GENERAL ISOTROPIC FLAGS ARE GENERAL (FOR GRASSMANNIAN SCHUBERT CALCULUS)

FRANK SOTTILE

Abstract. We show that general isotropic flags for odd-orthogonal and symplectic groups are general for Schubert calculus on the classical Grassmannian in that Schubert varieties defined by such flags meet transversally. This strengthens a result of Belkale and Kumar.

Schubert varieties $\Omega_I E_\bullet$ in a classical flag manifold G/P are given by a flag $E_\bullet$ and a Schubert condition I [3]. By Kleiman's Transversality Theorem [4], if the flags $E^1_\bullet, \ldots, E^s_\bullet$ are general, then any corresponding Schubert varieties intersect (generically) transversally in that they meet transversally along a Zariski dense open subset of every component of their intersection.

Oftentimes we do not have the luxury of general flags, yet need to show that the Schubert varieties meet transversally. It is often sufficient for their intersection to be *proper* (has the expected dimension or is empty). Belkale and Kumar [1] needed such a case where G was either $Sp(2n)$ or $SO(2n{+}1)$, the flags $E_\bullet$ were isotropic flags, and G/P was an isotropic Grassmannian which is naturally a subset of a classical Grassmannian Gr.

Proposition 1 (Belkale and Kumar [1])**.** *The intersection $\cap_{i=1}^s \Omega_{I^s} E^i_\bullet$ in Gr is proper when $E^1_\bullet, \ldots, E^s_\bullet$ are general isotropic flags and $I^1, \ldots, I^s$ are Schubert conditions for $Sp(2n)$ or $SO(2n{+}1)$*

We show that the intersection is in fact transverse.

Theorem 2. *The intersection $\cap_{i=1}^s \Omega_{I^i} E^i_\bullet$ in Gr is transverse when $E^1_\bullet, \ldots, E^s_\bullet$ are general isotropic flags for $Sp(2n)$ or $SO(2n{+}1)$.*

We use a case for Gr where the flags are not general, yet the corresponding Schubert varieties meet transversally. Let $f_1(t), \ldots, f_m(t)$ be a basis for the space of polynomials of degree less than m. These define a *rational normal curve* $\gamma\colon \mathbb{C} \to \mathbb{C}^m$ by

$$\gamma \;:\; t \;\longmapsto\; (f_1(t), \ldots, f_m(t))^T\,.$$

(We use column vectors and $(\cdots)^T$ denotes transpose.) For each $t \in \mathbb{C}$ and $i = 1, \ldots, m$, the i-plane osculating γ at $\gamma(t)$ is the linear span of $\gamma(t), \gamma'(t), \ldots, \gamma^{(i-1)}(t)$. These osculating planes form the *osculating flag $E_\bullet(t)$*. An intersection of Schubert varieties for Gr given by osculating flags consists of those linear series on $\mathbb{P}^1$ with at least some prescribed ramification. Eisenbud and Harris [2] showed that this intersection is proper.

Proposition 3. *The intersection $\cap_{i=1}^s \Omega_{I^i} E_\bullet(t_i)$ in Gr is proper if $t_1, \ldots, t_s \in \mathbb{C}$ are distinct.*

2000 *Mathematics Subject Classification.* 14M15, 14N15.

Key words and phrases. Schubert calculus, isotropic Schubert calculus, transversality.

Work of Sottile supported by NSF grant DMS-0701050.

This result is elementary—the codimension of the Schubert variety $\Omega_I E_\bullet(t)$ is the order of vanishing at t of the Wronskian of a general linear series in $\Omega_I E_\bullet(t)$. Considerably less elementary is the following result of Mukhin, Tarasov, and Varchenko [6, Corollary 6.3].

Proposition 4. *The intersection $\cap_{i=1}^s \Omega_{I^i} E_\bullet(t_i)$ in Gr is transverse if $t_1, \dots, t_s \in \mathbb{R}$ are distinct.*

Mukhin, Tarasov, and Varchenko proved this when the intersection is zero-dimensional, but the full statement follows from their result via a standard argument. Suppose that an intersection of Schubert varieties as in Proposition 4 has dimension $r(> 0)$ and let Z be any of its components. Let ι be the codimension 1 Schubert condition, so that $\Omega_\iota E_\bullet$ is a hypersurface in Gr. Let $u_1, \dots, u_r \in \mathbb{R}$ be distinct from $t_1, \dots, t_s$. Then the intersection

$$\bigcap_{i=1}^{s} \Omega_{I^i} E_\bullet(t_i) \ \cap \ \bigcap_{i=1}^{r} \Omega_\iota E_\bullet(u_i) \tag{5}$$

is zero-dimensional and therefore transverse. Since $\Omega_\iota E_\bullet$ meets every curve in Gr, the intersection $Z \cap \cap_{i=1}^r \Omega_\iota E_\bullet(u_i)$ is non-empty. Thus the intersection of Proposition 4 was transverse along Z, for otherwise the intersection (5) would not be transverse at points of Z.

Let $\langle\, ,\, \rangle$ be a non-degenerate alternating form on $\mathbb{C}^{2n}$ whose matrix $(\langle e_i, e_j \rangle)_{i,j=1,\dots,n}$ with respect to the standard ordered basis $e_1, \dots, e_{2n}$ is

$$\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} ,$$

where J is the anti-diagonal matrix $(1, \dots, 1)$ of size n. The *symplectic group $Sp(2n)$* is the group of linear transformations of $\mathbb{C}^{2n}$ which preserve $\langle\, ,\, \rangle$. In this ordered basis

$$\gamma(t) \ := \ \left(1,\, t,\, \frac{t^2}{2},\, \dots,\, \frac{t^n}{n!},\, -\frac{t^{n+1}}{(n+1)!},\, \frac{t^{n+2}}{(n+2)!},\, \dots,\, (-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!}\right)^T \tag{6}$$

is a rational normal curve whose osculating flag is *isotropic* in that $E_{2n-i}(t)$ annihilates $E_i(t)$ for $i < 2n$. We leave this as an exercise for the reader.

Similarly, let $\langle\, ,\, \rangle$ be a non-degenerate symmetric form on $\mathbb{C}^{2n+1}$ whose matrix is the anti-diagonal matrix $(1, \dots, 1)$ of size $2n+1$. The *special orthogonal group $SO(2n+1)$* is the group of linear transformations of $\mathbb{C}^{2n+1}$ of determinant 1 which preserve $\langle\, ,\, \rangle$. Then

$$\gamma(t) \ := \ \left(1,\, t,\, \frac{t^2}{2},\, \dots,\, \frac{t^n}{n!},\, -\frac{t^{n+1}}{(n+1)!},\, \frac{t^{n+2}}{(n+2)!},\, \dots,\, (-1)^{n}\frac{t^{2n}}{(2n)!}\right)^T \tag{7}$$

is a rational normal curve whose osculating flag is *isotropic* in that $E_{2n+1-i}(t)$ annihilates $E_i(t)$ for $i \leq 2n$.

Since it is an open condition on s-tuples of isotropic flags that Schubert varieties in Gr meet properly or meet transversally, Proposition 1 and Theorem 2 follow from Propositions 3 and 4, respectively. These rational normal curves (6) and (7) were introduced in [9] to study the analog of the Shapiro conjecture [8] for flag varieties for $Sp(2n)$ and $SO(2n+1)$, and the proof of the Shapiro conjecture [5] motivated Proposition 4.

These special osculating flags are better understood in terms of Lie theory. Let G be a semisimple complex Lie group with Lie algebra $\mathfrak{g}$. The adjoint action of G on the nilpotent elements of $\mathfrak{g}$ has finitely many orbits, with dense orbit consisting of *principal nilpotent* elements of $\mathfrak{g}$. Write exp: $\mathfrak{g} \to G$ for the exponential map. For a principal nilpotent $\eta \in \mathfrak{g}$, $\{\exp(t\eta) \mid t \in \mathbb{C}\}$ is the corresponding 1-parameter subgroup of G. It is natural to consider Schubert varieties defined by translates of a fixed flag by elements $\exp(t\eta)$.

The matrix $\eta \in \mathfrak{sl}_m$ with entries $1, 2, \ldots, m-1$ below its diagonal is principal nilpotent. Dale Peterson observed that the action of $\exp(t\eta)$ on the standard coordinate flag gives the osculating flag $E_\bullet(t)$ to the rational normal curve $\gamma(t) := (1, t, t^2, \ldots, t^{m-1})^T$. The osculating flags to (6) and (7) also arise from exponentiating principal nilpotents in $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n+1}$, respectively. These nilpotents have entries $1, \ldots, 1, -1, \ldots, -1$ below their diagonals with n 1s. We obtain flags osculating a rational normal curve because principal nilpotents are mapped to principal nilpotents under the inclusions $\mathfrak{sp}_{2n} \hookrightarrow \mathfrak{sl}_{2n}$ and $\mathfrak{so}_{2n+1} \hookrightarrow \mathfrak{sl}_{2n+1}$.

This is not the case for the even orthogonal groups, which explains their exclusion from Theorem 2. A principal nilpotent for $\mathfrak{so}_{2n}$ is the $2n \times 2n$ matrix η with 1 in positions $i, i+1$ and -1 in positions $2n-i, 2n-i+1$ for $i = 1, \ldots, n$ (it has $1, \ldots, 1, 0, -1, \ldots, -1$ below its diagonal) and also 1 in position $n-1, n+1$ and -1 in position $n, n+2$. As $\eta^{2n-1} = 0$, it is not a principal nilpotent for $\mathfrak{sl}_{2n}$, whose principal nilpotents N have $N^{2n-1} \neq 0$.

We point out a further limitation of this method. Proposition 3 becomes false if we replace a classical Grassmannian Gr by a general type A flag variety. Indeed, in the 8-dimensional manifold of flags $\{F_1 \subset F_3 \subset \mathbb{C}^5\}$ consisting of a 1-dimensional subspace lying in a 3-dimensional subspace in $\mathbb{C}^5$, the Schubert variety $\Omega_{32514}E_\bullet$ has codimension 5 and the Schubert variety $\Omega_{21435}E_\bullet$ has codimension 2. Consequently, if $E_\bullet$, $E'_\bullet$, and $E''_\bullet$ are general flags, then

$$\Omega_{32514}E_\bullet \cap \Omega_{21435}E'_\bullet \cap \Omega_{21435}E''_\bullet$$

is empty for dimension reasons. If however, $E_\bullet$, $E'_\bullet$, and $E''_\bullet$ osculate a rational normal curve, then the intersection is non-empty. This is shown in Section 3.3.6 of [7].

References

[1] P. Belkale and S. Kumar, *Eigencone, saturation and Horn problems for symplectic and odd orthogonal groups*, arXiv:0708.0398. J. Alg. Geom., to appear.

[2] D. Eisenbud and J. Harris, *Divisors on general curves and cuspidal rational curves*, Invent. Math. **74** (1983), no. 3, 371–418.

[3] Wm. Fulton and P. Pragacz, *Schubert varieties and degeneracy loci*, Lecture Notes in Mathematics, vol. 1689, Springer-Verlag, Berlin, 1998, Appendix J by the authors in collaboration with I. Ciocan-Fontanine.

[4] S. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297.

[5] E. Mukhin, V. Tarasov, and A. Varchenko, *The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz*, Annals of Math., to appear.

[6] ———, *Schubert calculus and representations of the general linear group*, Journal of the AMS, to appear.

[7] J. Ruffo, Y. Sivan, E. Soprunova, and F. Sottile, *Experimentation and conjectures in the real Schubert calculus for flag manifolds*, Experiment. Math. **15** (2006), no. 2, 199–221.

[8] F. Sottile, *Real Schubert calculus: polynomial systems and a conjecture of Shapiro and Shapiro*, Experiment. Math. **9** (2000), no. 2, 161–182.

[9] ———, *Some real and unreal enumerative geometry for flag manifolds*, Michigan Math. J. **48** (2000), 573–592, Dedicated to William Fulton on the occasion of his 60th birthday.

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA
E-mail address: sottile@math.tamu.edu
URL: www.math.tamu.edu/~sottile