PIERI'S FORMULA VIA EXPLICIT RATIONAL EQUIVALENCE

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ABSTRACT. Pieri's formula describes the intersection product of a Schubert cycle by a special Schubert cycle on a Grassmannian. We present a new geometric proof, exhibiting an explicit chain of rational equivalences from a suitable sum of distinct Schubert cycles to the intersection of a Schubert cycle with a special Schubert cycle. The geometry of these rational equivalences indicates a link to a combinatorial proof of Pieri's formula using Schensted insertion.

1. INTRODUCTION

Pieri's formula asserts that the product of a Schubert class and a special Schubert class is a sum of certain other Schubert classes, each with coefficient 1. This determines the multiplicative structure of the Chow ring of a Grassmann variety. Pieri's formula also arises in algebra, combinatorics, and representation theory, and has several proofs these contexts [13, p. 73][6, p. 463][5, p. 24]. Among the geometric proofs, perhaps the most vivid uses linear algebra to compute a triple intersection of Schubert varieties (*cf.* [9][7, p. 203][5, §9.4]) and then invokes (Poincaré) duality. Interestingly, Hodge [9] does not deduce Pieri's formula from this triple intersection, but rather gives an inductive proof based upon certain deformations in the Grassmannian. Laksov [12] uses Giambelli's formula and intersection-theoretic maps (a substitute for Hodge's deformations) in his inductive proof and Hiller [8] uses Borel's characteristic map and the Chevalley [3] formula. Recently, Pragacz and Ratajski [15, 16, 17, 18] have developed an approach valid for all G/P's, (G a classical algebraic group, and P a maximal parabolic) using Borel's characteristic map and divided differences [2, 4]. This is summarized in [14].

We present a new geometric proof of Pieri's formula, explicitly describing a sequence of deformations (inducing rational equivalence) that transform a general intersection of a Schubert variety with a special Schubert variety into a union of distinct Schubert varieties. This gives an understanding of the structure of rational equivalence on Grassmann varieties in terms of the combinatorics of the Bruhat order of the Schubert cellular decomposition. This proof enables one to determine some enumerative problems [24, §5] (those involving at most five Schubert varieties where at least three are special Schubert varieties) *without* reference to a Chow or cohomology ring, the traditional tool in enumerative geometry. Moreover, these deformations show that these enumerative problems may be solved over the real numbers [24]. The geometry of these deformations

Date: 14 January 1998.

1991 *Mathematics Subject Classification.* Primary. 14M15 Secondary. 05E10.

Key words and phrases. Pieri's formula, rational equivalence, Grassmannian, Schensted insertion.

Research supported in part by NSERC grant # OGP0170279.

Canad. J. Math., 49 (1997), 1281–1298.

is quite interesting and their form parallels a combinatorial proof of Pieri's formula [5, p. 24] using Schensted insertion [21].

Their explicit nature leads to homotopy continuation algorithms [1] for finding numerical solutions to enumerative problems involving any number of special Schubert conditions [10].

Let $\mathbf{G}_m V$ be the Grassmannian of m-dimensional subspaces of an n-dimensional vector space V over a field k. A decreasing sequence α of length m ($n \geq \alpha_1 > \cdots > \alpha_m \geq 1$) and a complete flag $F_\bullet$ in V together determine a Schubert subvariety $\Omega_\alpha F_\bullet$ of $\mathbf{G}_m V$. Special Schubert varieties Ω_L are those Schubert varieties given by the single condition that an m-plane intersect a given linear subspace L non-trivially. For any subscheme X of $\mathbf{G}_m V$, let $[X]$ be its cycle class in the Chow ring of $\mathbf{G}_m V$. Pieri's formula asserts

$$(1) \qquad [\Omega_\alpha F_\bullet] \cdot [\Omega_L] \;=\; \sum [\Omega_\gamma F_\bullet],$$

the sum over all sequences γ with $\gamma_1 \geq \alpha_1 > \gamma_2 \geq \cdots > \gamma_m \geq \alpha_m$ where $\sum \gamma_i - \alpha_i$ is equal to the codimension b of Ω_L. (L necessarily has codimension $m+b-1$ in V.) Let $\alpha * b$ denote this set of sequences. One may deduce Pieri's formula as follows (*cf.* [5, § 9.4]): Let $E_\bullet$ be a flag in general position with respect to $F_\bullet$ and L, and define γ^c by $\gamma^c_j := n+1-\gamma_{m+1-j}$. By (Poincaré) duality, Pieri's formula is equivalent to the statement that

$$(2) \qquad \Omega_\alpha F_\bullet \bigcap \Omega_L \bigcap \Omega_{\gamma^c} E_\bullet$$

is either a transverse intersection consisting of a single point or is empty, depending upon whether or not $\gamma \in \alpha * b$.

Indeed, $\Omega_\alpha F_\bullet \bigcap \Omega_{\gamma^c} E_\bullet = \emptyset$ unless $\alpha_j \leq \gamma_j$ for each j. If also $b = \sum_j \gamma_j - \alpha_j$, then there exists a subspace C of dimension $m + b - \sum_{j=2}^m \max\{0, \gamma_j - \alpha_{j-1} + 1\}$ such that if $H \in \Omega_\alpha F_\bullet \bigcap \Omega_{\gamma^c} E_\bullet$, then $H \subset C$. Hence (2) is empty unless $L \cap C \neq \emptyset$ and so $\gamma_j \leq \alpha_{j-1}$, hence $\gamma \in \alpha * b$. Moreover, in that case, $C = C_1 \oplus \cdots \oplus C_m$ and $H \in \Omega_\alpha F_\bullet \bigcap \Omega_{\gamma^c} E_\bullet$ implies that $\dim H \cap C_j = 1$. Since $L \cap C$ is spanned by the vector $f_1 \oplus f_2 \oplus \cdots \oplus f_m$, where $f_i \in C_i$, the intersection (2) is the singleton $\langle f_1, f_2, \ldots, f_m \rangle$. Examining local equations shows the intersection is transverse. Similar ideas lead to a proof of a Pieri-type formula for the flag manifold [22].

In contrast, Hodge [9] deforms the cycle $\Omega_\alpha F_\bullet \bigcap \Omega_L$ into a sum $A + B$ of cycles, where $A \subset \{H \in \mathbf{G}_m k^n | v \in H\} \simeq \mathbf{G}_{m-1} k^{n-1}$, with $v \in k^n$, and B comes from a cycle $B' \subset \mathbf{G}_m k^{n-1}$. He shows that both A and B' have the form $\Omega_{\alpha'} F'_\bullet \bigcap \Omega_{L'}$ and completes the proof by induction.

For our proof, let $Chow\, \mathbf{G}_m V$ be the Chow variety of $\mathbf{G}_m V$, let $Y_{\alpha,b}$ be the cycle $\sum_{\gamma \in \alpha * b} \Omega_\gamma F_\bullet$, and let $\mathcal{G} \subset Chow\, \mathbf{G}_m V$ be the set of cycles $\Omega_\alpha F_\bullet \bigcap \Omega_L$ for all L of a fixed dimension such that the intersection is generically transverse. We describe a partial compactification of $\mathcal{G}$ in $Chow\, \mathbf{G}_m V$ with $b+1$ rational strata, each an orbit of the Borel subgroup of $GL(V)$ stabilizing $F_\bullet$, hence consisting of isomorphic cycles. The 0th stratum is dense in $\mathcal{G}$ and cycles in the ith stratum have components X_β indexed by $\beta \in \alpha * i$, where X_β is a subvariety of $\Omega_\beta F_\bullet$. Passing from one stratum to the next, each component X_β deforms into some components of cycles in the next stratum. The 'history' of each component $\Omega_\gamma F_\bullet$ of $Y_{\alpha,b}$ through this process gives a chain in the Bruhat order of Schubert varieties, recording which component at each stage gave rise to $\Omega_\gamma F_\bullet$.

This leads to the following interpretation of Pieri's formula: The sum in (1) is over a certain set of chains in the Bruhat order which begin at α, with the chain ending at γ recording the history of the cycle $\Omega_\gamma F_\bullet$ in the sequence of deformations. In §4, we show how this is similar to a combinatorial proof of Pieri's formula based on Schensted insertion.

A chain in the Bruhat order is a standard skew tableau [5, 19]. Thus the Littlewood-Richardson rule for multiplying two Schubert classes has an interpretation as a sum over certain chains in the Bruhat order. A (as yet unknown) geometric proof of the Littlewood-Richardson rule for Grassmannians should provide an explanation for this, similar to what we give here for Pieri's formula.

Kleiman [11] proves that in characteristic zero, general subvarieties of a Grassmannian intersect generically transversally and gives a counterexample in positive characteristic. In §2, we work over an arbitrary field and give a precise determination (Theorem 2.4) of when a special Schubert variety meets a fixed Schubert variety generically transversally, and describe the components of such an intersection. The geometry of these components is interesting: while not an intersection of Schubert varieties, each component is 'birationally fibred' over such an intersection, with Schubert variety fibres. Such cycles are the key to our proof of Pieri's formula in §3; they are the components of the intermediate cycles in the deformations used to establish Pieri's formula.

2. Geometry of Pieri-type intersections

2.1. **Grassmann and Schubert varieties.** Let k be a fixed, but arbitrary, field and $m \leq n$ positive integers. For sets $U \subset W$ let $W - U$ be their set-theoretic difference. Let $V \simeq k^n$ be an n-dimensional vector space over k and $\mathbf{G}_m V$ be the Grassmannian of m-planes in V. A *complete flag* $F_\bullet$ in V is a sequence of subspaces

$$0 = F_{n+1} \subset F_n \subset \cdots \subset F_2 \subset F_1 = V$$

of V where $\dim F_j = n+1-j$. Let $\langle S \rangle$ denote the linear span of a subset S of V. We let $\binom{[n]}{m}$ be the set of all m-element subsets of $[n] := \{1, 2, \ldots, n\}$, considered as decreasing sequences α of length m: $n \geq \alpha_1 > \alpha_2 > \cdots > \alpha_m \geq 1$. A complete flag $F_\bullet$ and a sequence $\alpha \in \binom{[n]}{m}$ together determine a Schubert (sub)variety of $\mathbf{G}_m V$,

$$\Omega_\alpha F_\bullet \; := \; \{H \in \mathbf{G}_m V \mid \dim H \cap F_{\alpha_j} \geq j, \; 1 \leq j \leq m\}.$$

This variety has codimension $|\alpha| := \sum \alpha_i - i$. For example, let $E_\bullet$ be a complete flag in k^{10}. The Schubert subvariety $\Omega_{8531} E_\bullet$ of $\mathbf{G}_4 k^{10}$ is

$$\{H \mid \dim H \cap E_8 \geq 1, \; \dim H \cap E_5 \geq 2, \; \dim H \cap E_3 \geq 3\}.$$

A *special Schubert variety* consists of all m-planes H which intersect a single subspace F_{m+s} in the flag non-trivially, that is, $\Omega_{m+s,m-1,\ldots,2,1} F_\bullet$. We use a compact notation for special Schubert varieties. Let $L := F_{m+s}$, a subspace of dimension $n+1-m-s$, and define

$$\Omega_L \; := \; \Omega_{m+s,m-1,\ldots,2,1} F_\bullet.$$

Two subvarieties meet *generically transversally* if they intersect transversally along a dense subset of every component of their intersection. They meet *improperly* if the codimension of their (non-empty) intersection is less than the sum of their codimensions. A subspace L meets a flag $F_\bullet$ *properly* if it meets each subspace F_i properly.

To simplify some assertions and formulae, we adopt the convention that if γ is a decreasing sequence of length m with $\gamma_1 > n$, then $\Omega_\gamma F_\bullet = \emptyset$. Similarly, if the dimension of a subspace is asserted to be negative, we intend that subspace to be $\{0\}$. Also, $\dim\{0\} = -\infty$. We sometimes make no distinction between a subvariety and its fundamental cycle.

Let $\alpha \in \binom{[n]}{m}$ and r be a positive integer. Define $\alpha * r \subset \binom{[n]}{m}$ to be the set of those $\beta \in \binom{[n]}{m}$ with $\beta_1 \geq \alpha_1 > \beta_2 \geq \cdots > \beta_m \geq \alpha_m$ and $|\beta| = |\alpha| + r$. If $\beta \in \alpha * r$, set $j(\alpha,\beta) := \min\{i \mid \beta_i > \alpha_i\}$, the first index i where β_i differs from α_i. For $1 \leq j \leq m$, let δ^j be the Kronecker delta, the sequence with a 1 in the jth position and 0's elsewhere.

2.2. **The cycle $X_\beta(j, F_\bullet, L)$.** Central to the geometry of Pieri-type intersections are the components, $X_\beta(j, F_\bullet, L)$, of reducible intersections. These subvarieties are also components of cycles intermediate in deformations we use to establish Pieri's formula. Let $\beta \in \binom{[n]}{m}$, $1 \leq j \leq m$ be an integer, $F_\bullet$ a flag, and L a linear subspace in V. Define

$$X_\beta(j, F_\bullet, L) \;:=\; \{H \in \Omega_\beta F_\bullet \mid \dim H \cap F_{\beta_j} \cap L \geq 1\},$$

a subvariety of $\Omega_\beta F_\bullet \bigcap \Omega_L$.

2.3. **Example.** We illustrate this notion in $\mathbf{G}_4 k^{10}$. First note that

$$\Omega_{8631} E_\bullet \;=\; \{H \mid \dim H \cap E_8 \geq 1,\; \dim H \cap E_6 \geq 2,\; \text{and } \dim H \cap E_3 \geq 3\}.$$

Suppose $\Lambda \subset k^{10}$ has codimension $5 = 4 + 2 - 1$ (hence dimension 5) so that Ω_Λ has codimension 2 in $\mathbf{G}_4 k^{10}$. Then

$$X_{8631}(2, E_\bullet, \Lambda) \;=\; \{H \in \Omega_{8631} E_\bullet \mid \dim H \cap E_6 \cap \Lambda \geq 1\}.$$

This has dimension $0, 13, 14, 15$, or 16 depending upon whether $\dim E_6 \cap \Lambda$ is $0, 1, 2, 3$, or ≥ 4. (This is determined by considering the condition that a 2-dimensional subspace ('$H \cap E_6$') of E_6 meet $\Lambda \cap E_6$.) Since the expected dimension of $\Omega_{8631} E_\bullet \bigcap \Omega_\Lambda$ is 14, $X_{8631}(2, E_\bullet, \Lambda)$ is a proper subvariety of $\Omega_{8631} E_\bullet \bigcap \Omega_\Lambda$ if $\dim E_6 \cap \Lambda \leq 1$ and $\Omega_{8631} E_\bullet \bigcap \Omega_\Lambda$ has excess intersection if $\dim E_6 \cap \Lambda \geq 3$.

The following theorem generalizes these observations, giving precise conditions on L and $F_\bullet$ which determine whether $\Omega_\alpha F_\bullet \bigcap \Omega_L$ is improper, generically transverse, or irreducible. Moreover, it computes the components of the intersection in the crucial case of a generically transverse intersection with the maximal number of irreducible components.

2.4. **Theorem.** *Let $\alpha \in \binom{[n]}{m}$, $s > 0$, $F_\bullet$ be a complete flag in V, and $L \in \mathbf{G}_{n+1-m-s} V$.*

(1) *If, for some $1 \leq j \leq m$, $\dim F_{\alpha_j} \cap L > n + 2 - \alpha_j - j - s$ and $F_{\alpha_j} \cap L \neq \{0\}$, then $\Omega_\alpha F_\bullet \bigcap \Omega_L$ is improper. Otherwise, it is generically transverse.*

(2) *Suppose $\dim F_{\alpha_j} \cap L = n + 2 - \alpha_j - j - s$ for each $1 \leq j \leq m$. Let $M_\bullet$ be any flag satisfying $M_{\alpha_j} = F_{\alpha_j}$ and $M_{\alpha_j+1} \supset \langle F_{\alpha_{j-1}}, F_{\alpha_j} \cap L\rangle$, for $1 \leq j \leq m$. Then $\Omega_\alpha F_\bullet$ meets Ω_L generically transversally, and*

$$\Omega_\alpha F_\bullet \bigcap \Omega_L \;=\; \sum_{\beta \in \alpha * 1} X_\beta(j(\alpha,\beta), M_\bullet, L).$$

(3) *Suppose* $\dim F_{\alpha_j} \cap L < n+2-\alpha_j-j-s$ *for each* $1 \leq j < m$ *and* F_{α_m} *meets* L *properly, so that* $\dim F_{\alpha_m} \cap L = n+2-\alpha_m-m-s$. *Then* $\Omega_\alpha F_\bullet \bigcap \Omega_L$ *is irreducible.*

Note that $n+2-\alpha_j-j-s$, the critical dimension for $F_{\alpha_j} \cap L$ in this theorem, exceeds the expected dimension of $n+2-\alpha_j-m-s$ by $m-j$. Thus, it is not necessary that $F_\bullet$ and L meet properly for $\Omega_\alpha F_\bullet \bigcap \Omega_L$ to be generically transverse or even irreducible. However, it is necessary that F_{α_m} and L meet properly. Also, as the relative position of $F_\bullet$ and L becomes more degenerate, the intersection $\Omega_\alpha F_\bullet \bigcap \Omega_L$ 'branches' into components, one for each j such that $\dim F_{\alpha_j} \cap L = n+2-\alpha_j-j-s$, and it will attain excess intersection if $\dim F_{\alpha_j} \cap L > n+2-\alpha_j-j-s$, for even one j.

2.5. **Remark.** In the situation of Theorem 2.4(2), if $\beta \in \alpha * 1$ and $j(\alpha,\beta)=1$, then $\beta = \alpha+\delta^1$. Suppose further that $M_{\alpha_1} \cap L = M_{\alpha_1+s}$. Then

$$X_\beta(1, M_\bullet, L) \;=\; \Omega_{\alpha+s\delta^1} M_\bullet \;=\; \Omega_{\beta+(s-1)\delta^1} M_\bullet,$$

so we have

$$\Omega_\alpha F_\bullet \bigcap \Omega_L \;=\; \sum_{\substack{\beta\in\alpha*1\\ j(\alpha,\beta)=1}} \Omega_{\beta+(s-1)\delta^1} M_\bullet \;+\; \sum_{\substack{\beta\in\alpha*1\\ j(\alpha,\beta)>1}} X_\beta(j(\alpha,\beta), M_\bullet, L).$$

We prove Theorem 2.4 in §2.11. First, we study the varieties $X_\beta(j, F_\bullet, L)$. Let $\beta \in \binom{[n]}{m}$, $F_\bullet$ be a complete flag, and $1 \leq j \leq m$ an integer. The rational map from $\Omega_\beta F_\bullet$ to $\mathbf{G}_j F_{\beta_j}$ given by $H \longmapsto H \cap F_{\beta_j}$ is defined on the dense locus in $\Omega_\beta F_\bullet$ of those H where $\dim H \cap F_{\beta_j} = j$. The closure of the graph of this map is the variety

$$\widetilde{\Omega}^j_\beta F_\bullet \;:=\; \{(H,K) \in \Omega_\beta F_\bullet \times \mathbf{G}_j F_{\beta_j} \mid K \subset H \;\text{ and }\; \dim K \cap F_{\beta_i} \geq i,\; 1 \leq i \leq j\}.$$

In Lemma 2.7, we show that the projection to $\mathbf{G}_j F_{\beta_j}$ realizes $\widetilde{\Omega}^j_\beta F_\bullet$ as a fibre bundle with base and fibres themselves Schubert varieties. Let p be the projection to $\Omega_\beta F_\bullet$ and π the projection to $\mathbf{G}_j F_{\beta_j}$. For $K \subset V$, let $F_\bullet/K$ be the image of the flag $F_\bullet$ in V/K. Let $F_\bullet|_{\beta_j}$ be the flag

$$F_n \subset \cdots \subset F_{\beta_j+1} \subset F_{\beta_j}$$

and $\beta|_j \in \binom{[n+1-\beta_j]}{j}$ the sequence

$$\beta_1-\beta_j+1 > \cdots > \beta_{j-1}-\beta_j+1 > 1 = (\beta|_j)_j.$$

Unraveling this definition shows $(F_\bullet|_{\beta_j})_{(\beta|_j)_i} = F_{\beta_i}$, for $i \leq j$.

2.6. **Example.** Let $(H,K) \in \widetilde{\Omega}^2_{8631} E_\bullet$. Then $\dim H \cap E_3 \geq 3$, $K \subset H \cap E_6$ has dimension 2, and $\dim K \cap E_8 \geq 1$. If $\dim H \cap E_6 = 2$, so H is in the 'big cell' of $\Omega_{8631} E_\bullet$, then $K = H \cap E_6$ and H determines K uniquely. Also, any $K \in \mathbf{G}_2 E_6$ such that $\dim K \cap E_8 \geq 1$ may arise in this way, which shows

$$\pi\left(\widetilde{\Omega}^2_{8631} E_\bullet\right) \;=\; \Omega_{86} E_\bullet \;=\; \Omega_{31}(E_{10} \subset \cdots \subset E_6) \;=\; \Omega_{8631|_2} E_\bullet|_6.$$

We also see that

$$\frac{H}{K} \subset \frac{E_1}{K} \quad\text{and}\quad \dim\left(\frac{H}{K} \bigcap \frac{E_3}{K}\right) \;\geq\; 1,$$

which shows $H/K \in \Omega_{31}(E_\bullet/K)$.

2.7. **Lemma.** *Let $\beta \in \binom{[n]}{m}$, $F_\bullet$ be a flag, and $1 \le j \le m$. Then p is an isomorphism over the dense subset $\{H \in \Omega_\beta F_\bullet \mid \dim H \cap F_{\beta_j} = j\}$. Also, π exhibits $\widetilde{\Omega}^j_\beta F_\bullet$ as a fibre bundle with base $\Omega_{\beta|_j} F_\bullet|_{\beta_j}$ whose fibre over $K \in \Omega_{\beta|_j} F_\bullet|_{\beta_j}$ is the Schubert variety $\Omega_{\beta_{j+1}\cdots\beta_m} F_\bullet/K \subset \mathbf{G}_{m-j}V/K$. Moreover, each fibre of π meets the locus where p is an isomorphism.*

Proof: We describe the fibres of π. Note that Schubert varieties have a dual description:

$$H \in \Omega_\beta F_\bullet \iff \dim \frac{H}{H \cap F_{\beta_i}} \le m - i, \quad \text{for } 1 \le i \le m.$$

If $K \in \Omega_{\beta|_j} F_\bullet|_{\beta_j}$, then $K \subset F_{\beta_j} \subset F_{\beta_i}$, for $i > j$. Thus $(F_\bullet/K)_{\beta_i} = F_{\beta_i}/K$, for $i > j$. Hence, if H is in the fibre over K, then $H \in \Omega_\beta F_\bullet$ and $K \subset H$, so

$$\dim \frac{H/K}{H/K \cap (F_\bullet/K)_{\beta_i}} = \dim \frac{H}{H \cap F_{\beta_i}} \le m - i, \quad \text{for } j < i \le m.$$

Thus $H/K \in \Omega_{\beta_{j+1}\cdots\beta_m} F_\bullet/K$. The reverse implication is similar and the remaining assertions follow easily from the definitions. ∎

Reformulating the definition of $X_\beta(j, F_\bullet, L)$ in these terms gives a useful characterization:

2.8. **Corollary.** $X_\beta(j, F_\bullet, L) = p(\pi^{-1}(\Omega_{\beta|_j} F_\bullet|_{\beta_j} \bigcap \Omega_{F_{\beta_j} \cap L}))$.

Since the fibres of π meet the locus where p is an isomorphism, the map

$$p \;:\; \pi^{-1}(\Omega_{\beta|_j} F_\bullet|_{\beta_j} \bigcap \Omega_{F_{\beta_j} \cap L}) \longrightarrow X_\beta(j, F_\bullet, L)$$

is proper and birational. Thus, while $X_\beta(j, F_\bullet, L)$ is neither a Schubert variety nor an intersection of Schubert varieties, it is 'birationally fibred' over an intersection of Schubert varieties with Schubert variety fibres, and hence is intermediate between these extremes.

2.9. **Tangent spaces to Schubert varieties.** Let $H \in \mathbf{G}_m V$ and $K \in \mathbf{G}_{n-m} V$ be complementary subspaces, so $H \cap K = \{0\}$. The open set $U \subset \mathbf{G}_m V$ of those H' with $H' \cap K = \{0\}$ is identified with $\mathrm{Hom}(H, K)$ by $\phi \in \mathrm{Hom}(H, K) \mapsto \Gamma_\phi$, the graph of ϕ in $H \oplus K = V$. Thus we identify $T_H \mathbf{G}_m V$, the tangent space of $\mathbf{G}_m V$ at H, with $\mathrm{Hom}(H, V/H)$, as K is canonically isomorphic to V/H. The intersection of a Schubert variety $\Omega_\alpha F_\bullet$ containing H with this open set U can be used to determine whether $\Omega_\alpha F_\bullet$ is smooth at H and its tangent space at H. This gives the following description: If $H \in \mathbf{G}_m V$ and $\dim H \cap F_{\alpha_j} = j$ for $1 \le j \le m$, then $\Omega_\alpha F_\bullet$ is smooth at H and

$$T_H \Omega_\alpha F_\bullet = \{\phi \in \mathrm{Hom}(H, V/H) \mid \phi(H \cap F_{\alpha_j}) \subset (F_{\alpha_j} + H)/H, \ 1 \le j \le m\}.$$

Similarly, if $H \in \mathbf{G}_m V$, $L \in \mathbf{G}_{n+1-m-s} V$, and $\dim H \cap L = 1$, then Ω_L is smooth at H and the tangent space of Ω_L at H is

$$T_H \Omega_L = \{\phi \in \mathrm{Hom}(H, V/H) \mid \phi(H \cap L) \subset (L + H)/H\}.$$

Let P be the subgroup of $GL(V)$ stabilizing the partial flag $F_{\alpha_1} \subset F_{\alpha_2} \subset \cdots \subset F_{\alpha_m}$. The orbit $P \cdot L'$ consists of those L with $\dim F_{\alpha_j} \cap L = \dim F_{\alpha_j} \cap L'$ for $1 \leq j \leq m$. Similarly, $L \in \overline{P \cdot L'}$ if $\dim F_{\alpha_j} \cap L \geq \dim F_{\alpha_j} \cap L'$ for $1 \leq j \leq m$. If $P \cdot L = P \cdot L'$, then $\Omega_\alpha F_\bullet \bigcap \Omega_L \simeq \Omega_\alpha F_\bullet \bigcap \Omega_{L'}$. Thus P-orbits on $\mathbf{G}_{n+1-m-s}V$ determine the isomorphism type of Pieri-type intersections.

2.10. **Lemma.** *Suppose that $L, L' \in \mathbf{G}_{n+1-m-s}V$ with $L \in \overline{P \cdot L'}$. Then*

(1) $\dim \Omega_\alpha F_\bullet \bigcap \Omega_L \geq \dim \Omega_\alpha F_\bullet \bigcap \Omega_{L'}$.

(2) *If $\Omega_\alpha F_\bullet \bigcap \Omega_L$ is generically transverse, then $\Omega_\alpha F_\bullet \bigcap \Omega_{L'}$ is generically transverse.*

(3) *If $\Omega_\alpha F_\bullet \bigcap \Omega_L$ is generically transverse and irreducible, then $\Omega_\alpha F_\bullet \bigcap \Omega_{L'}$ is generically transverse and irreducible.*

Proof: Let $\psi : \mathbf{P}^1 \to \overline{P \cdot L'}$ be a map with $\psi(0) = L$ and $\psi(\mathbf{P}^1) \cap (P \cdot L') \neq \emptyset$. Then $\Omega_\alpha F_\bullet \bigcap \Omega_{\psi(t)}$ is isomorphic to $\Omega_\alpha F_\bullet \bigcap \Omega_{L'}$, for any $t \in \psi^{-1}(P \cdot L')$. The lemma follows by considering the subvariety of $\mathbf{P}^1 \times \mathbf{G}_m V$ whose fibre over $t \in \mathbf{P}^1$ is $\Omega_\alpha F_\bullet \bigcap \Omega_{\psi(t)}$. ∎

2.11. **Proof of Theorem 2.4:** Let $\alpha \in \binom{[n]}{m}$, $s > 0$, $F_\bullet$ be a complete flag, and $L \in \mathbf{G}_{n+1-m-s}V$. The conditions on L in statement (2), that $\dim F_{\alpha_j} \cap L = n+2-\alpha_j-j-s$ for each j, determine a P-orbit, which is the closure of any P-orbit $P \cdot L'$, where $\dim F_{\alpha_j} \cap L' \leq n+2-\alpha_j-j-s$ for each j. Thus (2) and Lemma 2.10(2) together imply that if $\dim F_{\alpha_j} \cap L \leq n+2-\alpha_j-j-s$ for each j, then $\Omega_\alpha F_\bullet \bigcap \Omega_L$ is generically transverse, proving the second part of (1).

For the first part of (1), suppose $\dim F_{\alpha_j} \cap L > n+2-\alpha_j-j-s$ and let $L' := F_{\alpha_j} \cap L \neq \{0\}$. Then L' has codimension at most $j+s-1$ in F_{α_j}. Hence $\Omega_{\alpha|_j} F_\bullet|_{\alpha_j} \bigcap \Omega_{L'} \neq \emptyset$ and so has codimension in $\Omega_{\alpha|_j} F_\bullet|_{\alpha_j}$ at most that of $\Omega_{L'}$ in $\mathbf{G}_j F_{\alpha_j}$, which is at most $s-1$. Thus

$$X_\alpha(j, F_\bullet, L) \;=\; p(\pi^{-1}(\Omega_{\alpha|_j} F_\bullet|_{\alpha_j} \bigcap \Omega_{L'}))$$

which has codimension less than s in $\Omega_\alpha F_\bullet = p(\pi^{-1}(\Omega_{\alpha|_j} F_\bullet|_{\alpha_j}))$. Hence $\Omega_\alpha F_\bullet \bigcap \Omega_L$ is improper, as $X_\alpha(j, F_\bullet, L) \subset \Omega_\alpha F_\bullet \bigcap \Omega_L$, proving (1).

We make a computation before proceeding with the rest of the proof. Suppose $\dim F_{\alpha_j} \cap L \leq n+2-\alpha_j-j-s$ for $1 \leq j \leq m$ and $F_{\alpha_m} \cap L \not\subset F_{\alpha_{m-1}}$. Then there exists $H \in \Omega_\alpha F_\bullet \bigcap \Omega_L$ with $\dim H \cap F_{\alpha_j} = j$ for $1 \leq j \leq m$, $\dim H \cap L = 1$, and $H \cap L \not\subset F_{\alpha_{m-1}}$: Inductively choose linearly independent vectors $f_j \in F_{\alpha_j}$ for $1 \leq j \leq m$ as follows. Let $f_1 \in F_{\alpha_1} - \{0\}$. Then for $1 < j < m$ suppose that $f_1, \ldots, f_{j-1}$ have been chosen. Since

$$\dim F_{\alpha_j} \cap \langle L, f_1, \ldots, f_{j-1}\rangle \;\leq\; n+2-\alpha_j-j-s+(j-1) \;<\; \dim F_{\alpha_j},$$

we can select a vector f_j in

$$F_{\alpha_j} - F_{\alpha_j} \bigcap \langle L, f_1, \ldots, f_{j-1}\rangle - F_{\alpha_{j-1}}.$$

Let $f_m \in F_m \cap L - F_{\alpha_{m-1}}$, and set $H := \langle f_1, \ldots, f_m\rangle$. Then $H \in \Omega_\alpha F_\bullet \bigcap \Omega_L$, $\dim H \cap F_{\alpha_j} = j$ for $1 \leq j \leq m$, $\dim H \cap L = 1$, and $H \cap L \not\subset F_{\alpha_{m-1}}$. Let X_m° be the set of all such H. For $H \in X_m^\circ$,

$$T_H \Omega_\alpha F_\bullet \bigcap T_H \Omega_L \;=\; \{\phi \in T_H \Omega_\alpha F_\bullet \mid \phi(H \cap L) \subset (F_{\alpha_m} \cap L + H)/H\}.$$

This has codimension in $T_H\Omega_\alpha F_\bullet$ equal to $\dim(F_{\alpha_m}+H)-\dim(F_{\alpha_m}\cap L+H)=s$. Thus $\Omega_\alpha F_\bullet$ and Ω_L meet transversally along X_m°.

We show (2). Suppose $\dim F_{\alpha_j}\cap L = n+2-\alpha_j-s$ for each $1\leq j\leq m$. Let $M_\bullet$ be any flag satisfying

$$M_{\alpha_j}=F_{\alpha_j} \qquad \text{and} \qquad M_{\alpha_j+1}\supset\langle F_{\alpha_{j-1}},\, F_{\alpha_j}\cap L\rangle, \quad j=1,\ldots,m.$$

Let $H\in\Omega_\alpha F_\bullet\bigcap\Omega_L$. Then there is some $1\leq j\leq m$ with $H\cap L\cap F_{\alpha_j}\not\subset F_{\alpha_{j-1}}$. Since $\dim H\cap F_{\alpha_{j-1}}\geq j-1$, we have $\dim H\cap\langle F_{\alpha_{j-1}},F_{\alpha_j}\cap L\rangle\geq j$ and so $\dim H\cap M_{\alpha_j+1}\geq j$. Thus $H\in\Omega_{\alpha+\delta^j}M_\bullet$ if $\alpha+\delta^j\in\binom{[n]}{m}$. But this is the case, as $\alpha_j+1<\alpha_{j-1}$, for otherwise dimensional considerations imply that $L\cap F_{\alpha_j}=L\cap F_{\alpha_{j-1}}\subset F_{\alpha_{j-1}}$. Let $\beta:=\alpha+\delta^j\in\alpha*1$. Then $j(\alpha,\beta)=j$ and $H\in X_\beta(j(\alpha,\beta),M_\bullet,L)$, since $H\in\Omega_\beta M_\bullet$ and $\dim H\cap L\cap M_{\beta_j}\geq 1$. Conversely, if $\beta\in\alpha*1$, then $\Omega_\beta M_\bullet\subset\Omega_\alpha F_\bullet$, so $X_\beta(j(\alpha,\beta),M_\bullet,L)\subset\Omega_\alpha F_\bullet\bigcap\Omega_L$. This shows

$$\Omega_\alpha F_\bullet\bigcap\Omega_L \;=\; \sum_{\beta\in\alpha*1}X_\beta(j(\alpha,\beta),M_\bullet,L).$$

We claim this intersection is generically transverse. Let $\beta\in\alpha*1$ and $j:=j(\alpha,\beta)$. Then $X_\beta(j,M_\bullet,L)$ has an open subset X_j° consisting of those H with $\dim H\cap F_{\alpha_i}=i$ for $1\leq i\leq m$, $\dim H\cap L=1$, and $H\cap L\subset F_{\alpha_j}$ but $H\cap L\not\subset F_{\alpha_j-1}$. As with X_m° above, X_j° is nonempty, so it is a dense open subset of $X_\beta(j,M_\bullet,L)$. For $H\in X_j^\circ$,

$$T_H\Omega_\alpha F_\bullet\bigcap T_H\Omega_L \;=\; \{\phi\in T_H\Omega_\alpha F_\bullet\mid\phi(H\cap L)\subset(L\cap F_{\alpha_j}+H)/H\}.$$

Since $\dim(F_{\alpha_j}+H)-\dim(L\cap F_{\alpha_j}+H)=s$, this has codimension s in $T_H\Omega_\alpha F_\bullet$, showing that $\Omega_\alpha F_\bullet$ and Ω_L meet transversally along X_j°, a dense subset of $X_\beta(j(\alpha,\beta),M_\bullet,L)$.

By Lemma 2.10(3), it suffices to prove a special case of (3):

(3)′ *If F_{α_m} meets L properly, and for $1\leq j<m$, $\dim F_{\alpha_j}\cap L=n+2-\alpha_j-j-(s+1)$, then $\Omega_\alpha F_\bullet\bigcap\Omega_L$ is irreducible.*

These conditions imply $F_{\alpha_m}\cap L\not\subset F_{\alpha_{m-1}}$. In the notation of §2.5, let $L':=F_{\alpha_{m-1}}\cap L$, $F_\bullet':=F_\bullet|_{\alpha_{m-1}}$, and $\alpha':=\alpha|_{m-1}$. Consider

$$X_\alpha(m-1,F_\bullet,L) \;=\; p(\pi^{-1}(\Omega_{\alpha'}F_\bullet'\bigcap\Omega_{L'})).$$

For $j\leq m-1$,

$$\begin{aligned}\dim F_{\alpha_j}\cap L' &= n+2-\alpha_j-j-(s+1)\\ &= \dim F_{\alpha_{m-1}}+2-\alpha_j'-j-(s+1),\end{aligned}$$

so L' and $F_\bullet'$ satisfy the conditions of (2) for the pair $\alpha', s+1$. Thus $\Omega_{\alpha'}F_\bullet'\bigcap\Omega_{L'}$ is generically transverse, which implies that $X_\alpha(m-1,F_\bullet,L)$ has codimension $s+1$ in $\Omega_\alpha F_\bullet$ and hence is a proper subvariety of $\Omega_\alpha F_\bullet\bigcap\Omega_L-X_\alpha(m-1,F_\bullet,L)$. Since X_m° is dense in $\Omega_\alpha F_\bullet\bigcap\Omega_L-X_\alpha(m-1,F_\bullet,L)$, this establishes (3)′. ∎

3. Construction of explicit rational equivalences

Theorem 2.4 shows that for L in a dense subset of $\mathbf{G}_{n+1-m-b}V$, the intersection $\Omega_\alpha F_\bullet \bigcap \Omega_L$ is generically transverse and irreducible. We use Theorem 2.4(2) to study such a cycle as L 'moves out of' this set, ultimately deforming it into the cycle $\sum_{\gamma\in\alpha*b}\Omega_\gamma F_\bullet$.

3.1. **Families and Chow varieties.** Suppose $\Sigma \subset (\mathbf{P}^1-\{0\})\times \mathbf{G}_m V$ has equidimensional fibres over $\mathbf{P}^1-\{0\}$. Then its Zariski closure $\overline{\Sigma}$ in $\mathbf{P}^1\times\mathbf{G}_m V$ has equidimensional fibres over $\mathbf{P}^1$. Denote the fibre of $\overline{\Sigma}$ over 0 by $\lim_{t\to 0}\Sigma_t$, where Σ_t is the fibre of Σ over $t\in\mathbf{P}^1-\{0\}$. The association of a point t of $\mathbf{P}^1$ to the fundamental cycle of the fibre $\overline{\Sigma}_t$ determines a morphism $\mathbf{P}^1\to \mathit{Chow}\,\mathbf{G}_m V$. Moreover, if Σ is defined over k, then so is the map $\mathbf{P}^1\to\mathit{Chow}\,\mathbf{G}_m V$ ([20], §I.9).

3.2. **The cycle $Y_{\alpha,r}(F_\bullet, L)$.** In §2.2, we defined the components $X_\beta(j,F_\bullet,L)$ of the cycles intermediate between $\Omega_\alpha F_\bullet\bigcap\Omega_L$ and $\sum_{\gamma\in\alpha*b}\Omega_\gamma F_\bullet$. Here, we define the intermediate cycles, $Y_{\alpha,r}(F_\bullet,L)$, which are parameterized by subspaces L in certain Schubert cells $U_{\alpha,s}F_\bullet$ of $\mathbf{G}_{n+1-m-s}V$. Let $U_{\alpha,s}F_\bullet$ be the set of those $L\in\mathbf{G}_{n+1-m-s}V$ such that

(1) $F_{\alpha_1}\cap L = F_{\alpha_1+s}$, and
(2) $F_{\alpha_j}\cap L = F_{\alpha_j+1}\cap L$, and has dimension $n+2-\alpha_j-j-s$, for $1\le j\le m$.

These conditions are consistent and determine $\dim F_i\cap L$ for $1\le i\le n$. For example,

$$\alpha_j < i < \alpha_{j-1} \implies \dim F_i\cap L \;=\; \dim F_i + 1 - j - (s-1). \tag{3.2}$$

Thus $U_{\alpha,s}F_\bullet$ is a single Schubert cell of $\mathbf{G}_{n+1-m-s}V$. Specifically, $U_{\alpha,s}F_\bullet$ is the dense cell of $\Omega_\beta F_\bullet$, where $\beta\in\binom{[n]}{n+1-m-s}$ is defined as follows: If $\alpha_1\le n+1-s$, then $\beta=[n]-\alpha-\{\alpha_1+1,\ldots,\alpha_1+s-1\}$. Otherwise, β is the smallest $n+1-m-s$ integers in $[n]-\alpha$.

For $\beta\in\alpha*r$, recall that $j(\alpha,\beta)=\min\{i\mid\alpha_i<\beta_i\}$. If $L\in U_{\alpha,s}F_\bullet$, define the cycle

$$Y_{\alpha,r}(F_\bullet,L) \;:=\; \sum_{\substack{\beta\in\alpha*r\\ j(\alpha,\beta)=1}}\Omega_{\beta+(s-1)\delta^1}F_\bullet \;+\; \sum_{\substack{\beta\in\alpha*r\\ j(\alpha,\beta)>1}} X_\beta(j(\alpha,\beta),F_\bullet,L).$$

Let $\mathcal{G}_{\alpha,s,r}F_\bullet\subset\mathit{Chow}\,\mathbf{G}_m V$ be the set of these cycles $Y_{\alpha,r}(F_\bullet,L)$ for $L\in U_{\alpha,s}F_\bullet$. Since $U_{\alpha,s}F_\bullet$ is a Schubert cell, $\mathcal{G}_{\alpha,s,r}F_\bullet$ is an orbit of the Borel subgroup stabilizing $F_\bullet$ and hence is rational.

3.3. **Example.** The cell $U_{8531,\,2}E_\bullet\subset\mathbf{G}_5k^{10}$ consists of those Λ with

(1) $E_{10}\subset\Lambda$, so $E_8\cap\Lambda=E_9\cap\Lambda$ has dimension 1,
(2) $E_5\cap\Lambda=E_6\cap\Lambda$ has dimension 3,
(3) $E_3\cap\Lambda=E_4\cap\Lambda$ has dimension 4, and
(4) $\Lambda\subset E_2$.

In this case, the sequence $(\dim(E_j\cap\Lambda))_j$ is $(5\underline{5}4\underline{4}3\underline{3}21\underline{1}1)$. Hence, for $\Lambda\in U_{8531,\,2}E_\bullet$,

$$\begin{aligned} Y_{8531,\,1}(E_\bullet,\Lambda) &= \Omega_{10\,531}E_\bullet + X_{8631}(2,E_\bullet,\Lambda)+X_{8541}(3,E_\bullet,\Lambda)+X_{8532}(4,E_\bullet,\Lambda)\\ &= \Omega_{8531}E_\bullet\bigcap\Omega_\Lambda. \end{aligned}$$

The second line is a consequence of Remark 2.5. To see the first, suppose $H \in \Omega_{8531}E_\bullet \bigcap \Omega_\Lambda$, then $H \cap \Lambda$ meets a unique largest of $E_8 \subset E_5 \subset E_3 \subset E_1$, which gives four cases:

(1) $H \cap \Lambda \subset E_8$. Hence $E_{10} \subset H$ so $H \in \Omega_{10\,531}E_\bullet$.
(2) $H \cap \Lambda$ meets $E_5 - E_8$. Thus $H \cap \Lambda$ meets $E_6 - E_8$, so $\dim H \cap E_6 \geq 2$ and $H \cap E_6$ meets Λ, hence $H \in X_{8631}(2, E_\bullet, \Lambda)$.
(3) $H \cap \Lambda$ meets $E_3 - E_5$. Thus $H \cap \Lambda$ meets $E_4 - E_5$, so $\dim H \cap E_4 \geq 3$ and $H \cap E_4$ meets Λ, hence $H \in X_{8541}(3, E_\bullet, \Lambda)$.
(4) $H \cap \Lambda$ meets $E_1 - E_3$. Thus $H \cap \Lambda$ meets $E_2 - E_3$, so $H \subset E_2$ hence $H \in X_{8532}(4, E_\bullet, \Lambda)$.

3.4. **Remark.** Suppose $L \in U_{\alpha,s}F_\bullet$, then by Remark 2.5,

$$\begin{aligned}\Omega_\alpha F_\bullet \bigcap \Omega_L &= \sum_{\substack{\beta\in\alpha*1\\ j(\alpha,\beta)=1}} \Omega_{\beta+(s-1)\delta^1}F_\bullet + \sum_{\substack{\beta\in\alpha*1\\ j(\alpha,\beta)>1}} X_\beta(j(\alpha,\beta), F_\bullet, L)\\ &= Y_{\alpha,1}(F_\bullet, L).\end{aligned}$$

The following lemma parameterizes our explicit rational equivalences. It is identical to Lemma 6.1 of [23].

3.5. **Lemma.** *Let $l \leq n$ and let $M_\bullet$ be a complete flag in $M \simeq k^a$. Suppose L_∞ is a hyperplane containing M_l but not M_{l-1}. Then there exists a pencil of hyperplanes L_t, for $t \in \mathbf{P}^1$, such that if $t \neq 0$, then L_t contains M_l but not M_{l-1} and, for each $i \leq l-1$, the family of codimension $i+1$ planes induced by $M_i \cap L_t$ for $t \neq 0$ has fibre M_{i+1} over 0.*

Proof: Let $e_1, \ldots, e_a$ be a basis of M such that $M_i := \langle e_i, \ldots, e_a\rangle$ and $L_\infty = \langle e_1, \ldots, e_{l-2}, M_l\rangle$. Define

$$L_t := \langle M_l, te_j + e_{j+1} \,|\, 1 \leq j \leq l-2\rangle.$$

For $t \neq 0$ and $1 \leq i \leq l-1$, $M_i \cap L_t = \langle M_l, te_j + e_{j+1} \,|\, i \leq j \leq l-2\rangle$ and so has dimension $n-i$. The fibre of this family at $t = 0$ is $\langle M_l, e_{j+1} \,|\, i \leq j \leq l-2\rangle = M_{i+1}$.
∎

In §3.10, we prove the following theorem.

3.6. **Theorem.** *Let $\alpha \in \binom{[n]}{m}$, s,r be positive integers and $F_\bullet$ a flag in V. Let $M \in U_{\alpha,s-1}F_\bullet$ and define $M_\bullet$ to be the flag in M consisting of the subspaces in $F_\bullet \cap M$.*

Let $L_\infty \subset M$ be any hyperplane containing F_{α_1+s} but not F_{α_1+s-1}. Suppose L_t is the family of hyperplanes of M given by Lemma 3.5. Then

(1) *For $t \neq 0$, $L_t \in U_{\alpha,s}F_\bullet$.*
(2) $\lim_{t\to 0} Y_{\alpha,r}(F_\bullet, L_t) = Y_{\alpha,r+1}(F_\bullet, M)$.

In the invocation of Lemma 3.5 in this theorem, we have $a = n+2-m-s$ and $l = \alpha_1 - m + 2$, so that $M_l = F_{\alpha_1+s}$.

3.7. **Example.** Let $e_1,\ldots,e_{10}$ be a basis for k^{10} and suppose $E_j = \langle e_j,\ldots,e_{10}\rangle$. Then let

$$M \;:=\; \langle e_2,e_4,e_6,e_7,e_9,e_{10}\rangle \;\in\; U_{8531,\,1}E_{\bullet}.$$

Set $\Lambda_\infty := \langle e_2,e_4,e_6,e_7,e_{10}\rangle$, and, for $t\in\mathbf{P}^1$, define

$$\Lambda_t := \langle te_2+e_4, te_4+e_6, te_6+e_7, te_7+e_9, e_{10}\rangle.$$

For $t\neq 0$, $\Lambda_t\in U_{8531,\,2}E_{\bullet}$. We compute $\lim_{t\to 0} Y_{8531,\,1}(E_{\bullet},\Lambda_t)$, which is

$$\Omega_{10\,531}E_{\bullet} + \lim_{t\to 0} X_{8631}(2,E_{\bullet},\Lambda_t) + \lim_{t\to 0} X_{8541}(3,E_{\bullet},\Lambda_t) + \lim_{t\to 0} X_{8532}(4,E_{\bullet},\Lambda_t).$$

For $t\neq 0$, consider the component

$$X_{8631}(2,E_{\bullet},\Lambda_t) \;=\; \{H\in\Omega_{8631}E_{\bullet} \mid \dim H\cap E_6\cap\Lambda_t\geq 1\}.$$

When $t\neq 0$, $\{K\in\Omega_{68}E_{\bullet} \mid \dim K\cap\Lambda_t\geq 1\}$ is irreducible. To describe this as $t\to 0$, let $\lambda := \lim_{t\to 0}(\Lambda_t\cap E_6) = \langle e_7,e_9,e_{10}\rangle$. Then $\{K\in\Omega_{68}E_{\bullet} \mid \dim K\cap\lambda\geq 1\}$ has two components:

$$\{K\subset\langle\lambda,E_8\rangle = E_7\} = \Omega_{87}E_{\bullet} \quad\text{and}\quad \{K\mid K\supset\lambda\cap E_8 = E_9\} = \Omega_{96}E_{\bullet}.$$

Thus, since

$$X_{8631}(2,E_{\bullet},\Lambda_t) = p(\pi^{-1}(\{K\in\Omega_{68}E_{\bullet} \mid \dim K\cap\Lambda_t\geq 1\})),$$

we see that

$$\lim_{t\to 0} X_{8631}(2,E_{\bullet},\Lambda_t) \;=\; \Omega_{8731}E_{\bullet} + \Omega_{9631}E_{\bullet}.$$

Similarly,

$$\begin{aligned}
\lim_{t\to 0} X_{8541}(3,E_{\bullet},\Lambda_t) \;&=\; \Omega_{8641}E_{\bullet} + \Omega_{9541}E_{\bullet}\\
\lim_{t\to 0} X_{8532}(4,E_{\bullet},\Lambda_t) \;&=\; \Omega_{8542}E_{\bullet} + \Omega_{8632}E_{\bullet} + \Omega_{9532}E_{\bullet}.
\end{aligned}$$

These Schubert varieties, plus $\Omega_{10\,531}E_{\bullet}$, are the summands of $Y_{8531,\,2}(E_{\bullet},M) = \sum_{\gamma\in 8531*2}\Omega_\gamma E_{\bullet}$, which proves

$$\lim_{t\to 0} Y_{8531,\,1}(E_{\bullet},\Lambda_t) \;=\; \sum_{\gamma\in 8531*2}\Omega_\gamma E_{\bullet}.$$

Since $Y_{8531,\,1}(E_{\bullet},\Lambda_t) = \Omega_{8531}E_{\bullet}\bigcap\Omega_{\Lambda_t}$, this proves this instance of Pieri's formula.

3.8. **Theorem. [Pieri's Formula]** *Let $\alpha\in\binom{[n]}{m}$, $F_{\bullet}$ be a complete flag in V, and $K\in\mathbf{G}_{n+1-m-b}V$ be a subspace which meets $F_{\bullet}$ properly. Then the cycle $\Omega_\alpha F_{\bullet}\bigcap\Omega_K$, a generically transverse intersection, is rationally equivalent to $\sum_{\gamma\in\alpha*b}\Omega_\gamma F_{\bullet}$. Thus, in the Chow ring $A^*\mathbf{G}_mV$ of $\mathbf{G}_mV$,*

$$[\Omega_\alpha F_{\bullet}]\cdot[\Omega_K] \;=\; \sum_{\gamma\in\alpha*b}[\Omega_\gamma F_{\bullet}].$$

Moreover, let $\mathcal{G}\subset Chow\,\mathbf{G}_mV$ be the set of cycles arising as generically transverse intersections of the form $\Omega_\alpha F_{\bullet}\bigcap\Omega_K$ for $K\in\mathbf{G}_{n+1-m-b}V$. Then one may give $b+1$ explicit rational deformations inducing this rational equivalence, where the cycles at the ith stage are of the form $Y_{\alpha,i}(F_{\bullet},M)$, with $M\in U_{\alpha,b+1-i}F_{\bullet}$, and all are within $\overline{\mathcal{G}}$.

Hodge [9] also described deformations of $\Omega_\alpha F_{\bullet}\bigcap\Omega_K$ into a sum of distinct Schubert cycles. However, these are not contained the Zariski closure of $\mathcal{G}$, and there could be as many as $\max\{m, n-m\}$ deformations. Theorem 3.8 uses fewer ($b \leq \max\{m, n-m\}$) deformations and the structure of the deformations reflects the Bruhat order on Schubert cells.

3.9. **Proof of Pieri's formula using Theorem 3.6.** Let $b > 0$, and $\alpha \in \binom{[n]}{m}$. For $1 \leq i \leq b$, let $U_i := U_{\alpha, b+1-i} F_{\bullet}$ and $\mathcal{G}_i := \mathcal{G}_{\alpha, b+1-i, i} F_{\bullet}$. Let $U_0 \subset \mathbf{G}_{n+1-m-b} V$ be the (dense) set of those L which meet F_{α_m} properly and for $1 \leq j < m$, $\dim F_{\alpha_j} \cap L < n+2-\alpha_j - j - b$. By Theorem 2.4, if $L \in \mathbf{G}_{n+1-m-b} V$, then $\Omega_\alpha F_{\bullet} \bigcap \Omega_L$ is generically transverse and irreducible if and only if $L \in U_0$. Let $\mathcal{G}_0 \subset \mathit{Chow}\, \mathbf{G}_m V$ be the set of cycles $\Omega_\alpha F_{\bullet} \bigcap \Omega_L$ for $L \in U_0$.

Let $L \in U_b$ and consider the cycle $Y_{\alpha,b}(F_{\bullet}, L) \in \mathcal{G}_b$:

$$Y_{\alpha,b}(F_{\bullet}, L) \;=\; \sum_{\substack{\beta \in \alpha * b \\ j(\alpha,\beta)=1}} \Omega_\beta F_{\bullet} \;+\; \sum_{\substack{\beta \in \alpha * b \\ j(\alpha,\beta)>1}} X_\beta(j(\alpha,\beta), F_{\bullet}, L).$$

We claim $Y_{\alpha,b}(F_{\bullet}, L) = \sum_{\beta \in \alpha * b} \Omega_\beta F_{\bullet}$, the cycle $Y_{\alpha,b} F_{\bullet}$ of the Introduction. It suffices to show $X_\beta(j(\alpha,\beta), F_{\bullet}, L) = \Omega_\beta F_{\bullet}$ for $\beta \in \alpha * b$ with $j(\alpha,\beta) > 1$. Suppose $j = j(\alpha,\beta) > 1$, then

$$X_\beta(j, F_{\bullet}, L) \;=\; p(\pi^{-1}(\Omega_{\beta|_j} F_{\bullet}|_{\beta_j} \bigcap \Omega_{F_{\beta_j} \cap L})).$$

By Formula (3.2), $\dim F_{\beta_j} \cap L = \dim F_{\beta_j} - j + 1$, as $\alpha_j < \beta_j < \alpha_{j-1}$ and $s = 1$. So $\Omega_{F_{\beta_j} \cap L} = \mathbf{G}_j F_{\beta_j}$, since any j-plane in F_{β_j} meets $F_{\beta_j} \cap L$ non-trivially. Thus $X_\beta(j(\alpha,\beta), F_{\bullet}, L) = \Omega_\beta F_{\bullet}$, by the definition of p and π in §2.5.

Let $\mathcal{G} \subset \mathit{Chow}\, \mathbf{G}_m V$ be the set of all cycles $\Omega_\alpha F_{\bullet} \cap \Omega_L$, where $L \in \mathbf{G}_{n+1-m-b} V$ and the intersection is generically transverse. Then by Theorem 2.4 and Remark 3.4, both $\mathcal{G}_0$ and $\mathcal{G}_1$ are subsets of $\mathcal{G}$. Arguing as in the proof of Lemma 2.10 shows $\mathcal{G} \subset \overline{\mathcal{G}_0}$. Theorem 3.6 implies $\mathcal{G}_i \subset \overline{\mathcal{G}_{i-1}}$ for $2 \leq i \leq b$, so in particular, $Y_{\alpha,b} F_{\bullet} \in \mathcal{G}_b \subset \overline{\mathcal{G}}$. Since $\mathcal{G}_0$, and hence $\overline{\mathcal{G}}$, is rational, $Y_{\alpha,b} F_{\bullet}$ is rationally equivalent to any cycle in $\mathcal{G}$, including $\Omega_\alpha F_{\bullet} \bigcap \Omega_K$, proving Pieri's formula.

More explicitly, one may construct a sequence of parameterized rational curves $\phi_i : \mathbf{P}^1 \to \overline{\mathcal{G}}$ for $1 \leq i \leq b$ witnessing this rational equivalence. For $2 \leq i \leq b$, select subspaces $M_i \in U_i$ and pencils $L_{i,t}$ of hyperplanes of M_i by downward induction on i as follows: Choose $M_b \in U_b$. Given $M_i \in U_i$, let $L_{i,t}$ be a pencil of hyperplanes of M_i as in Theorem 3.6, let $M_{i-1} := L_{i,\infty}$, and continue. Then for each i, if $t \neq 0$, $L_{i,t} \in U_{i-1}$. Define $\Sigma_i \subset \mathbf{P}^1 \times \mathbf{G}_m V$ to be the family whose fibre over $t \in \mathbf{P}^1 - \{0\}$ is the variety $Y_{\alpha,i-1}(F_{\bullet}, L_{i,t})$.

Let $\psi : \mathbf{P}^1 \to \overline{U_0} = \mathbf{G}_{n+1-m-s} V$ be a map with $\psi(0) = M_1 := L_{2,\infty}$, $\psi(\infty) = K$, and $\psi^{-1}(U_0) = \mathbf{P}^1 - \{0\}$. Let $\Sigma_1 \subset \mathbf{P}^1 \times \mathbf{G}_m V$ be the family whose fibre over $t \in \mathbf{P}^1$ is $\Omega_\alpha F_{\bullet} \bigcap \Omega_{\psi(t)}$, a generically transverse intersection which is irreducible for $t \neq 0$, by Theorem 2.4. Then for $1 \leq i \leq b$, $\Sigma_i \subset \mathbf{P}^1 \times \mathbf{G}_m V$ is a family with equidimensional generically reduced fibres over $\mathbf{P}^1$.

For $1 \leq i \leq b$, let $\phi_i : \mathbf{P}^1 \to \overline{\mathcal{G}_{i-1}}$ be the map associated to the family Σ_i, as in §3.1. Then $\phi_i(0) = \phi_{i+1}(\infty) \in \mathcal{G}_i$ and $\phi_i(t) \in \mathcal{G}_{i-1}$ for $t \neq 0$, by Theorem 3.6. Thus these parameterized rational curves give a chain of rational equivalences between $\Omega_\alpha F_{\bullet} \bigcap \Omega_K$ and $Y_{\alpha,b} F_{\bullet}$. ▮

Let $\beta \in \alpha * r$ and $\gamma \in \alpha * (r+1)$. If $\gamma \in \beta * 1$ with $j(\alpha, \gamma) = j(\beta, \gamma)$, write $\beta \prec_\alpha \gamma$. For example, if $\alpha = 8531$ and $\beta = 8631 \in \alpha * 1$, then those $\gamma \in \alpha * 2$ with $\beta \prec_\alpha \gamma$ are 9631 and 8731. Note these index the summands of $\lim_{t\to 0} X_{8631}(2, E_{\bullet}, \Lambda_t)$ in the example following Theorem 3.6.

3.10. **Proof of Theorem 3.6.** Let $t \neq 0$. Recall that L_t contains the subspace F_{α_1+s} of $M_{\bullet}$, but not F_{α_1+s-1}. Since $M \in U_{\alpha,s-1}F_{\bullet}$, we have $F_{\alpha_1} \cap M = F_{\alpha_1+s-1}$, but $F_{\alpha_1} \cap L_t = F_{\alpha_1+s}$, thus $F_i \cap L_t$ is a hyperplane of $F_i \cap M$ for any $i \leq \alpha_1$. Then $L_t \in U_{\alpha,s}F_{\bullet}$, for $t \neq 0$, as

(1) $F_{\alpha_1} \cap L_t = F_{\alpha_1+s}$.
(2) For $1 \leq j \leq m$, $F_{\alpha_j} \cap M = F_{\alpha_j+1} \cap M$. So $F_{\alpha_j} \cap L_t = F_{\alpha_j+1} \cap L_t$. Moreover, $\dim F_{\alpha_j} \cap L_t = \dim F_{\alpha_j} \cap M - 1$, which is $n + 2 - \alpha_j - j - s$.

Suppose $t \neq 0$ and recall that

$$Y_{\alpha,r}(F_{\bullet}, L_t) \;=\; \sum_{\substack{\beta\in\alpha*r\\ j(\alpha,\beta)=1}} \Omega_{\beta+(s-1)\delta^1}F_{\bullet} \;+\; \sum_{\substack{\beta\in\alpha*r\\ j(\alpha,\beta)>1}} X_\beta(j(\alpha,\beta), F_{\bullet}, L_t).$$

This defines a family $\Sigma \subset (\mathbf{P}^1 - \{0\}) \times \mathbf{G}_m V$ with equidimensional (actually isomorphic) fibres over $\mathbf{P}^1 - \{0\}$. We establish Theorem 3.6, showing the fibre of $\overline{\Sigma}$ at 0 is $Y_{\alpha,r+1}(F_{\bullet}, M)$ by examining each component of $Y_{\alpha,r}(F_{\bullet}, L_t)$ separately, then assembling the result.

Let $\beta \in \alpha * r$. Consider a component of $Y_{\alpha,r}(F_{\bullet}, L_t)$ in the first summand, so $j(\alpha,\beta) = 1$. Then $\gamma := \beta + \delta^1$ is the unique sequence satisfying $\beta \prec_\alpha \gamma$. In this case, $\Omega_{\beta+(s-1)\delta^1}F_{\bullet} = \Omega_{\gamma+(s-2)\delta^1}F_{\bullet}$.

Now consider a component in the second sum, so $j = j(\alpha,\beta) > 1$. Let $\beta' := \beta|_j$, $F_{\bullet}' := F_{\bullet}|_{\beta_j}$, and $L_t' := F_{\beta_j} \cap L_t$. For $t \neq 0$, Corollary 2.8 gives

$$X_\beta(j(\alpha,\beta), F_{\bullet}, L_t) \;=\; p(\pi^{-1}(\Omega_{\beta'}F_{\bullet}' \textstyle\bigcap \Omega_{L_t'})).$$

As $\alpha_j < \beta_j < \alpha_{j-1}$, $\dim L_t' = \dim F_{\beta_j} + 1 - j - (s-1)$, by formula (3.2). For $1 \leq i < j$, $\beta_i = \alpha_i$ and so $\dim L_t' \cap F_{\beta_i} = n + 2 - \beta_i - i - s$. Thus, by Theorem 2.4(3), $\Omega_{\beta'}F_{\bullet}' \bigcap \Omega_{L_t'}$ is generically transverse and irreducible. We study the 'limit' of these cycles as $t \to 0$, in the sense of §3.1. Define $L' := \lim_{t\to 0} L_t' = \lim_{t\to 0} F_{\beta_j} \cap L_t$, which is $F_{\beta_j+1} \cap M$, by Lemma 3.5. Then

(1) $F_{\alpha_1} \cap L' = F_{\alpha_1} \cap M = F_{\alpha_1+s-1}$.
(2) For $1 \leq i \leq j$, $F_{\beta_i} \cap L' = F_{\beta_i+1} \cap L'$. This follows for $i = j$ because we have $L' \subset F_{\beta_j+1} \subset F_{\beta_j}$ and for $i < j$, because $\beta_i = \alpha_i$ and $F_{\alpha_i} \cap M = F_{\alpha_i+1} \cap M$. Moreover, for $1 \leq i \leq j$, $\dim F_{\beta_i} \cap L' = n + 2 - \beta_i - i - (s-1)$.

Thus $L' \in U_{\beta',s-1}F_{\bullet}'$ so $\Omega_{\beta'}F_{\bullet}' \bigcap \Omega_{L'}$ is generically transverse, by Theorem 2.4(1). So,

$$\lim_{t\to 0} X_\beta(j(\alpha,\beta), F_{\bullet}, L_t) \;=\; p(\pi^{-1}(\Omega_{\beta'}F_{\bullet}' \textstyle\bigcap \Omega_{L'})).$$

But $\langle F_{\beta_{i-1}}, F_{\beta_i} \cap L\rangle \subset F_{\beta_i+1}$, since $L' \in U_{\beta',s-1}F_{\bullet}'$. By Remark 2.5,

$$\Omega_{\beta'}F_{\bullet}' \textstyle\bigcap \Omega_{L'} \;=\; \displaystyle\sum_{\substack{\gamma'\in\beta'*1\\ j(\beta',\gamma')=1}} \Omega_{\gamma'+(s-2)\delta^1}F_{\bullet} \;+\; \sum_{\substack{\gamma'\in\beta'*1\\ j(\beta',\gamma')>1}} X_{\gamma'}(j(\beta',\gamma'), F_{\bullet}', L').$$

And so $\lim_{t\to 0} X_\beta(j(\alpha,\beta), F_\bullet, L_t)$ is the cycle

$$\sum_{\substack{\gamma'\in\beta'*1\\ j(\beta',\gamma')=1}} p(\pi^{-1}(\Omega_{\gamma'+(s-2)\delta^1}F'_\bullet)) \;+\; \sum_{\substack{\gamma'\in\beta'*1\\ j(\beta',\gamma')>1}} p(\pi^{-1}(X_{\gamma'}(j(\beta',\gamma'), F'_\bullet, L'))).$$

We simplify this expression, beginning with the first sum. Let $\gamma' \in \beta' * 1$ satisfy $j(\beta',\gamma') = 1$. Then by Lemma 2.7, $p(\pi^{-1}(\Omega_{\gamma'+(s-2)\delta^1}F_\bullet))$ equals $\Omega_{\gamma+(s-2)\delta^1}F_\bullet$, where $\gamma := \beta+\delta^1$ is the unique sequence with $\beta \prec_\alpha \gamma$ and $j(\alpha,\gamma)=1$.

Consider terms in the second sum, those for which $\gamma' \in \beta' * 1$ with $j(\beta',\gamma') > 1$. Then $p(\pi^{-1}(X_{\gamma'}(j(\beta',\gamma'), F'_\bullet, L')))$ is the subvariety of $\Omega_\beta F_\bullet$ consisting of those H such that there exists $K \subset H$ with $\dim K = j$, $K \in \Omega_{\gamma'}F'_\bullet$, and $\dim K \cap F'_{\gamma'_{j(\beta',\gamma')}} \cap L' \geq 1$.

Let $\gamma := \beta + \delta^{j(\beta',\gamma')}$, the unique sequence with $\beta \prec_\alpha \gamma$ and $j(\alpha,\gamma) = j(\beta',\gamma')$. Then, as $\gamma_{j(\alpha,\gamma)} > \beta_j$, the definition of $F'_\bullet$ implies $F'_{\gamma'_{j(\beta',\gamma')}} = F_{j(\alpha,\gamma)} \subset F_{\beta_j+1}$. Since $L' = F_{\beta_j+1}\cap M$, we see that $F'_{\gamma'_{j(\beta',\gamma')}} \cap L' = F_{\gamma_{j(\alpha,\gamma)}} \cap M$. Thus if

$$H \;\in\; p(\pi^{-1}(X_{\gamma'}(j(\beta',\gamma'), F'_\bullet, L'))),$$

then $H \in \Omega_\gamma F_\bullet$ and $\dim H \cap F_{\gamma_{j(\alpha,\gamma)}} \cap M \geq 1$, so $H \in X_\gamma(j(\alpha,\gamma), F_\bullet, M)$. The reverse inclusion,

$$X_\gamma(j(\alpha,\gamma), F_\bullet, M) \;\subset\; p(\pi^{-1}(X_{\gamma'}(j(\beta',\gamma'), F'_\bullet, L'))),$$

is similar.

This shows that $\lim_{t\to 0} X_\beta(j(\alpha,\beta), F_\bullet, L_t)$ is the cycle

$$\sum_{\substack{\beta\prec_\alpha\gamma\\ j(\alpha,\gamma)=1}} \Omega_{\gamma+(s-2)\delta^1}F_\bullet \;+\; \sum_{\substack{\beta\prec_\alpha\gamma\\ j(\alpha,\gamma)>1}} X_\gamma(j(\alpha,\gamma), F_\bullet, L). \tag{3.10}$$

The sets $\{\gamma \mid \beta \prec_\alpha \gamma\}$ for $\beta \in \alpha * r$ partition the set $\alpha * (r+1)$. Thus

$$\lim_{t\to 0} Y_{\alpha,r}(F_\bullet, L_t) \;=\; \sum_{\substack{\gamma\in\alpha*(r+1)\\ j(\alpha,\gamma)=1}} \Omega_{\gamma+(s-2)\delta^1}F_\bullet \;+\; \sum_{\substack{\gamma\in\alpha*(r+1)\\ j(\alpha,\gamma)>1}} X_\beta(j(\alpha,\gamma), F_\bullet, M),$$

which is $Y_{\alpha,r+1}(F_\bullet, M)$. ∎

4. Link to Schensted insertion

The set $\binom{[n]}{m}$ has a partial order, called the *Bruhat order*: $\alpha \leq \beta$ if and only if $\Omega_\beta F_\bullet \subset \Omega_\alpha F_\bullet$. Combinatorially, this is $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for $1 \leq i \leq m$.

We interpret the behavior of the components $X_\beta(j(\alpha,\beta)F_\bullet, L)$ of the intermediate cycles $Y_{\alpha,i-1}(F_\bullet, L)$ in our proof of Pieri's formula (§3.9) as the branching of a certain subtree of $\binom{[n]}{m}$ with root α. This tree arises similarly in a combinatorial proof of Pieri's formula for Schur polynomials using Schensted insertion [5, p. 24]. We assume familiarity with the notions of Young tableaux and Schensted insertion as found in [5, 19]. To simplify this discussion, assume further that $n > \alpha_1 + b$.

Each rational equivalence of §3.9 is induced by a family Σ_i over $\mathbf{P}^1$ with generic fibre in $\mathcal{G}_{i-1}$ and special fibre in $\mathcal{G}_i$. The components of cycles in $\mathcal{G}_{i-1}$ are indexed by

$\beta \in \alpha * (i-1)$, with βth component $\Omega_{\beta+(b+1-i)\delta^1} F_\bullet$, if $j(\alpha, \beta) = 1$, and $X_\beta(j(\alpha,\beta), F_\bullet, L)$ otherwise. In passing to $\mathcal{G}_i$ via ϕ_i, the component $\Omega_{\beta+(b+1-i)\delta^1} F_\bullet$ is unchanged, but reindexed: $\Omega_{\gamma+(b-i)\delta^1} F_\bullet$, where $\gamma := \beta + \delta^1$ is the unique sequence in $\alpha * i$ with $\beta \prec_\alpha \gamma$. By equation (3.10), the other components become

$$\sum_{\substack{\beta \prec_\alpha \gamma \\ j(\alpha,\gamma)=1}} \Omega_{\gamma+(b-i)\delta^1} F_\bullet \;+\; \sum_{\substack{\beta \prec_\alpha \gamma \\ j(\alpha,\gamma)>1}} X_\gamma(j(\alpha,\gamma), F_\bullet, M_i).$$

Thus the component of the generic fibre of Σ_i indexed by $\beta \in \alpha * (i-1)$ becomes a sum of components indexed by $\{\gamma \in \alpha * i \mid \beta \prec_\alpha \gamma\}$ at the special fibre.

This suggest defining a tree $\mathcal{T}_{\alpha,b}$ whose branching represents the 'branching' of components of $Y_{\alpha,i-1}(F_\bullet, L)$ in these deformations. Let $\mathcal{T}_{\alpha,b} \subset \binom{[n]}{m}$ be the tree with vertex set $\bigcup\{\alpha * i \mid 0 \le i \le b\}$ and covering relation $\beta \prec_\alpha \gamma$. This is a tree as $\alpha * i$ is partitioned by the sets $\{\gamma \mid \beta \prec_\alpha \gamma\}$ for $\beta \in \alpha * (i-1)$.

For a decreasing m-sequence α, let $\lambda(\alpha)$ be the partition $(\alpha_1 - m, \alpha_2 - m + 1, \ldots, \alpha_m - 1)$. The association $\alpha \longleftrightarrow \lambda(\alpha)$ gives an order isomorphism between the set of decreasing m-sequences and the set of partitions of length at most m. This transfers notions for sequences into corresponding notions for partitions.

To a (semi-standard) Young tableau T with entries among $1, \ldots, m$, associate a monomial x^T in the variables $x_1, x_2 \ldots, x_m$: The exponent of x_i in x^T is the number of occurrences of i in T. This exponent vector is called the *content* of T. The Schur polynomial s_λ is $\sum x^T$, the sum over all tableaux T of shape λ. There is surjective homomorphism from the algebra of Schur polynomials to the Chow ring of $\mathbf{G}_m V$ defined by:

$$s_\lambda \longmapsto \begin{cases} [\Omega_\alpha F_\bullet] & \text{if } \lambda = \lambda(\alpha) \text{ for some } \alpha \in \binom{[n]}{m} \\ 0 & \text{otherwise} \end{cases}.$$

Special Schur polynomials are indexed by partitions $(b, 0, \ldots, 0)$ with a single row.

Schensted insertion gives a combinatorial proof of Pieri's formula, providing a content-preserving bijection between the set of pairs (S, T) of tableaux where S has shape λ and T has shape $(b, 0, \ldots, 0)$ and the set of all tableaux whose shape is in $\lambda * b$: Insert the reading word of T into S. The resulting tableau has shape $\mu \in \lambda * b$.

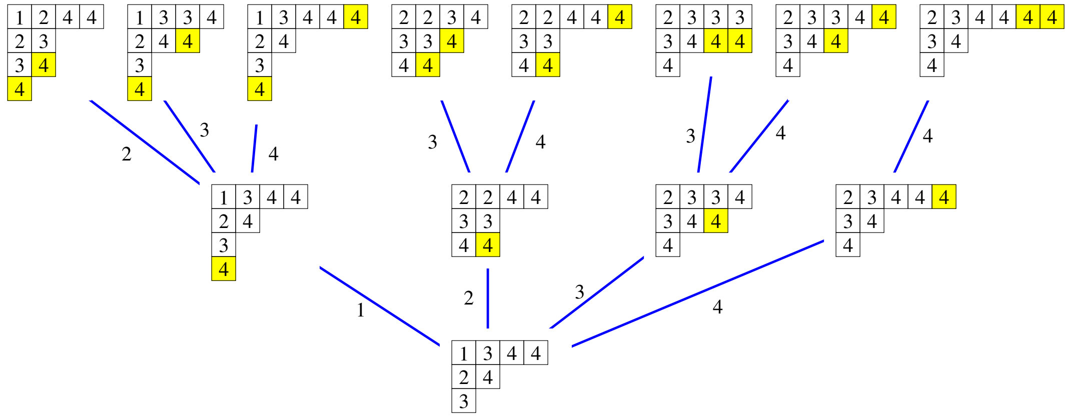
Consider the tableau S of shape $\lambda(8531) = 421$.

$$\begin{array}{|c|c|c|c|}\hline 2&3&4&4\\ \hline 3&4 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

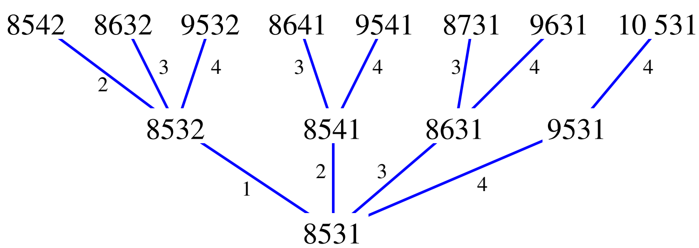
Schensted insertion of 1,2,3, respectively, 4 into S gives the following tableaux:

$$\begin{array}{|c|c|c|c|}\hline 1&3&4&4\\ \hline 2&4 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2&2&4&4\\ \hline 3&3 \\ \cline{1-2} 4&4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2&3&3&4\\ \hline 3&4&4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 2&3&4&4&4\\ \hline 3&4 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

If we insert the sequences $12, 13, 14, 23, 24, 33, 34$, and 44 in to S, we obtain all possible sequences of shapes. This may be displayed as a tree of tableaux, where the edges are labeled by the integer inserted:



Converting the shapes into sequences, we obtain the tree $\mathcal{T}_{8531,\,2}$:



This is exactly the branching of components in the example in §§3.3 and 3.7.

Let $\lambda = \lambda^0, \lambda^1, \ldots, \lambda^b = \mu$ be the sequence of shapes resulting from the insertion of successive entries of T into S. Since T is a single row, it is a property of the insertion algorithm that $\lambda^i \prec_\lambda \lambda^{i+1}$, and so this sequence is a chain in the tree $\mathcal{T}_{\alpha,b}$.

The totality of these insertions for all such pairs of tableaux gives all chains in $\mathcal{T}_{\alpha,b}$. Thus the 'branching' of shapes during Schensted insertion is identical to the branching of components in the rational equivalences of §3.9. We feel this relation to combinatorics is one of the more intriguing aspects of our proof of Pieri's formula and that similar ideas may yield a geometric proof of the Littlewood-Richardson rule.

Acknowledgment

I thank the anonymous referee for his helpful comments and suggestions, in particular the suggestion to include an illustrative running example and to discuss other geometric proofs of Pieri's formula.

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