HOPF ALGEBRAS AND EDGE-LABELED POSETS

NANTEL BERGERON AND FRANK SOTTILE

Abstract. Given a finite graded poset with labeled Hasse diagram, we construct a quasi-symmetric generating function for chains whose labels have fixed descents. This is a common generalization of a generating function for the flag f-vector defined by Ehrenborg and of a symmetric function associated to certain edge-labeled posets which arose in the theory of Schubert polynomials. We show this construction gives a Hopf morphism from an incidence Hopf algebra of edge-labeled posets to the Hopf algebra of quasi-symmetric functions.

To the memory of Gian-Carlo Rota

Journal of Algebra **216** (1999) pp. 641-651.

Joni and Rota [8], and later Schmitt [13] construct Hopf algebras from partially ordered sets, giving a global algebraic framework for studying partially ordered sets. For every graded partially ordered set, Ehrenborg [4] defines a quasi-symmetric generating function for its flag f-vector. He shows that this induces a Hopf morphism from the Hopf algebra of graded posets to the Hopf algebra of quasi-symmetric functions.

Edge-labeled posets are finite graded partially ordered sets, the edges of the Hasse diagrams of which are labeled with integers. Following the construction of Stanley's symmetric function [14], we associate with each such poset a quasi-symmetric generating function for maximal chains whose sequence of edge labels has fixed descents. We show that this reduces to Ehrenborg's function in an important special case and induces a Hopf morphism from the Hopf algebra of edge-labeled posets to the Hopf algebra of quasi-symmetric functions.

While studying structure constants for Schubert polynomials, we defined a symmetric function for any edge-labeled poset with a certain symmetry [3], giving a unified construction of skew Schur functions, Stanley symmetric functions, and skew Schubert functions. We show that this symmetric function equals the quasi-symmetric generating function defined here.

Date: 29 April 1999.

1991 *Mathematics Subject Classification.* 06A07, 16W30, 05E05.

Key words and phrases. edge-labeled poset, Hopf algebra, quasi-symmetric function, incidence algebra.

First author supported in part by NSERC and CRM grants.

Second author supported in part by NSERC grant OGP0170279.

1. Edge-labeled posets. A *poset* P is a finite partially ordered set with maximal element $\hat{1}$ and minimal element $\hat{0}$. For $x \leq y$ in P, let $[x,y] := \{z \mid x \leq z \leq y\}$. A poset P is *graded* of rank $\mathrm{rk}(P) := n$ if every maximal chain has length n. Let $R(P)$ be the set of maximal chains in a poset P. The rank $\mathrm{rk}(x)$ of $x \in P$ is $\mathrm{rk}[\hat{0}, x]$.

We say that $x \lessdot y$ is a *cover* if $[x,y] = \{x,y\}$. An *edge-labeled poset* is a graded poset whose covers are labeled with integers. The sequence of labels in a maximal chain is its *word*. The *descent set* $D(\rho)$ of a maximal chain ρ with word $w_1 \cdot w_2 \cdots w_n$ ($n = \mathrm{rk}(P)$) is

$$D(\rho) \;=\; \{j \mid w_j > w_{j+1}\}.$$

For $I, J \subseteq \{1, \ldots, \mathrm{rk}(P){-}1\}$, define

$$\begin{aligned} d_I(P) &= |\{\rho \in R(P) \mid D(\rho) = I\}| \\ f_J(P) &= |\{\rho \in R(P) \mid D(\rho) \subseteq J\}| \quad = \quad \sum_{I \subseteq J} d_I(P). \end{aligned}$$

By inclusion-exclusion, we have

$$d_I(P) \;=\; \sum_{J \subseteq I} (-1)^{|I-J|} f_J(P).$$

Ehrenborg and Readdy [5] noted $d_I(P)$ is an analog of the rank-selected Möbius invariant, for edge-labeled posets.

We sometimes use compositions α of n in place of subsets I of $\{1, \ldots, n{-}1\}$ to index these numbers, and we wish to go back and forth between these two indexing schemes. Given a subset $I = \{i_1 < i_2 < \cdots < i_k\}$ of $\{1, \ldots, n-1\}$, define a composition $\alpha(I) := (i_1, i_2 - i_1, \ldots, n - i_k)$ of n. Likewise, given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, define a subset $I(\alpha)$ so that $\alpha(I(\alpha)) = \alpha$. The length, $\ell(\alpha)$, of $\alpha = (\alpha_1, \ldots, \alpha_k)$ is k. Let $C(n)$ be the set of compositions of n.

2. Quasi-symmetric functions. Gelfand *et al.* [6] define the graded Hopf algebra $\mathcal{NC}_\bullet$ of non-commutative symmetric functions to be the free associative algebra with one generator S_i of degree i for each $i = 1, 2, \ldots$ The graded Hopf dual of $\mathcal{NC}_\bullet$ is the algebra $\mathcal{Q}_\bullet$ of quasi-symmetric functions [7], which consists of all formal power series of bounded degree in commuting indeterminates $x_1, x_2, \ldots$ which are quasi-symmetric: the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ depends only upon α and not on $i_1, \ldots, i_k$, if $i_1 < i_2 < \cdots < i_k$. Thus $\mathcal{Q}_\bullet$ has a basis of *monomial quasi-symmetric functions* M_α defined by

$$M_\alpha \;:=\; \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

The basis of $\mathcal{NC}_\bullet$ dual to the M_α are the *quasi-Schur functions* $S^\alpha := S_{\alpha_1} S_{\alpha_2} \cdots S_{\alpha_k}$, where $\ell(\alpha) = k$. Thus, for any edge-labeled poset P, the linear map $\psi_P : \mathcal{NC}_\bullet \to \mathbb{Z}$ given by

$$\psi_P(S^\alpha) \;=\; \begin{cases} f_{I(\alpha)}(P) & \text{if } \alpha \in C(\mathrm{rk}(P)) \\ 0 & \text{otherwise} \end{cases}$$

defines a quasi-symmetric function F_P. It follows that if P is an edge-labeled poset of rank n, then

$$F_P \;=\; \sum_{\alpha\in C(\mathrm{rk}(P))} f_{I(\alpha)}(P) M_\alpha. \tag{1}$$

This function is our main object of study.

Another basis of $\mathcal{Q}_\bullet$ is the *fundamental quasi-symmetric functions* $F_{I,n}$. For any subset I of $\{1,\ldots,n-1\}$, define

$$F_{I,n} \;:=\; \sum_{\substack{j_1\leq j_2\leq\cdots\leq j_n\\ i\in I\Rightarrow j_i<j_{i+1}}} x_{j_1}x_{j_2}\cdots x_{j_n}.$$

One checks that

$$\begin{aligned} F_{I,n} \;&=\; \sum_{I\subseteq J\subseteq\{1,\ldots,n-1\}} M_{\alpha(J)}\\ M_\alpha \;&=\; \sum_{I(\alpha)\subseteq J} (-1)^{|J-I(\alpha)|} F_{J,n} \end{aligned} \tag{2}$$

The *ribbon Schur functions* R_α form a basis of $NC_\bullet$ dual to the $F_{I,n}$, and there is a change of basis between the S^α and the R_α analogous to (2). The expressions relating $f_I(P)$ to $d_I(P)$ and $F_{I,n}$ to M_α give

$$F_P \;=\; \sum_{I\subseteq\{1,\ldots,n-1\}} d_I(P) F_{I,n} \tag{3}$$

$$\;=\; \sum_{\rho\in R(P)} F_{D(\rho),n}. \tag{4}$$

This last expression shows that F_P is a generalization of Stanley's (quasi-)symmetric function F_w [14, Equation (1)], introduced to study reduced decompositions of elements w of the symmetric group. To see this, let P be the interval $[1,w]$ in the weak order on the symmetric group, with the label of a cover $u \lessdot v$ the integer i, where $(i,i+1) = vu^{-1}$. Then $R([1,w])$ is the set of reduced decompositions of w, and our definition (4) for $F_{[1,w]}$ coincides with Stanley's definition of F_w.

3. Incidence Hopf algebras. See [11, 17] for more on Hopf algebras. Let $\mathcal{P}$ be a class of graded posets closed under taking subintervals and products. The (reduced) *incidence coalgebra* [8, 13] $\mathcal{IP}$ of $\mathcal{P}$ is the graded free abelian group generated by isomorphism classes of posets in $\mathcal{P}$ with grading induced by the rank of a poset and coproduct by

$$\Delta(P) \;=\; \sum_{x\in P} [\hat{0},x]\otimes[x,\hat{1}].$$

The augmentation is given by projecting onto the degree 0 component. The product of posets induces an algebra structure on $\mathcal{IP}$ with identity the class of a one element poset.

We say that edge-labeled posets P and Q are *label-equivalent* if there is an isomorphism $P \xrightarrow{\sim} Q$ preserving the numbers f_I of subintervals. A map preserving the relative order of the edge labels is such a function, but there are others. Suppose now that $\mathcal{P}$ is a class of edge-labeled posets. The *incidence coalgebra* $\mathcal{IP}$ of $\mathcal{P}$ is the graded free abelian group on label-equivalence classes in $\mathcal{P}$, with coproduct and augmentation as before.

To define an algebra structure on $\mathcal{IP}$, we first form the product $P \times Q$ of edge-labeled posets P and Q. Recall that a cover $(p,q) \lessdot (p',q')$ in $P \times Q$ has one of two forms: either $p = p'$ and $q \lessdot q'$ is a cover in Q, or $p \lessdot p'$ is a cover in P and $q = q'$. Label a cover $(p,q) \lessdot (p',q')$ in $P \times Q$ by the label of the corresponding cover in P or Q.

Proposition 1 (Lemma 3.9 of [3])**.** *Suppose that P and Q are edge-labeled posets with distinct sets of edge labels. Then for any composition α of* $\mathrm{rk}(P) + \mathrm{rk}(Q)$,

$$f_{I(\alpha)}(P \times Q) \;=\; \sum_{\beta+\gamma=\alpha} f_{I(\beta)}(P) \cdot f_{I(\gamma)}(Q),$$

where β ranges over compositions of $\mathrm{rk}(P)$ *and γ over compositions of* $\mathrm{rk}(Q)$*, and addition of compositions is component-wise.*

Let $x, y \in \mathcal{IP}$ be label-equivalence classes of edge-labeled posets. Then xy is the equivalence class with representative $P \times Q$, where P is a representative of x, Q is a representative of y, and P, Q have disjoint sets of edge labels. This product is independent of choices, by Proposition 1. It is also commutative and compatible with the coproduct, so $\mathcal{IP}$ is a graded bialgebra and hence has a unique antipode [4, Lemma 2.1]. We summarize these facts.

Theorem 2. *Let $\mathcal{P}$ be a class of edge-labeled posets closed under taking subintervals and products. Then, with the above definitions, $\mathcal{IP}$ is a commutative graded Hopf algebra.*

We give our main theorem.

Theorem 3. *Let $\mathcal{P}$ be a class of edge-labeled posets closed under subintervals and products. Then the map $\Phi : \mathcal{IP} \to \mathcal{Q}_{\bullet}$ induced by*

$$P \in \mathcal{P} \;\longmapsto\; F_P \in \mathcal{Q}_{\bullet}$$

is a morphism of graded Hopf algebras.

Proof. The expression (4) shows that F_P is a generalization of Stanley's symmetric function F_w. In fact, the proof [14, Theorem 3.4] that $F_{w \times v} = F_w \cdot F_v$ also shows the corresponding fact for F_P: If P, Q are edge-labeled posets with disjoint sets of edge labels, then $F_{P \times Q} = F_P \cdot F_Q$. Thus Φ is an algebra morphism.

We show it is a coalgebra morphism. For a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ and integer $0 \leq j \leq k (= \ell(\alpha))$, define (possibly empty) compositions $\alpha_{\leq j}$ and $\alpha_{>j}$:

$$\begin{aligned} \alpha_{\leq j} &:= (\alpha_1, \ldots, \alpha_j) \\ \alpha_{>j} &:= (\alpha_{j+1}, \ldots, \alpha_k) \end{aligned}$$

The coalgebra structure on $\mathcal{Q}_\bullet$ is given by

$$\Delta M_\alpha \;=\; \sum_{j=0}^{\ell(\alpha)} M_{\alpha_{\leq j}} \otimes M_{\alpha_{>j}}.$$

For an edge-labeled poset P and composition α of $\mathrm{rk}(P)$, let $f_\alpha(P) = f_{I(\alpha)}(P)$. Then, for any $1 \leq j \leq k$, the following identity is straightforward.

$$f_\alpha(P) \;=\; \sum_{\substack{x \in P \\ \mathrm{rk}(x)=I(\alpha)_j}} f_{\alpha_{\leq j}}[\hat{0},x] \cdot f_{\alpha_{>j}}[x,\hat{1}]. \tag{5}$$

Using equations (1) and (5), we have

$$\begin{aligned}
\Delta F_P \;&=\; \sum_{\alpha \in C(\mathrm{rk}(P))} f_\alpha(P) \Delta M_\alpha \;=\; \sum_{\alpha \in C(\mathrm{rk}(P))} f_\alpha(P) \sum_{j=0}^{\ell(\alpha)} M_{\alpha_{\leq j}} \otimes M_{\alpha_{>j}} \\
&=\; \sum_{\alpha \in C(\mathrm{rk}(P))} \sum_{j=0}^{\ell(\alpha)} \sum_{\substack{x \in P \\ \mathrm{rk}(x)=I(\alpha)_j}} f_{\alpha_{\leq j}}[\hat{0},x] M_{\alpha_{\leq j}} \otimes f_{\alpha_{>j}}[x,\hat{1}] M_{\alpha_{>j}} \\
&=\; \sum_{x \in P} \left(\sum_{\beta \in C(\mathrm{rk}([\hat{0},x]))} f_\beta[\hat{0},x] M_\beta \right) \otimes \left(\sum_{\gamma \in C(\mathrm{rk}([x,\hat{1}]))} f_\gamma[x,\hat{1}] M_\gamma \right),
\end{aligned}$$

which we recognize as $F_{\Delta P}$. ∎

Example 4. A Boolean poset is the poset of subsets of a finite set of integers in which a cover $X \lessdot Y$ is labeled by the integer $X \backslash Y$. The one element chain $x := (\hat{0} < \hat{1})$ is the unique primitive element in any non-trivial (reduced) incidence Hopf algebra of edge-labeled posets. (All labelings of x are equivalent.) This primitive element generates the commutative subalgebra $\mathbb{Z}[x]$, which is the incidence Hopf algebra for the class $\mathcal{B}$ of Boolean posets (algebras) with a standard labeling for a lattice of order ideals, as the Boolean poset of subsets of $\{1, 2, \ldots, n\}$ is the lattice of order ideals of the antichain $\{1, 2, \ldots, n\}$ [15, Example 3.13.3]. Moreover, the map $\Phi : \mathcal{IB}(= \mathbb{Z}[x]) \to \mathcal{Q}_\bullet$ is an isomorphism onto the subalgebra generated by $h_1 = F_x$, which is a subalgebra of symmetric functions.

4. Rank-selected posets.

Let P be a graded poset and I be a subset of $\{1, \ldots, \mathrm{rk}(P){-}1\}$. The *rank-selected poset* $P(I)$ is the induced subposet of P consisting of all elements of P with rank in I, together with $\hat{0}$ and $\hat{1}$. Set $\varphi_I(P)$ to be the number of maximal chains in $P(I)$. These numbers $\varphi_I(P)$ constitute the *flag f-vector* of P. Ehrenborg's quasi-symmetric generating function E_P for the flag f-vector satisfies

$$E_P \;=\; \sum_I \varphi_I(P) M_{\alpha(I)}.$$

An edge-labeled poset P is *R-labeled* if every interval has a unique increasing chain. For these posets, $\varphi_I(P) = f_I(P)$, and so $E_P = F_P$. Similarly, the numbers $d_I(P)$ are the rank-selected Möbius invariant for R-labeled posets P [15, Section 3.13], and the η and ν functions of Ehrenborg-Readdy [5] (for edge-labeled posets) reduce to the zeta and Möbius functions for R-labeled posets.

More generally, we regard $f_I(P)$ as an extension of the notion of flag f-vector. Suppose P is an edge-labeled poset and $I \subseteq \{1,\ldots,\mathrm{rk}(P)-1\}$. Let $P(I)_{\mathrm{wt}}$ be the rank selected poset as before, but with every cover $x \lessdot y$ in $P(I)_{\mathrm{wt}}$ weighted by the number of chains with increasing labels in the interval $[x,y]$ of P. A maximal chain in $P(I)_{\mathrm{wt}}$ has weight given by the product of the weights of its covers. Then $f_I(P)$ counts these weighted maximal chains of $P(I)_{\mathrm{wt}}$, and so F_P is a weighted version of E_P.

Another connection between these theories is given by Stanley [16] and concerns a relative version of E_P and the flag f-vector. Let Γ be a set of (not necessarily maximal) chains of a poset P that is closed under taking subchains. The relative flag f-vector $\varphi_I(P/\Gamma)$ counts chains in the rank-selected poset $P(I)$ that are not in Γ, and $E_{P/\Gamma}$ is the quasi-symmetric generating function for $\varphi_I(P/\Gamma)$.

An edge-labeled poset is *relative R-labeled* if each interval has at most one increasing chain, and all subintervals of an interval with an increasing chain also have an increasing chain. If P is relative R-labeled, and Γ is the set of chains $\hat{0} = t_0 < t_1 < \cdots < t_r = \hat{1}$ for which there is an i where the interval $[t_{i-1}, t_i]$ does not have an increasing chain, then Stanley shows that $\varphi_I(P/\Gamma) = f_I(P)$, so that $E_{P/\Gamma} = F_P$.

Interestingly, the labeled posets whose study led us to consider F_P are all relative R-labeled. These are intervals in the weak order [14], the k-Bruhat order [1], and the Grassmannian Bruhat order [2], all on the symmetric group.

5. Symmetric edge-labeled posets. For more on symmetric functions, see [10]. For a composition α, let $\lambda(\alpha)$ be the partition obtained by listing the components of α in decreasing order. For a partition μ, the *monomial symmetric function* m_μ is

$$m_\mu \ := \sum_{\alpha \,:\, \lambda(\alpha)=\mu} M_\alpha .$$

These form a basis for the algebra of symmetric functions. From Equation (1), we deduce the following fact.

Theorem 5. *The function F_P is symmetric if and only if for every composition α of* $\mathrm{rk}(P)$*, the number $f_\alpha(P)$ depends only upon $\lambda(\alpha)$.*

An edge-labeled poset is *symmetric* if $f_\alpha(P)$ depends only upon $\lambda(\alpha)$. Symmetric posets arose in the study of Schubert polynomials [3], where we defined a symmetric function S_P for each symmetric poset. This provided a common definition of Stanley symmetric functions, skew Schur functions, and skew Schubert functions. For these, the posets were intervals in, respectively, the weak order on the symmetric group, Young's lattice, and the Grassmannian Bruhat order [2]. The labeling for the weak order was described in Section 2. In Young's lattice, a cover $\mu \lessdot \lambda$ has a unique index

i with $\mu_i < \lambda_i$, and we label that cover with the the integer $i-\lambda_i$. The Grassmannian Bruhat order is a common generalization of both of these labeled posets, and we refer the reader to [1] for details.

We will show S_P is just the function F_P. A quasi-symmetric generating function construction of skew Schur functions was given in [7], which is essentially the same as given here. While Gessel uses a poset labeling different from that used in [3], the two are label-equivalent in a strong sense—a maximal chain in either labeling has the same descent set.

The algebra $\Lambda_\bullet$ of symmetric functions has several distinguished bases besides the m_λ. These include the complete symmetric functions h_λ and the Schur functions S_λ. These bases are related by the Cauchy formula, an element in the graded completion of $\bigoplus_n \Lambda_n(x) \otimes \Lambda_n(y)$:

$$\prod_{i,j}(1-x_iy_j)^{-1} \;=\; \sum_\lambda h_\lambda(x)m_\lambda(y) \;=\; \sum_\lambda S_\lambda(x)S_\lambda(y)$$

Suppose P is a symmetric edge-labeled poset. Define a linear map $\chi_P : \Lambda_\bullet \to \mathbb{Z}$ by

$$\chi_P(h_\mu) \;=\; \begin{cases} f_\mu(P) & \text{if } \mu \text{ is a partition of } \mathrm{rk}(P), \\ 0 & \text{otherwise.} \end{cases}$$

We define $c^P_\lambda := \chi_P(S_\lambda)$, which generalizes the Littlewood-Richardson numbers, as $c^P_\lambda = c^\nu_{\mu,\lambda}$ when P is the interval $[\mu,\nu]$ in Young's lattice with a natural labeling of covers [3].

Since $\Lambda_\bullet$ is a self-dual Hopf algebra (with $\{h_\mu, m_\mu\}$ and $\{S_\mu, S_\mu\}$ pairs of dual bases), χ_P gives a symmetric function S_P. From the Cauchy formula, we see that

$$\begin{aligned} S_P &= \chi_P \otimes 1_{\Lambda_\bullet(y)}\left(\prod_{i,j}(1-x_iy_j)^{-1}\right) \\ &= \sum_{\lambda\vdash \mathrm{rk}(P)} f_\lambda(P)\, m_\lambda \\ &= \sum_{\lambda\vdash \mathrm{rk}(P)} c^P_\lambda S_\lambda. \end{aligned}$$

Theorem 6. *Let P be a symmetric edge-labeled poset. Then $S_P = F_P$.*

Proof.

$$\begin{aligned} F_P &= \sum_{\alpha\in C(\mathrm{rk}(P))} f_\alpha(P) M_\alpha \\ &= \sum_{\mu\vdash \mathrm{rk}(P)} f_\mu(P) \sum_{\alpha\,:\,\lambda(\alpha)=\mu} M_\alpha \\ &= \sum_{\mu\vdash \mathrm{rk}(P)} f_\mu(P) m_\mu \;=\; S_P. \end{aligned}$$ ∎

Remark 7. The definition of F_P in terms of the linear map ψ_P (Section 2) mimics the Cauchy identity construction of S_P above. The Cauchy identity for $NC_\bullet$ and $\mathcal{Q}_\bullet$ is an element in the graded completion of $\bigoplus_n NC_n \otimes \mathcal{Q}_n$ [6, Section 6]:

$$\sum_\alpha R_\alpha \otimes F_{I(\alpha)} \;=\; \sum_\alpha S^\alpha \otimes M_\alpha.$$

Thus F_P is just $\psi_P \otimes 1_{\mathcal{Q}_\bullet}$ applied to this element. Here S^α is the analog of the homogeneous symmetric function and $\psi_P(S^\alpha) = f_{I(\alpha)}(P)$.

Remark 8. In many cases when F_P is symmetric, the symmetric function F_P is known to be the Frobenius characteristic of a representation of the symmetric group $\mathcal{S}_{\mathrm{rk}(P)}$ on the linear span of maximal chains of P. For example, if P is the Boolean poset of subsets of $[n]$, then $F_P = (h_1)^n$, which is the Frobenius characteristic of the right regular representation of $\mathcal{S}_n$. This is the action of $\mathcal{S}_n$ on maximal chains induced by permuting the factors of $P = (\hat{0} < \hat{1})^n$.

Similarly, F_P is the Frobenius characteristic of a representation if P is an interval in either the weak order on the symmetric group [9] or Young's lattice [12]. If P is an interval in either the k-Bruhat order or the Grassmannian Bruhat order, then F_P is known to be Schur-positive, by geometry. For such intervals P, it would be interesting to construct a $\mathcal{S}_{\mathrm{rk}(P)}$-representation on the linear span of the maximal chains of P with Frobenius characteristic F_P. Considering rank 3 intervals in these orders shows that this representation cannot arise from a permutation action of $\mathcal{S}_{\mathrm{rk}(P)}$ on the maximal chains of P.

Acknowledgments

We are indebted to Richard Stanley, who suggested studying chains with fixed descents using quasi-symmetric functions, and pointed out the connection to relative R-labelings. We also thank Richard Ehrenborg for helpful conversations and Sarah Witherspoon whose Hopf-algebraic advice was crucial at the start of this project.

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Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, CANADA

E-mail address, Nantel Bergeron: bergeron@mathstat.yorku.ca
URL, Nantel Bergeron: http://www.math.yorku.ca/bergeron

Department of Mathematics, University of Wisconsin, Van Vleck Hall, 480 Lincoln Drive, Madison, Wisconsin 53706-1388, USA

E-mail address, Frank Sottile: sottile@math.wisc.edu
URL, Frank Sottile: http://www.math.wisc.edu/~sottile