence was addressed in Theorem 5.1. The existence of such curves is also a consequence of Theorem 2.3, as the ramification indices of cusps and flexes are special, in the sense of that theorem.

Since Theorem 2.3 is an existence result and gives no information about the geometry of the resulting curve, it is very instructive to look at specific examples coming from constructions. As an example, consider the possible necklaces of such curves, which is interesting only for two or three cusps. There are three possible necklaces of maximally

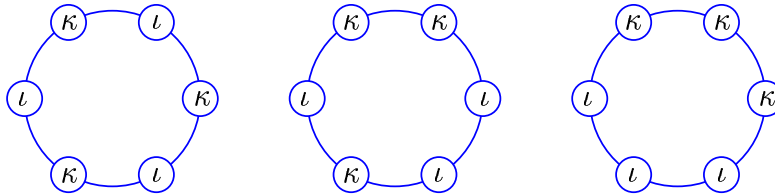
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<sup>†</sup>See [www.math.umass.edu/~sottile/pages/inflected](http://www.math.umass.edu/~sottile/pages/inflected)

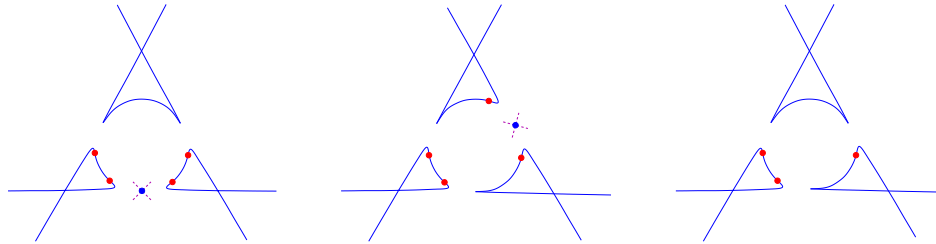
inflected quintics with 2 cusps and 5 flexes,



and three possible necklaces with 3 cusps and 3 flexes.

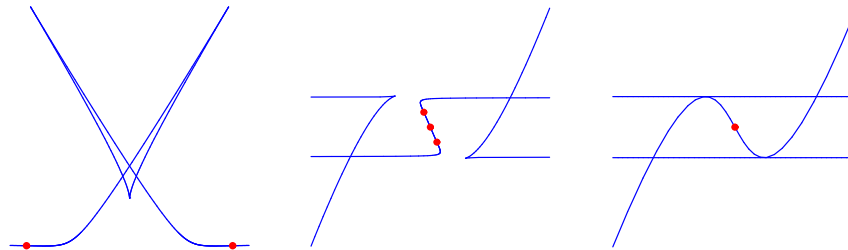


The leftmost necklace shown for both values of  $\kappa$  is that of the corresponding quintic in Figure 4.1. We can realize three of the remaining four necklaces using the variant of the construction of Theorem 4.1 where we use a different local model for the perturbation of a tacnode into a cusp, as explained in Remark 4.3. This gives the three quintics displayed below



which correspond to the remaining two necklaces of curves with two cusps, and the second necklace for curves with three cusps.

What is missing is a quintic with three consecutive cusps. There are six maximally inflected plane quintics with cusps at  $\infty, \pm 3$  and flexes at  $0, \pm 1$ , which we have computed using symbolic methods. One of these six is particularly interesting. The two pictures on the left below are two different views of this curve. In the first, we put one flex at infinity, and in the second, one cusp at infinity.



The second view of this curve suggests that it is a perturbation of the singular and reducible curve shown on the right.

**6.2. Solitary points of quintics with cusps and flexes.** In this section, we address the finer classification of Question 5.7 concerning the possible numbers of solitary points and nodes in a maximally inflected quintic having flexes and cusps with a given necklace. Here, the situation is quite intricate and our knowledge at present is not definitive. It

is, however, based upon extensive experimental numerical evidence. Briefly, for quintics with either zero or one cusp, all possibilities occur, but for most necklaces with two or three cusps, we have not yet seen all possibilities for the numbers of solitary points. We have also observed a fascinating global phenomenon, related to (but weaker than) the very strong classification result we obtained for quartics in Theorem 5.6.

Let us recall the results of Corollary 3.3 for quintics with flexes and cusps.

A rational quintic plane curve with  $\kappa$  cusps and  $9 - 2\kappa$  flexes has genus  $g_\kappa = 6 - \kappa$ . Maximally inflected quintics with four cusps ( $\kappa = 4$ ) are dual to maximally inflected quartics with four flexes and one cusp, and were discussed from different points of view in Remarks 3.6, 3.7, and 3.8. In particular, there we showed the impossibility of such a quintic with no solitary points and no nodes. In Figure 6.1, we give examples of the two remaining possibilities for maximally quintics with four cusps and one flex. The quintic

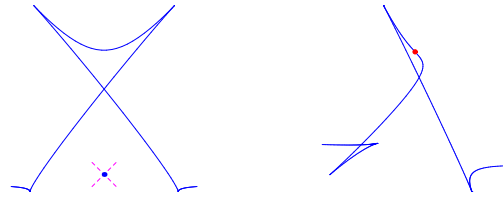


FIGURE 6.1. Quintics with four cusps

on the left has its flex at infinity.

Consider quintics with three or fewer cusps. By Corollary 3.3, the number  $\delta$  of solitary points of such a quintic satisfies

$$3 - \kappa \leq \delta \leq 6 - \kappa =: g_\kappa,$$

or,  $3 \leq \delta + \kappa \leq 6$ . The number  $c$  of complex nodes is an even number between 0 and  $g_\kappa - \delta$ , and the number  $\eta$  of nodes satisfies  $\eta + c = g_\kappa - \delta$ . Thus the possibilities for these numbers for quintics are in bijection with the cells of the diagram

		$\delta + \kappa$			
		3	4	5	6
$c$	0				
	2				

with the first and second rows corresponding to the values of 0 and 2 for  $c$ , and the columns correspond to the possible values for  $\delta + \kappa$ , from 3 on the left to 6 on the right. The first column of Table 6.1 lists the possible necklaces for quintics with at most three cusps. The filled cells of the diagrams in its second column indicate the observed cardinalities of solitary points, complex nodes, and nodes.

The third column of Table 6.1 records an interesting global phenomenon we have observed. Recall from Section 1.2 that for a given collection of ramification data  $\alpha^1, \dots, \alpha^n$ , there will be the same number  $N := N(\alpha^1, \dots, \alpha^n)$  of rational curves having that ramification at specified points. We observe that that the number of these  $N$  curves having a given number of solitary points appears to depend only upon the necklace, and not on the placement of the ramification. We record this in the third column, with the columns corresponding to the possible values of  $\delta + \kappa$  between 3 and 6, from left to right. The last two columns of Table 6.1 give the number  $N(\alpha^1, \dots, \alpha^n)$  of rational quintics of that type

Necklace	Observed $\delta, \eta, c$	$\delta + \kappa$				$N$	Number Tested
		3	4	5	6		
$\kappa\kappa\kappa\ell\ell$		1	3	2	0	6	985
$\kappa\kappa\ell\kappa\ell\ell$		2	3	1	0	6	986
$\kappa\ell\kappa\ell\kappa\ell$		5	0	0	1	6	986
$\kappa\kappa\ell\ell\ell\ell$		3	5	3	0	11	731
$\kappa\ell\kappa\ell\ell\ell$		5	4	1	1	11	731
$\kappa\ell\ell\kappa\ell\ell$		4	5	2	0	11	731
$\kappa\ell\ell\ell\ell\ell$		7	9	4	1	21	1000
$\ell\ell\ell\ell\ell\ell$		12	18	9	3	42	11,416

TABLE 6.1. Observed numbers of nodes and solitary points of quintics

having a given choice of ramification, and the number of choices of ramification for which we computed all  $N(\alpha^1, \dots, \alpha^n)$  quintics and determined their numbers of solitary points, complex nodes, and real nodes.

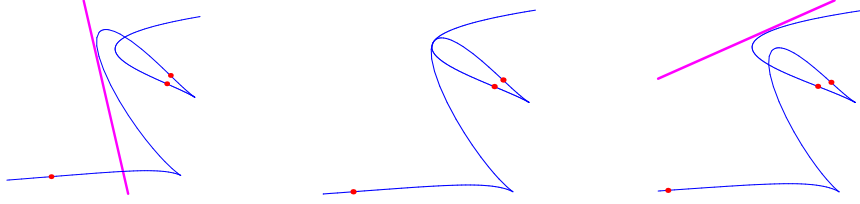
We make the observation in the third column of Table 6.1 more precise in the following conjecture.

**Conjecture 6.1.** The number of solitary points in a maximally inflected plane quintic is a deformation invariant.

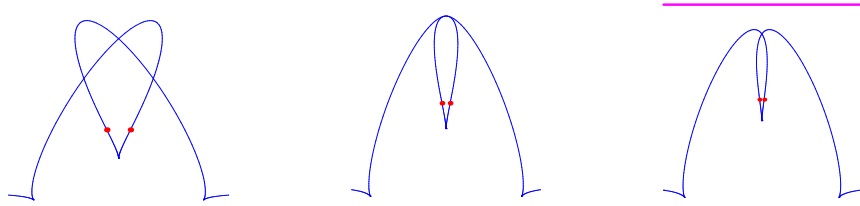
In particular, if Conjecture 2.6 (concerning non-degeneracy) held for quintics, then the number of curves having a given ramification and given number of solitary points depends only on the necklace of the ramification, that is, only on the relative positions of the points of ramification. This is similar to the conclusion of Theorem 5.6 for quartics, but it is a weaker phenomenon, as the topology of the embedding of a quintic can change under an isotopy. (We give an example below.) If Conjecture 2.6 held for quintics, then similar arguments as given in the proof of Theorem 5.6 concerning isolating solitary points by bitangents and Bézout’s theorem may suffice to prove Conjecture 6.1.

An instructive example is provided by quintics whose ramification consists of three cusps at  $\pm 1$  and  $\infty$  and three flexes at  $0$  and  $\pm t$ , where  $1 < t < \infty$ . For these the cusps and flexes alternate, and for a given choice of ramification, there are 6 curves. Letting  $t$  vary, we have 6 families of curves, which are deformations of the curves at any value of  $t$ . All the curves in one family have three solitary points, while those in the other five families have no solitary points. Two of these five families consist of curves with three nodes. When  $t = 3$ , each curve in the remaining three families has two smooth branches with a point of contact of order three. The other curves in two of these three families each have a single node and two complex nodes. The latter two families differ only by reparameterization of their curves. We display curves from one of them at the values  $t = 2.9, 3$ , and  $3.1$ . All three have a cusp at infinity. The line joining the complex nodes is

drawn in the first and third pictures. Despite appearances, it is not tangent to the curve. If it were tangent, then it would have intersection number at least 6 with the curve, which contradicts Bézout's theorem.



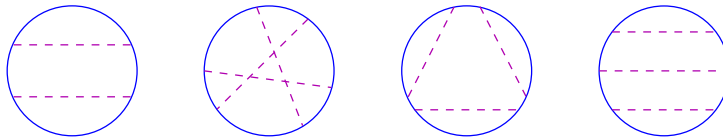
The curves in the third family have one node and two complex nodes when  $t > 3$ , but three nodes for  $t < 3$ . We display curves from this family at the values  $t = \frac{5}{2}$ , 3, and  $\frac{7}{2}$ . The horizontal line in the third picture is the real line meeting the two complex nodes, and all three have a flex at infinity.



We remark that these curves, like those shown in Section 4 and in Figure 6.2 below, were drawn using the computer algebra systems Maple and Singular [11]. We first computed the centers of projection giving all curves with a given ramification, formulating and solving the problem in local coordinates for the Grassmannian. Given the centers, we computed parameterizations and then set up and solved the equations for the double points. Those whose embedding we needed to study, we plotted, and for those displayed in this paper, we created postscript files which were then further edited to enhance important features, such as flexes and solitary points.

**6.3. Chord diagrams of quintics and embeddings.** We classify the possible chord diagrams of maximally inflected quintics and discuss the possible topology of the image as a subset of  $\mathbb{RP}^2$  (ignoring possible solitary points). Since the proof we give does not use flexes, it also classifies chord diagrams of quintics with at most three nodes.

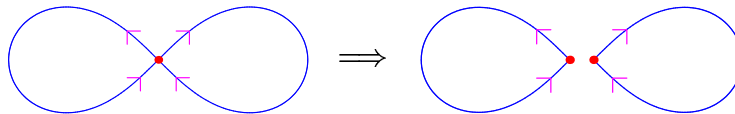
**Theorem 6.2.** *There are 6 possibilities for the chord diagram of a maximally inflected quintic in  $\mathbb{RP}^2$ . Here are the four with 2 or more chords.*



**Proof.** By Corollary 3.3, a maximally inflected quintic can have at most three nodes. Recall that the degree of any (piecewise smooth) closed curve in  $\mathbb{RP}^2$  is well-defined modulo 2, and this degree is additive, again modulo 2. Consider an image of  $\mathbb{RP}^1$  in  $\mathbb{RP}^2$  with a node. Under an orientation from  $\mathbb{RP}^1$ , the node has two incoming arcs and two outgoing arcs. We may split the curve into two pieces at the node by joining an incoming arc of



one branch with the outgoing arc of the other branch. We illustrate this splitting below.



The first case to consider is that of a quintic with two nodes. In a chord diagram of such a quintic, the two chords cannot cross. If they did, then split the quintic at a node. The resulting two closed curves have exactly one point of intersection (the other node), and so each piece necessarily has odd degree. It follows that their union, the original quintic, has even degree, a contradiction.

Suppose now that we have a quintic with three nodes. The argument we just gave forbids any chord diagrams containing a chord that meets exactly one other chord, and thus the only possibilities are as claimed. Concluding, we get only one chord diagram with 2 chords (two non-intersecting chords) and three chord diagrams with 3 chords (three pairwise intersecting chords, three sides of a hexagon, and three parallel chords).



Recall that a pseudoline is a closed curve in  $\mathbb{RP}^2$  which has odd degree and whose complement is connected (it is a one-sided curve). An oval is a two-sided closed curve which necessarily has even degree. We enumerate the possible topological embeddings of  $\mathbb{RP}^1$  given by a maximally inflected quintic, beginning with the classification of chord diagrams.

If there is a single node, then the curve must look like a pseudoline with a loop. Splitting the curve at the node and applying the Jordan Curve Theorem to the loop in the complement of the pseudoline shows that the loop is two-sided. Thus there is a unique possible topological embedding. Such a maximally inflected quintic is shown in Figure 6.2(a). We do not draw the solitary points.

Now consider a quintic with two nodes whose chord diagram (necessarily) consists of two non-intersecting chords. We split the curve at both nodes to obtain three closed curves that meet only at the nodes. Deforming them slightly away from the nodes, we see that at most one is a pseudoline (for any two pseudolines meet), and so exactly one is a pseudoline, as the original curve was a quintic. As before, the other components are then 2-sided ovals. These ovals cannot be nested. A line through a solitary point or a singular ramification point (a maximally inflected quintic must have one such point) that meets the nest has intersection number at least 6 with the quintic, contradicting Bézout's theorem.

Thus there are two possibilities. Either the pseudoline meets both ovals (see Figure 6.2(b)), or the pseudoline meets only one oval, which then meets the other (see Figure 6.2(c)). As before the ovals cannot be nested. Each open circle in Figure 6.2(c) represents two flexes that have nearly merged to create a planar point.

Suppose now that there are three pairwise intersecting chords. If we split the curve at *one* node, we obtain a pseudoline and an oval, which have two additional points of intersection. There is only one possibility for this configuration, and we have already seen it in the last pictures in each of Sections 6.1 and 6.2. We display yet another such curve below in Figure 6.2(d).

If the three chords are three sides of a hexagon, we may split the curve at each of its nodes to obtain a pseudoline and three loops. One piece meets the other three, and

these three are disjoint from each other. If the pseudoline meets the three loops, there are two possibilities for the disposition of the three loops along the pseudoline. Either they alternate (as shown in Figure 6.2(e) or in Figure 4.1) or they do not. We forbid the possibility of non-alternating loops in Proposition 6.3 below. We have not yet observed such a quintic where the pseudoline meets one loop, which then meets the other two loops. A schematic for this is Figure 6.3(a).

Lastly, we have the possibility of three parallel chords. As before, we split the curve at each of its nodes to obtain one pseudoline and three ovals, and the ovals cannot be nested. Two components meet two others, and two meet exactly one other. Either the pseudoline either meets two ovals (See Figure 6.2(f)), or it meets only one. This last case has not yet been observed, but we provide a schematic of it in Figure 6.3(b)

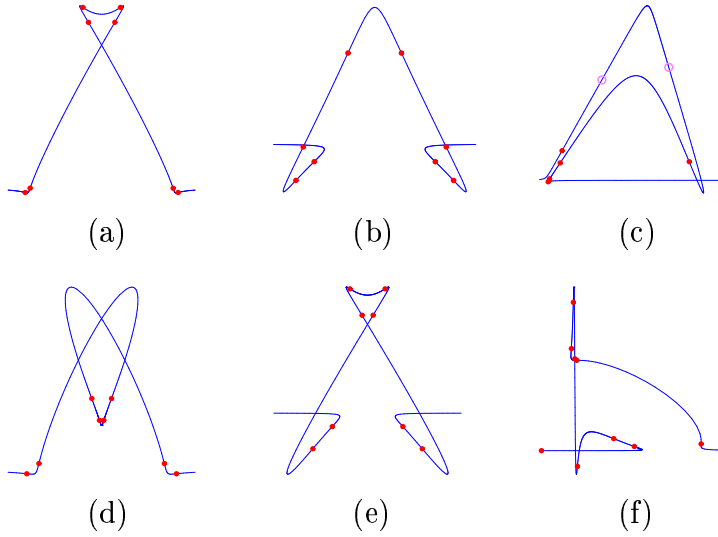


FIGURE 6.2. Maximally inflected quintics realizing different embeddings



FIGURE 6.3. Possible embeddings for quintics that have not been observed

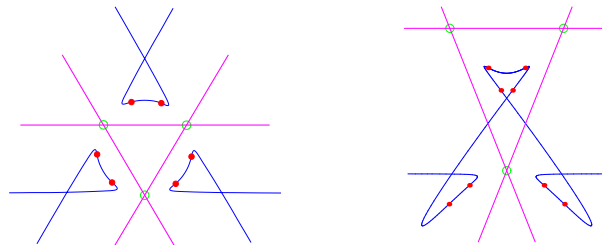
**Proposition 6.3.** *A maximally inflected quintic consisting of a pseudoline with three loops must have the loops alternating.*


**Proof.** Suppose that we have a maximally inflected quintic whose embedding consists of a pseudoline with three loops attached to the pseudoline. Such a curve has three nodes. By Corollary 3.2, a maximally inflected quintic with three nodes may have only flexes, planar points, cusps, or a point with ramification sequence  $(0, 1, 5)$ . Let  $\kappa$  be its number of cusps, which is at most 3—otherwise Corollary 3.3 restricts the number of nodes to be

at most 2. Since it must have at least  $3 - \kappa$  solitary points and  $6 - \kappa$  double points in all, the curve has  $3 - \kappa$  solitary points, or 3 solitary points plus cusps.

Gudkov's extension [13] of Brusotti's Theorem [3] allows us to independently smooth the cusps and nodes to obtain a smooth quintic. We smooth each cusp as in the local model of Figure 3.2, smooth each node to detach its loop from the pseudoline, and smooth each solitary point to obtain an oval. Thus we obtain a smooth plane quintic with 6 ovals, which is an  $M$ -curve. Furthermore, the three ovals arising from solitary points or cusps are distinguished.

Pick a point inside each of the three distinguished ovals, connect each pair by a line and study the configuration of these lines with respect to the quintic. By Bézout's theorem, each of these bisecants intersects the one-sided component at one point and its two ovals at two points each. Hence, the configuration does not depend on the choice of the points. Since all  $M$ -quintics are deformation equivalent, i.e., belong to one connected family of nonsingular quintics (see [20], a little correction of the proof is given in [5]), the configuration does not depend on the choice of the  $M$ -quintic either. Examining any of the various known examples, one finds the configuration like the picture below, where we have drawn the lines joining the solitary points of a rational quintic (the two pictures are for similar curves in different projections), leaving the smoothing to the imagination of the reader. These solitary points are indicated by open circles.



The bisecants form four triangles in  $\mathbb{RP}^2$ , with a distinguished triangle not meeting the pseudoline and containing no ovals (i.e., no ovals arising from the loops). The pseudoline divides each of the other triangles into two components, with the oval in the component adjacent to that edge of the distinguished triangle which is not intersected by the pseudoline. This configuration just described is equivalent to the statement that the loops alternate along the pseudoline. 

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