(FORMULAS OF BRION, LAWRENCE, AND VARCHENKO ON RATIONAL GENERATING FUNCTIONS FOR CONES)

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ABSTRACT. We discuss and give elementary proofs of results of Brion and of Lawrence–Varchenko on the lattice-point enumerator generating functions for polytopes and cones. This largely expository note contains a new proof of Brion's Formula using irrational decompositions, and a generalization of the Lawrence–Varchenko formula.

1. INTRODUCTION

Our aim in this paper is to illustrate two gems of discrete geometry, namely formulas of Michel Brion [6] and of James Lawrence [14] and Alexander N. Varchenko [15], which at first sight seem hard to believe, and which—even after some years of studying them—still provoke a slight feeling of mystery in us. Let us start with some examples.

Suppose we would like to list all positive integers. Although there are many, we may list them compactly in the form of a generating function:

$$(1.1) \qquad x^1 + x^2 + x^3 + \cdots \; = \; \sum_{k>0} x^k \; = \; \frac{x}{1-x} \; .$$

Let us list, in a similar way, all integers less than or equal to 5:

$$(1.2) \qquad \cdots + x^{-1} + x^0 + x^1 + x^2 + x^3 + x^4 + x^5 \; = \; \sum_{k \leq 5} x^k \; = \; \frac{x^5}{1-x^{-1}} \; .$$

Adding the two rational function right-hand sides leads to a miraculous cancellation

$$(1.3) \qquad \frac{x}{1-x} + \frac{x^5}{1-x^{-1}} \; = \; \frac{x}{1-x} + \frac{x^6}{x-1} \; = \; \frac{x-x^6}{1-x} \; = \; x + x^2 + x^3 + x^4 + x^5 \; .$$
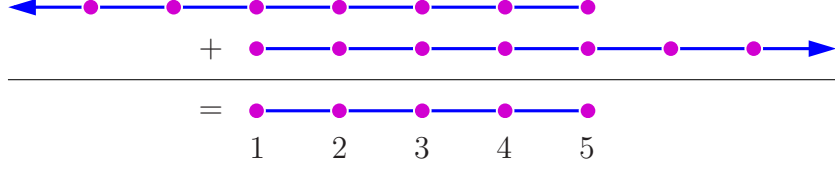
This sum of rational functions representing two *infinite* series collapses into a polynomial representing a *finite* series. This is a one-dimensional instance of a theorem due to Michel Brion. We can think of (1.1) as a function listing the integer points in the ray $[1, \infty)$ and of (1.2) as a function listing the integer points in the ray $(-\infty, 5]$. The respective rational

generating functions add up to the polynomial (1.3) that lists the integer points in the interval $[1, 5]$. Here is a picture of this arithmetic.

Let us move up one dimension. Consider the quadrilateral $\mathcal{Q}$ with vertices $(0, 0)$, $(2, 0)$, $(4, 2)$, and $(0, 2)$.

The analog of the generating functions (1.1) and (1.2) are the generating functions of the cones at each vertex generated by the edges at that vertex. For example, the two edges touching the origin generate the nonnegative quadrant, which has the generating function

$$\sum_{m,n\geq 0} x^m y^n = \sum_{m\geq 0} x^m \cdot \sum_{n\geq 0} y^n = \frac{1}{(1-x)} \cdot \frac{1}{(1-y)} = \frac{1}{(1-x)(1-y)} .$$

The two edges incident to $(0, 2)$ generate the cone $(0, 2) + \mathbb{R}_{\geq 0}(0, -2) + \mathbb{R}_{\geq 0}(4, 0)$, with the generating function

$$\sum_{m\geq 0, n\leq 2} x^m y^n = \frac{y^2}{(1-x)(1-y^{-1})} .$$

The third such *vertex cone*, at $(4, 2)$, is $(4, 2) + \mathbb{R}_{\geq 0}(-4, 0) + \mathbb{R}_{\geq 0}(-2, -2)$, which has the generating function
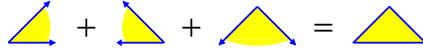
$$\frac{x^4 y^2}{(1-x^{-1})(1-x^{-1}y^{-1})} .$$

Finally, the fourth vertex cone is $(2, 0) + \mathbb{R}_{\geq 0}(2, 2) + \mathbb{R}_{\geq 0}(-2, 0)$, with the generating function

$$\frac{x^2}{(1-xy)(1-x^{-1})} .$$

Inspired by our one-dimensional example above, we add those four rational functions:

$$\frac{1}{(1-x)(1-y)} + \frac{y^2}{(1-x)(1-y^{-1})} + \frac{x^4 y^2}{(1-x^{-1})(1-x^{-1}y^{-1})} + \frac{x^2}{(1-xy)(1-x^{-1})}$$

$$\begin{aligned} = \quad & y^2 + xy^2 + x^2y^2 + x^2y^2 + x^4y^2 \\ & + y \; + xy \; + x^2y \; + x^3y \\ & + 1 \; + x \; \; + x^2 \quad . \end{aligned}$$

The sum of rational functions again collapses to a polynomial, which encodes precisely those integer points that are contained in the quadrilateral $\mathcal{Q}$.
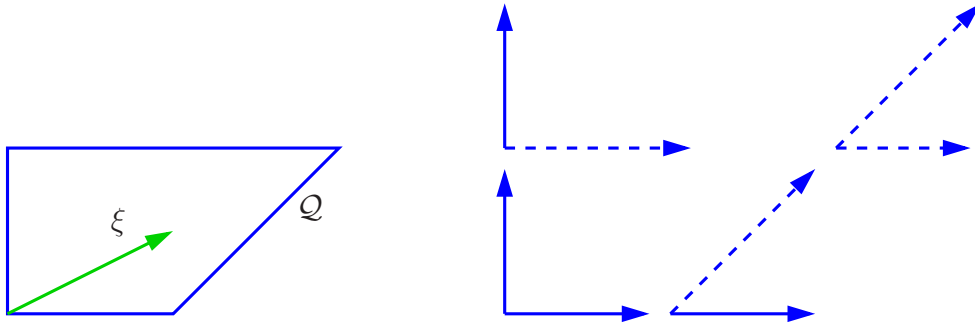
Brion's Theorem says that this magic happens for any polytope $\mathcal{P}$ in any dimension d, provided that $\mathcal{P}$ has rational vertices. (More precisely, the edges of $\mathcal{P}$ have rational directions.) The vertex cone $\mathcal{K}_{\mathbf{v}}$ at vertex $\mathbf{v}$ is the cone with apex $\mathbf{v}$ and generators the edge directions emanating from $\mathbf{v}$. The generating function

$$\sigma_{\mathcal{K}_{\mathbf{v}}}(x) \ := \ \sum_{\mathbf{m} \in \mathcal{K}_{\mathbf{v}} \cap \mathbb{Z}^d} x^{\mathbf{m}}$$

for such a cone is a rational function (again, provided that $\mathcal{P}$ has rational vertices). Here we abbreviate $x^{\mathbf{m}}$ for $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$. Brion's Formula says that the rational functions representing the integer points in each vertex cone sum up to the polynomial $\sigma_{\mathcal{P}}(x)$ encoding the integer points in $\mathcal{P}$:

$$\sigma_{\mathcal{P}}(x) \ = \ \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \sigma_{\mathcal{K}_{\mathbf{v}}}(x) \ .$$

A second theorem, which shows a similar collapse of generating functions of cones, is due (separately) to James Lawrence and to Alexander Varchenko. We illustrate it with the example of the quadrilateral $\mathcal{Q}$. Choose a direction vector ξ which is not perpendicular to any edge of $\mathcal{Q}$, for example we could take $\xi = (2,1)$. Now at each vertex v of $\mathcal{Q}$, we form a (not necessarily closed) cone generated by the edge directions m as follows. If $m \cdot \xi > 0$, then we take its nonnegative span, and if $m \cdot \xi < 0$, we take its negative span.

For example, the edge directions at the origin are along the positive axes and so this cone is again the nonnegative quadrant. At the vertex $(2,0)$ the edge directions are $(-2,0)$ and $(2,2)$. The first has negative dot product with ξ and the second has positive dot product, and so we obtain the half-open cone $(2,0)+\mathbb{R}_{<0}(-2,0)+\mathbb{R}_{\geq 0}(2,2) = (2,0)+\mathbb{R}_{>0}(2,0)+\mathbb{R}_{\geq 0}(2,2)$. At the vertex $(4,2)$ both edge directions have negative dot product with ξ and we get the open cone $(4,2) + \mathbb{R}_{>0}(0,4) + \mathbb{R}_{>0}(2,2)$, and at the vertex $(0,2)$ we get the half-open cone $(0,2) + \mathbb{R}_{\geq 0}(2,0) + \mathbb{R}_{>0}(0,2)$. The respective generating functions are

$$\frac{1}{(1-x)(1-y)} \ , \quad \frac{x^3}{(1-x)(1-xy)} \ , \quad \frac{x^6 y^3}{(1-xy)(1-y)} \ , \quad \text{and} \quad \frac{y^3}{(1-x)(1-y)} \ .$$

Now we add them with signs according to the parity of the number of negative $(m \cdot \xi < 0)$ edge directions m at the vertex. In our example, we obtain

$$
\begin{aligned}
&\frac{1}{(1-x)(1-y)} - \frac{x^3}{(1-x)(1-xy)} + \frac{x^6 y^3}{(1-xy)(1-y)} - \frac{y^3}{(1-x)(1-y)} \\
&=\; y^2 + xy^2 + x^2 y^2 + x^2 y^2 + x^4 y^2 \\
&\quad + y \; + xy \; + x^2 y \; + x^3 y \\
&\quad + 1 \; + x \; \; + x^2 \quad .
\end{aligned}
$$

This sum of rational functions again collapses to the polynomial that encodes the integer points in $\mathcal{Q}$. This should be clear here, for the integer points in the nonnegative quadrant are counted with a sign $\pm$, depending upon the cone in which they lie, and these coefficients cancel except for the integer points in the polytope $\mathcal{Q}$.

The identity illustrated by this example works for any *simple* polyope—a d-polytope where every vertex meets exactly d edges. Given a simple polytope, choose a direction vector $\xi \in \mathbb{R}^d$ that is not perpendicular to any edge direction. Let $E_{\mathbf{v}}^+(\xi)$ be the edge directions $\mathbf{w}$ at a vertex $\mathbf{v}$ with $\mathbf{w} \cdot \xi > 0$ and $E_{\mathbf{v}}^-(\xi)$ be those with $\mathbf{w} \cdot \xi < 0$. Define the cone

$$
\mathcal{K}_{\xi,\mathbf{v}} := \mathbf{v} \;+\; \sum_{\mathbf{w} \in E_{\mathbf{v}}^+(\xi)} \mathbb{R}_{\geq 0}\mathbf{w} \;+\; \sum_{\mathbf{w} \in E_{\mathbf{v}}^-(\xi)} \mathbb{R}_{<0}\mathbf{w} \,.
$$

This is the analog of the cones in our previous example. The Lawrence–Varchenko Formula says that adding the rational functions of these cones with appropriate signs gives the polynomial $\sigma_{\mathcal{P}}(x)$ encoding the integer points in $\mathcal{P}$:

$$
\sigma_{\mathcal{P}}(x) \;=\; \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} (-1)^{|E_{\mathbf{v}}^-(\xi)|}\, \sigma_{\mathcal{K}_{\xi,\mathbf{v}}}(x) \,.
$$

Here, $\sigma_{\mathcal{K}_{\xi,\mathbf{v}}}(x)$ is the generating function encoding the integer points in the cone $\mathcal{K}_{\xi,\mathbf{v}}$. An interesting feature of this identity, which also distinguishes it from Brion's Formula, is that the power series generating functions have a common region of convergence. Also, it holds without any restriction that the polytope be rational. In the general case, the generating functions of the cones are holomorphic functions, which we can add, as they have a common domain (the common region of convergence).

2. Proofs

Brion's original proof of his formula [6] used the Lefschetz-Riemann-Roch theorem in equivariant K-theory [3] applied to a singular toric variety. Fortunately for us, the remarkable formulas of Brion and of Lawrence–Varchenko now have easy proofs, based on counting.

Let us first consider an example based on the cone $\mathcal{K} = \mathbb{R}_{\geq 0}(0,1) + \mathbb{R}_{\geq 0}(2,1)$. The open circles in the picture on the left in Figure 1 represent the semigroup $\mathbb{N}(0,1) + \mathbb{N}(2,1)$, which is a proper subsemigroup of the integer points $\mathcal{K} \cap \mathbb{Z}^2$ in $\mathcal{K}$. The picture on the right shows how translates of the fundamental half-open parallelepipied $\mathcal{P}$ by this subsemigroup cover $\mathcal{K}$. This gives the formula
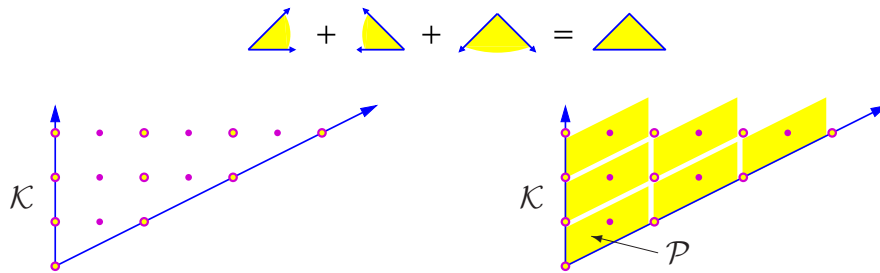
FIGURE 1. Tiling a simple cone by translates of its fundamental parallelepiped.

$$\sigma_{\mathcal{K}}(x) \;=\; \sigma_{\mathcal{P}}(x) \cdot \sum_{m,n \geq 0} x^m (x^2 y)^n \;=\; \frac{1 + xy}{(1 - x)(1 - x^2 y)} \;,$$

as the fundamental parallelepiped $\mathcal{P}$ contains two integer points, the origin and the point $(1, 1)$.

A simple rational cone in $\mathbb{R}^d$ has the form

$$\mathcal{K} \;:=\; \left\{ \mathbf{v} + \sum_{i=1}^{d} \lambda_i \mathbf{w}_i \mid \lambda_i \in \mathbb{R}_{\geq 0} \right\} \;=\; \mathbf{v} + \sum_{i=1}^{d} \mathbb{R}_{\geq 0} \mathbf{w}_i \;,$$

where $\mathbf{w}_1, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$ are linearly independent. This cone is tiled by the $(\mathbb{N}\mathbf{w}_1 + \cdots + \mathbb{N}\mathbf{w}_d)$-translates of the half-open parallelepiped

$$\mathcal{P} \;:=\; \left\{ \mathbf{v} + \sum_{i=1}^{d} \lambda_i \mathbf{w}_i \mid 0 \leq \lambda_i < 1 \right\} \;.$$

The generating function for $\mathcal{P}$ is the polynomial

$$\sigma_{\mathcal{P}}(x) \;=\; \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} x^{\mathbf{m}} \;,$$

and so the generating function for $\mathcal{K}$ is

$$\sigma_{\mathcal{K}}(x) \;=\; \sum_{\alpha \in \mathbb{N}\mathbf{w}_1 + \cdots + \mathbb{N}\mathbf{w}_d} x^{\alpha} \cdot \sigma_{\mathcal{P}}(x) \;=\; \frac{\sigma_{\mathcal{P}}(x)}{(1 - x^{\mathbf{w}_1}) \cdots (1 - x^{\mathbf{w}_d})} \;,$$

which is a rational function. This formula and its proof do not require that the apex $\mathbf{v}$ be rational, but only that the generators $\mathbf{w}_i$ of the cone be linearly independent vectors in $\mathbb{Z}^d$.

A rational cone $\mathcal{K}$ with apex $\mathbf{v}$ and generators $\mathbf{w}_1, \ldots, \mathbf{w}_n \in \mathbb{Z}^d$ has the form

$$\mathcal{K} \;=\; \mathbf{v} + \mathbb{R}_{\geq 0} \mathbf{w}_1 + \cdots + \mathbb{R}_{\geq 0} \mathbf{w}_n \;.$$

If there is a vector $\xi \in \mathbb{R}^d$ with $\xi \cdot \mathbf{w}_i > 0$ for $i = 1, \ldots, n$, then $\mathcal{K}$ is *strictly convex*. A fundamental result on convexity [2, Lemma VIII.2.3] is that $\mathcal{K}$ may be decomposed into simple cones $\mathcal{K}_1, \ldots, \mathcal{K}_l$ having pairwise disjoint interiors, each with apex $\mathbf{v}$ and generated by d of the generators $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of $\mathcal{K}$. We would like to add the generating functions for each cone $\mathcal{K}_i$ to obtain the generating function for $\mathcal{K}$. However, some of the cones may have lattice points in common, and some device is needed to treat the subsequent overcounting.

An elegant way to do this is to avoid the overcounting altogether by translating all the cones [4]. We explain this. There exists a short vector $\mathbf{s} \in \mathbb{R}^d$ such that

$$(2.1) \qquad \mathcal{K} \cap \mathbb{Z}^d \;=\; (\mathbf{s} + \mathcal{K}) \cap \mathbb{Z}^d \,,$$

and no facet of any cone $\mathbf{s}+\mathcal{K}_1, \ldots, \mathbf{s}+\mathcal{K}_l$ contains any integer points. This gives the disjoint *irrational decomposition*

$$\mathcal{K} \cap \mathbb{Z}^d \;=\; (\mathbf{s} + \mathcal{K}_1) \cap \mathbb{Z}^d \sqcup \cdots \sqcup (\mathbf{s} + \mathcal{K}_l) \cap \mathbb{Z}^d \,,$$
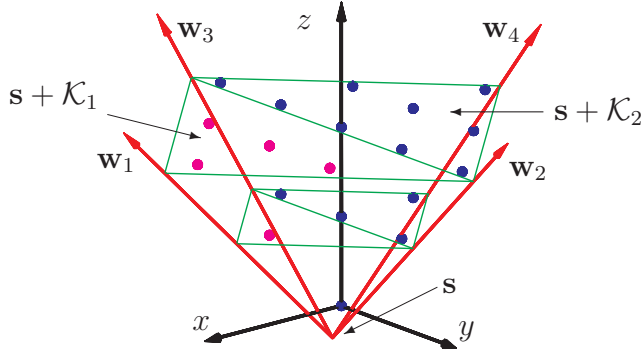
and so

$$(2.2) \qquad \sigma_{\mathcal{K}}(x) \;=\; \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^d} x^{\mathbf{m}} \;=\; \sum_{i=1}^{l} \sigma_{\mathbf{s}+\mathcal{K}_i}(x)$$

is a rational function.

For example, suppose that $\mathcal{K}$ is the cone in $\mathbb{R}^3$ with apex the origin and generators

$$\mathbf{w}_1 \;=\; (1,0,1), \quad \mathbf{w}_2 \;=\; (0,1,1), \quad \mathbf{w}_3 \;=\; (0,-1,1), \quad \text{and} \quad \mathbf{w}_4 \;=\; (-1,0,1)\,.$$

If we let $\mathcal{K}_1$ be the simple cone with generators $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ and $\mathcal{K}_2$ be the simple cone with generators $\mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$, then $\mathcal{K}_1$ and $\mathcal{K}_2$ decompose $\mathcal{K}$ into simple cones. If $\mathbf{s} = (\frac{1}{8}, 0, -\frac{1}{3})$, then (2.1) holds, and no facet of $\mathbf{s}+\mathcal{K}_1$ or of $\mathbf{s}+\mathcal{K}_2$ contains any integer points. We display these cones, together with their integer points having z-coordinate 0, 1, or 2.
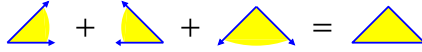

The cone $\mathbf{s} + \mathcal{K}_1$ contains the 5 magenta points shown with positive first coordinate, while $\mathbf{s} + \mathcal{K}_2$ contains the other displayed points. Their integer generating functions are

$$\sigma_{\mathbf{s}+\mathcal{K}_1}(x) \;=\; \frac{x + xz}{(1 - yz)(1 - y^{-1}z)(1 - xz)} \,,$$

$$\sigma_{\mathbf{s}+\mathcal{K}_2}(x) \;=\; \frac{1 + z}{(1 - yz)(1 - y^{-1}z)(1 - x^{-1}z)} \,, \quad \text{and}$$

$$\sigma_{\mathcal{K}}(x) \;=\; \frac{(1 + x)(1 - z^2)}{(1 - yz)(1 - y^{-1}z)(1 - xz)(1 - x^{-1}z)} \,.$$

Then $\sigma_{\mathbf{s}+\mathcal{K}_1}(x) + \sigma_{\mathbf{s}+\mathcal{K}_2}(x) = \sigma_{\mathcal{K}}(x)$, as

$$(x + xz)(1 - x^{-1}z) + (1 + z)(1 - xz) \;=\; 1 + x - z^2 - xz^2 \;=\; (1 + x)(1 - z^2)\,.$$

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While the cones that appear in the Lawrence–Varchenko formula are all simple, and those in Brion's formula are strictly convex, we use yet more general cones in their proof. A rational (closed) halfspace is the convex subset of $\mathbb{R}^d$ defined by

$$\{x \in \mathbb{R}^d \mid \mathbf{w} \cdot x \geq b\},$$

where $\mathbf{w} \in \mathbb{Z}^d$ and $b \in \mathbb{R}$. Its boundary is the rational hyperplane $\{x \in \mathbb{R}^d \mid \mathbf{w} \cdot x = b\}$. A (closed) cone $\mathcal{K}$ is the interection of finitely many closed halfspaces whose boundary hyperplanes have some point in common. We assume this intersection is irredundant. The *apex* of $\mathcal{K}$ is the intersection of these boundary hyperplanes, which is an affine linear subspace.

The generating function for the integer points in $\mathcal{K}$ is the formal Laurent series

$$(2.3) \qquad S_{\mathcal{K}} \; := \; \sum_{\mathbf{m} \in \mathcal{K}} x^{\mathbf{m}} \, .$$

This formal series makes sense as a rational function only if $\mathcal{K}$ is strictly convex, that is, if its apex is a single point. Otherwise, the apex is a rational affine linear subspace L, and the cone $\mathcal{K}$ is stable under translation by any integer vector $\mathbf{w}$ which is parallel to L. If $\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^d$, then the series $S_{\mathcal{K}}$ contains the series

$$x^{\mathbf{m}} \cdot \sum_{n \in \mathbb{Z}} x^{n\mathbf{w}}$$

as a subsum. As this converges only for $x = 0$, the series $S_{\mathcal{K}}$ converges only for $x = 0$.

We relate these formal Laurent series to rational functions. The product of a formal series and a polynomial is another formal series. Thus the additive group $\mathbb{C}[[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]]$ of formal Laurent series is a module over the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ of Laurent polynomials. The space PL of *polyhedral Laurent series* is the $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$-submodule of $\mathbb{C}[[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]]$ generated by the set of formal series

$$\{S_{\mathcal{K}} \mid \mathcal{K} \text{ is a simple rational cone}\} \, .$$

Since any rational cone may be triangulated by simple cones, PL contains the integer generating series of all rational cones.

Let $\mathbb{C}(x_1, \ldots, x_d)$ be the field of rational functions on $\mathbb{C}^d$, which is the quotient field of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$. According to Ishida [10], the proof of the following theorem is due to Brion.

Theorem 2.4. *There is a unique homomorphism of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$-modules*

$$\varphi : \;\; \mathrm{PL} \; \longrightarrow \; \mathbb{C}(x_1, \ldots, x_d) \, ,$$

such that $\varphi(S_{\mathcal{K}}) = \sigma_{\mathcal{K}}$ for every simple cone $\mathcal{K}$ in $\mathbb{R}^d$.

Proof. Given a simple rational cone $\mathcal{K} = v + \langle \mathbf{w}_1, \ldots, \mathbf{w}_d \rangle$ with fundamental parallelepiped $\mathcal{P}$, we have

$$\prod_{i=1}^{d} (1 - x^{\mathbf{w}_i}) \cdot S_{\mathcal{K}} \; = \; \sigma_{\mathcal{P}}(x) \, .$$

Hence, for each $S \in \mathrm{PL}$, there is a nonzero Laurent polynomial $g \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ such that $gS = f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$. If we define $\varphi(S) := f/g \in \mathbb{C}(x_1, \ldots, x_d)$, then $\varphi(S)$ is independent of the choice of g. This defines the required homomorphism. $\qquad \square$

The map φ takes care of the nonconvergence of the generating series $S_{\mathcal{K}}$ when $\mathcal{K}$ is not strictly convex.

Lemma 2.5. *If a rational polyhedral cone $\mathcal{K}$ is not strictly convex, then $\varphi(S_{\mathcal{K}}) = 0$.*

Proof. Let $\mathcal{K}$ be a rational polyhedral cone that is not strictly convex. Then there is a nonzero vector $\mathbf{w} \in \mathbb{Z}^d$ such that $\mathbf{w} + \mathcal{K} = \mathcal{K}$, and so $x^{\mathbf{w}} \cdot S_{\mathcal{K}} = S_{\mathcal{K}}$. Thus $x^{\mathbf{w}} \varphi(S_{\mathcal{K}}) = \varphi(S_{\mathcal{K}})$. Since $1 - x^{\mathbf{w}}$ is not a zero-divisor in $\mathbb{C}(x_1, \ldots, x_d)$, we conclude that $\varphi(S_{\mathcal{K}}) = 0$. $\square$

We now establish Brion's Formula, first for a simplex, and then use irrational decomposition for the general case. A simplex is the intersection of $d+1$ halfspaces, one for each facet.

For a face F of the simplex $\mathcal{P}$, let $\mathcal{K}_F$ be the tangent cone to F, which is the intersection of the halfspaces corresponding to the $d - \dim(F)$ facets containing F. Let $\emptyset$ be the empty face of $\mathcal{P}$, which has dimension -1. Its tangent cone is $\mathcal{P}$.

Theorem 2.6. *If P is a simplex, then*

$$(2.7) \qquad\qquad 0 \; = \; \sum_F (-1)^{\dim(F)} S_{\mathcal{K}_F} \, ,$$

the sum over all faces of P.

Proof. Consider the coefficient of $x^{\mathbf{m}}$ for some $\mathbf{m} \in \mathbb{Z}^d$ in the sum on the right. Then $\mathbf{m}$ lies in the tangent cone $\mathcal{K}_F$ to a unique face F of minimal dimension, as P is a simplex. The coefficient of $x^{\mathbf{m}}$ in the sum becomes

$$\sum_{G \supseteq F} (-1)^{\dim(G)} \, .$$

But this vanishes, as every interval in the face poset of P is a Boolean lattice. $\square$

Now we apply the evaluation map φ of Theorem 2.4 to the formula (2.7). Lemma 2.5 implies that $\varphi(S_{\mathcal{K}_F}) = 0$ except when $F = \emptyset$ or F is a vertex, and then $\varphi(S_{\mathcal{K}_F}) = \sigma_{\mathcal{K}_F}(x)$. This gives

$$0 \; = \; -\sigma_{\mathcal{P}}(x) + \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \sigma_{\mathcal{K}_{\mathbf{v}}}(x) \, ,$$

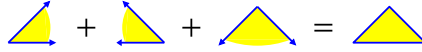which is Brion's Formula for simplices.

Just as for rational cones, every polytope may be decomposed into simplices having pairwise disjoint interiors, using only the vertices of $\mathcal{P}$.

$$\mathcal{P} \; = \; \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_l \, .$$

Then there exists a small real number $\epsilon > 0$ and a short vector $\mathbf{s}$ such that if we set

$$\mathcal{P}' \; := \; \mathbf{s} + (1+\epsilon)\mathcal{P} \quad \text{and} \quad \mathcal{P}'_i \; := \; \mathbf{s} + (1+\epsilon)\mathcal{P}_i \quad \text{for } i = 1, \ldots, l \, ,$$

then $\mathcal{P}' \cap \mathbb{Z}^d = \mathcal{P} \cap \mathbb{Z}^d$, and no hyperplane supporting any facet of any simplex $\mathcal{P}'_i$ meets $\mathbb{Z}^d$. If we write $\mathcal{K}(\mathcal{Q})_{\mathbf{w}}$ for the tangent cone to a polytope $\mathcal{Q}$ at a vertex $\mathbf{w}$, then for $\mathbf{v}$ a vertex of

$\mathcal{P}$ with $\mathbf{v}' = (1+\epsilon)\mathbf{v} + \mathbf{s}$ the coresponding vertex of $\mathcal{P}'$, we have $\mathcal{K}(\mathcal{P}')_{\mathbf{v}'} \cap \mathbb{Z}^d = \mathcal{K}(\mathcal{P})_{\mathbf{v}} \cap \mathbb{Z}^d$ and so this is an irrational decomposition. Then
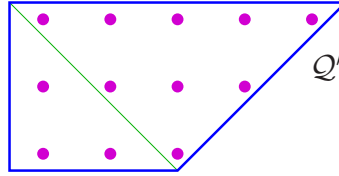
$$
\begin{aligned}
\sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \sigma_{\mathcal{K}(\mathcal{P})_{\mathbf{v}}}(x) &= \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}'} \sigma_{\mathcal{K}(\mathcal{P}')_{\mathbf{v}}}(x) \\
&= \sum_{i=1}^{l} \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}'_i} \sigma_{\mathcal{K}(\mathcal{P}'_i)_{\mathbf{v}}}(x) \\
&= \sum_{i=1}^{l} \sigma_{\mathcal{P}_i}(x) = \sigma_{\mathcal{P}'}(x) = \sigma_{\mathcal{P}}(x) \ .
\end{aligned}
$$

The second equality holds because the vertex cones $\mathcal{K}(\mathcal{P}'_i)_{\mathbf{v}}$ form an irrational decomposition of the vertex cone $\mathcal{K}(\mathcal{P}')_{\mathbf{v}}$, and because the same is true for the polytopes. This completes our proof of Brion's Formula.

Consider the quadrilateral $\mathcal{Q}$, which may be triangulated by adding an edge between the vertices $(2,0)$ and $(0,2)$. Let $\epsilon = \frac{1}{4}$ and $\mathbf{s} = (-\frac{1}{2}, -\frac{1}{4})$. Then $(1+\epsilon)\mathcal{Q} + \mathbf{s}$ has vertices

$$
(-\tfrac{1}{2}, -\tfrac{1}{4}), \quad (2, -\tfrac{1}{4}), \quad (-\tfrac{1}{2}, 2+\tfrac{1}{4}), \quad (4+\tfrac{1}{2}, 2+\tfrac{1}{4}) \ .
$$

We display the resulting irrational decomposition.

We use the map φ to deduce a very general form of the Lawrence–Varchenko formula. Let $\mathcal{P}$ be a simple polytope, and for each vertex $\mathbf{v}$ of $\mathcal{P}$ choose a vector $\xi_{\mathbf{v}}$ that is not perpendicular to any edge direction at $\mathbf{v}$. Form the cone $\mathcal{K}_{\xi_{\mathbf{v}}, \mathbf{v}}$ as before. Then we have

$$
(2.8) \qquad \sigma_{\mathcal{P}}(x) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} (-1)^{|E_{\mathbf{v}}^-(\xi_{\mathbf{v}})|} \sigma_{\mathcal{K}_{\xi_{\mathbf{v}}, \mathbf{v}}}(x) \ .
$$

Brion's formula is the special case when each vector $\xi_{\mathbf{v}}$ points into the interior of the polytope. We establish (2.8) by showing that the sum on the right does not change when any of the vectors $\xi_{\mathbf{v}}$ are rotated.

Pick a vertex $\mathbf{v}$ and vectors ξ, ξ' that are not perpendicular to any edge direction at $\mathbf{v}$ such that $\xi \cdot \mathbf{w}$ and $\xi \cdot \mathbf{w}'$ have the same sign for all except one edge direction $\mathbf{m}$ at $\mathbf{v}$. Then $\mathcal{K}_{\xi, \mathbf{v}}$ and $\mathcal{K}_{\xi', \mathbf{v}}$ are disjoint and their union is the (possibly) half-open cone $\mathcal{K}$ generated by the edge directions $\mathbf{w}$ at $\mathbf{v}$ such that $\xi \cdot \mathbf{w}$ and $\xi' \cdot \mathbf{w}$ have the same sign, but with apex the affine line $\mathbf{v} + \mathbb{R}\mathbf{m}$. Thus we have the identity of rational formal series

$$
S_{\mathcal{K}_{\xi, \mathbf{v}}} - S_{\mathcal{K}} = -S_{\mathcal{K}_{\xi', \mathbf{v}}} \ .
$$

Applying the evaluation map φ gives

$$
\sigma_{\mathcal{K}_{\xi, \mathbf{v}}}(x) = -\sigma_{\mathcal{K}_{\xi', \mathbf{v}}}(x) \ ,
$$

which proves the claim, and the generalized Lawrence–Varchenko formula (2.8).
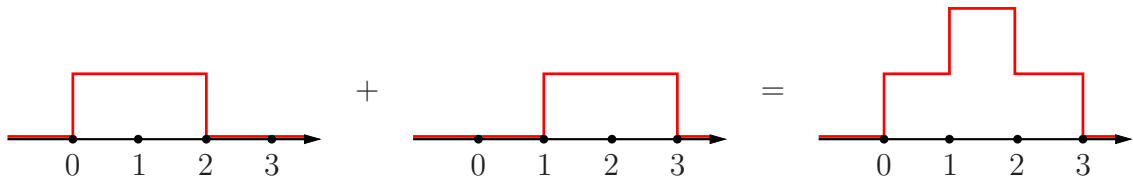
3. Valuations

Valuations provide a conceptual approach to these ideas. Once the theory is set up, both Brion's Formula and the Lawrence–Varchenko Formula are easy corollaries of duality being a valuation. We are indebted to Sasha Barvinok who pointed out this correspondence to the second author during a coffee break at the 2005 Park City Mathematical Institute. Let us explain.
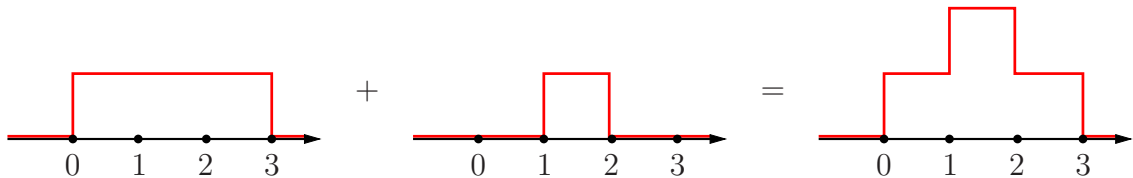
Consider the vector space of all functions $\mathbb{R}^d \to \mathbb{R}$. Let $\mathcal{V}$ be the subspace which is generated by indicator functions of polyhedra:

$$[\mathcal{P}] : \ x \ \mapsto \ \begin{cases} 1 & \text{if } x \in \mathcal{P} \\ 0 & \text{if } x \notin \mathcal{P} \end{cases} \ .$$

We add these functions point-wise. For example, if $d = 1$, and $\mathcal{P} = [0, 2]$, $\mathcal{Q} = [1, 3]$, then $[\mathcal{P}] + [\mathcal{Q}]$ takes the value 1 along $[0, 1)$ and $(2, 3]$, the value 2 along $[1, 2]$, and vanishes everywhere else.

Already this simple example shows that our generators do not form a basis: they are linearly dependent. For $\mathcal{P}' = [0, 3]$ and $\mathcal{Q}' = [1, 2]$, we get the same sum.
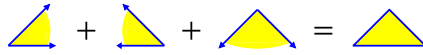

But this is the only thing that can happen.

Theorem 3.1 ([17, 9]). *The linear space of relations among the indicator functions $[\mathcal{P}]$ of convex polyhedra is generated by the relations $[\mathcal{P}] + [\mathcal{Q}] = [\mathcal{P} \cup \mathcal{Q}] + [\mathcal{P} \cap \mathcal{Q}]$ where $\mathcal{P}$ and $\mathcal{Q}$ run over polyhedra for which $\mathcal{P} \cup \mathcal{Q}$ is convex.*

A *valuation* is a linear map $\nu \colon \mathcal{V} \to V$, where V is some vector space. Some standard examples are

V	$\nu(\mathcal{P})$
$\mathbb{R}^d$	$\mathrm{vol}(\mathcal{P})$
PL	$S_\mathcal{P}(x)$
$\mathbb{C}(x_1, \ldots, x_d)$	$\sigma_\mathcal{P}(x)$
$\mathbb{R}^d$	1

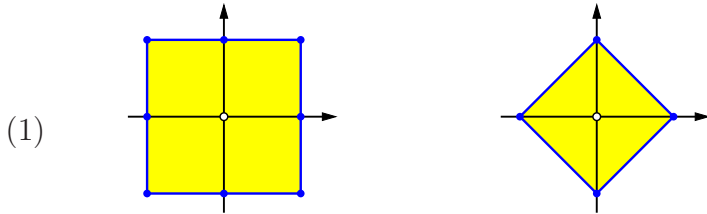That $\sigma_\mathcal{P}(x)$ is a valuation is a Theorem of Khovanskii-Pukhlikov [11] and of Lawrence [13]. The last example is called the Euler characteristic. This valuation is surprisingly useful. For example, it can be used to prove Theorem 3.2 below.
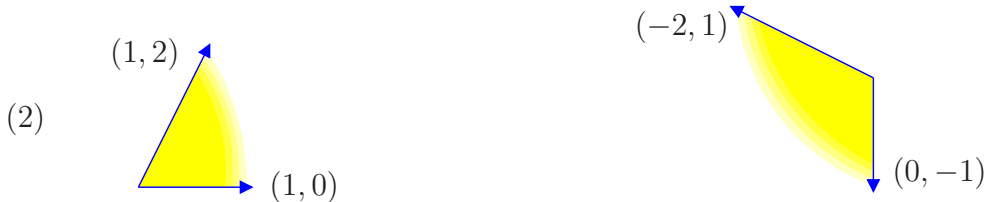
The most interesting valuation for us comes from the polar construction. The *polar* $\mathcal{P}^\vee$ of a polyhedron $\mathcal{P}$ is the polyhedron given by

$$\mathcal{P}^\vee \ := \ \{x \mid \langle x, y\rangle \le 1 \text{ for all } y \in \mathcal{P}\}.$$

It is instructive to work through some examples.
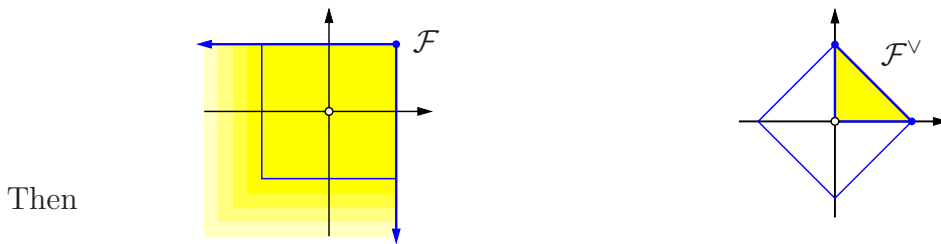
(1)

The polar of the square   ...   is the diamond.

(2)

The polar of a cone $\mathcal{K}$ ...   is the cone $\mathcal{K}^\vee := \{x \mid \langle x, y\rangle \le 0 \text{ for all } y \in \mathcal{K}\}$.

(3) Suppose that $\mathcal{P}$ is a polytope whose interior contains the orign and $\mathcal{F}$ is a face of $\mathcal{P}$.

Then

the polar of the tangent cone $\mathcal{K}_\mathcal{F}$ ...   is the convex hull of the origin together with the dual face
$$\mathcal{F}^\vee := \{x \in \mathcal{P}^\vee \mid \langle x, y\rangle = 1\},$$
which is a pyramid over $\mathcal{F}^\vee$.

For this last remark, note that if $x \in \mathcal{F}^\vee$ and $y \in \mathcal{K}_\mathcal{F}$, then $\langle x, y\rangle \le \langle \mathcal{F}^\vee, \mathcal{F}\rangle = 1$. Conversely, if $x \in \mathcal{K}_\mathcal{F}^\vee$, then $\langle x, .\rangle$ is maximized over $\mathcal{K}_\mathcal{F}$ at $\mathcal{F}$ by example (2), and it is at most 1 there.

In these examples, the polar of the polar is the original polyhedron. This happens if and only if the original polyhedron contains the origin.

(4) The polar of the interval $[1, 2]$ is the interval $[0, 1/2]$, but the polar of $[0, 1/2]$ is $[0, 2]$.

Now, we come to the main theorem of this section.

Theorem 3.2 (Lawrence [13])**.** *The assignment* $[\mathcal{P}] \mapsto [\mathcal{P}^\vee]$ *defines a valuation.*

This innocent result has powerful consequences. Suppose that $\mathcal{P}$ is a polytope whose interior contains the orign. Then we can cover $\mathcal{P}^\vee$ by pyramids $\mathrm{conv}(0, \mathcal{F}^\vee)$ over the codimension-one faces $\mathcal{F}^\vee$ of $\mathcal{P}^\vee$. The indicator functions of $\mathcal{P}$ and the cover differ by indicator functions
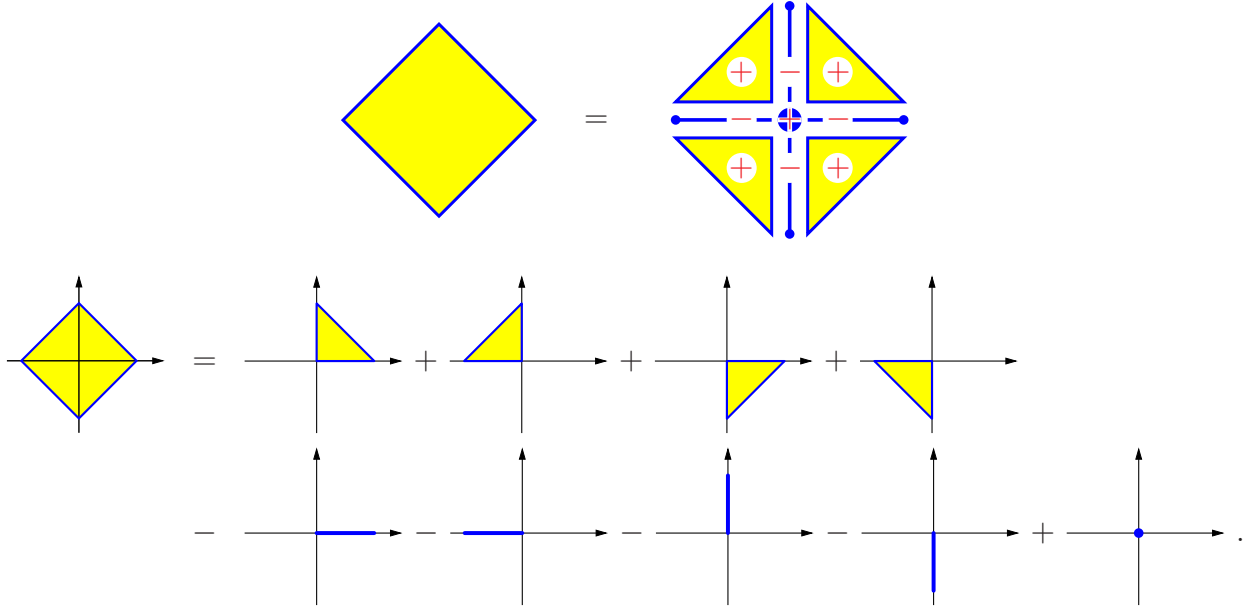
of pyramids of smaller dimension.

$$(3.3) \qquad [\mathcal{P}^\vee] \;=\; \sum_{\mathcal{F}^\vee} [\mathrm{conv}(0, \mathcal{F}^\vee)] \pm \text{ lower dimensional pyramids.}$$

The Euler–Poincaré formula for general polytopes organizes this inclusion-exclusion, giving the exact expression

$$[\mathcal{P}^\vee] \;=\; \sum (-1)^{\mathrm{codim}\,\mathcal{F}^\vee + 1} [\mathrm{conv}(0, \mathcal{F}^\vee)]\,.$$

We illustrate this when $\mathcal{P}$ is the square.

If we apply polarity to (3.3), we get the Briachon-Gram Theorem [5, 8].

$$(3.4) \qquad [\mathcal{P}] \;=\; \sum_{v \text{ vertex}} [\mathcal{K}_v] \pm \text{ tangent cones of faces of positive dimension.}$$
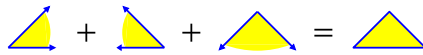
This is essentially the indicator function version of Theorem 2.6, but for general polytopes. If we now apply the valuation σ, and recall that σ evaluates to zero on cones that are not strictly convex, we obtain Brion's Formula.

Next, suppose that we are given a generic direction vector ξ. On a face $\mathcal{F}$ of $\mathcal{P}$, the dot product with ξ achieves its maximum at a vertex $v_\xi(\mathcal{F})$. For a vertex v of $\mathcal{P}$, we set
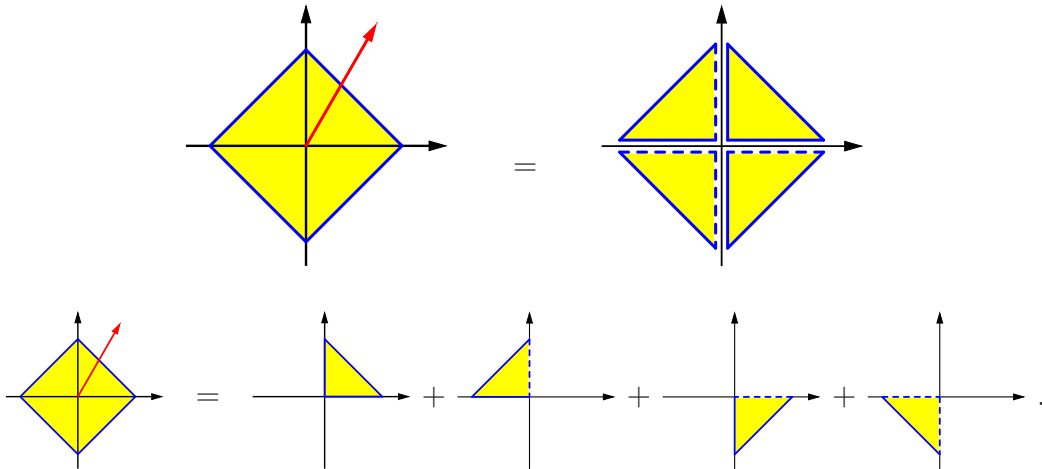
$$\mathcal{F}_\xi^\vee(v) \;:=\; \bigcup_{\mathcal{F}\,:\,v_\xi(\mathcal{F})=v} \mathrm{relint}\,\mathcal{F}^\vee.$$

(The relative interior, relint($\mathcal{P}$), of a polyhedron $\mathcal{P}$ is the topological interior when considered as a subspace of its affine hull.) In words, we attach the relative interior of a low-dimensional pyramid $\mathrm{conv}(0, \mathcal{F}^\vee)$ to the full-dimensional pyramid $\mathrm{conv}(0, v^\vee)$ which we see when we look in the ξ-direction from $\mathrm{conv}(0, \mathcal{F}^\vee)$. In this way, we obtain an honest decomposition

$$(3.5) \qquad [\mathcal{P}^\vee] \;=\; \sum_v [\mathrm{conv}(0, \mathcal{F}_\xi^\vee(v))]\,.$$

For the polar of the square, this is

To compute the polar of the half-open polyhedron $\mathrm{conv}(0, \mathcal{F}_\xi^\vee(v))$, we have to write its indicator function $[\mathrm{conv}(0, \mathcal{F}_\xi^\vee(v))]$ as a linear combination of indicator functions of (closed) polyhedra. If $\mathcal{P}$ is a simple polytope, then all the dual faces $\mathcal{F}^\vee$ are simplices. It turns out that the polar of $\mathrm{conv}(0, \mathcal{F}_\xi^\vee(v))$ is precisely the forward tangent cone $\mathcal{K}_{\xi,v}$ at the vertex v. So the Lawrence–Varchenko formula is just the polar of (3.5).

This gives a fairly general principle to construct Brion-type formulas: Choose a decomposition of (the indicator function of) $\mathcal{P}^\vee$, and then polarize. We invite the reader to set up their own equations this way.

4. An Application

Brion's Formula shows that certain data of a *polytope*—the list of its integer points encoded in a generating function—can be reduced to *cones*. We have already seen how to construct the generating function $\sigma_\mathcal{K}(x)$ for a simple cone $\mathcal{K}$. General cones can be composed from simple ones via triangulation and either irrational decomposition or inclusion-exclusion. Given a rational polytope $\mathcal{P}$, Brion's Formula allows us to write the possibly huge polynomial $\sigma_\mathcal{P}(x)$ as a sum of rational functions, which stem from (triangulations of) the vertex cones. *A priori* it is not clear that this rational-function representation of $\sigma_\mathcal{P}(x)$ is any shorter than the original polynomial. That this is indeed possible is due to the *signed decomposition* theorem of Barvinok [1].

To state Barvinok's Theorem, we call a rational d-cone $\mathcal{K} = \mathbf{v} + \sum_{i=1}^d \mathbb{R}_{\geq 0}\mathbf{w}_i$ *unimodular* if $\mathbf{w}_1, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$ generate the integer lattice $\mathbb{Z}^d$. The significance of a unimodular cone $\mathcal{K}$ for us is that its fundamental (half-open) parallelepiped contains precisely one integer point, $\mathbf{p}$, and so the generating function of $\mathcal{K}$ has a very simple and short form

$$\sigma_\mathcal{K}(x) \;=\; \frac{x^\mathbf{p}}{(1 - x^{\mathbf{w}_1}) \cdots (1 - x^{\mathbf{w}_d})} \;.$$

In fact, the description length of this is proportional to the description of the cone, $\mathcal{K}$.
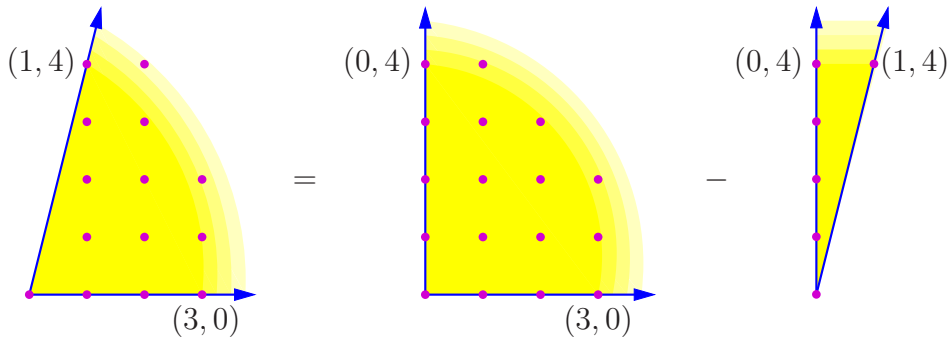
Theorem 4.1 (Barvinok)**.** *For fixed dimension d, the generating function $\sigma_\mathcal{K}$ for any rational cone $\mathcal{K}$ in $\mathbb{R}^d$ can be decomposed into generating functions of unimodular cones in polynomial*

time; that is, there is a polynomial-time algorithm and (polynomially many) unimodular cones $\mathcal{K}_j$ *such that* $\sigma_{\mathcal{K}}(x) = \sum_j \epsilon_j \sigma_{\mathcal{K}_j}(x)$*, where* $\epsilon_j \in \{\pm 1\}$*.*
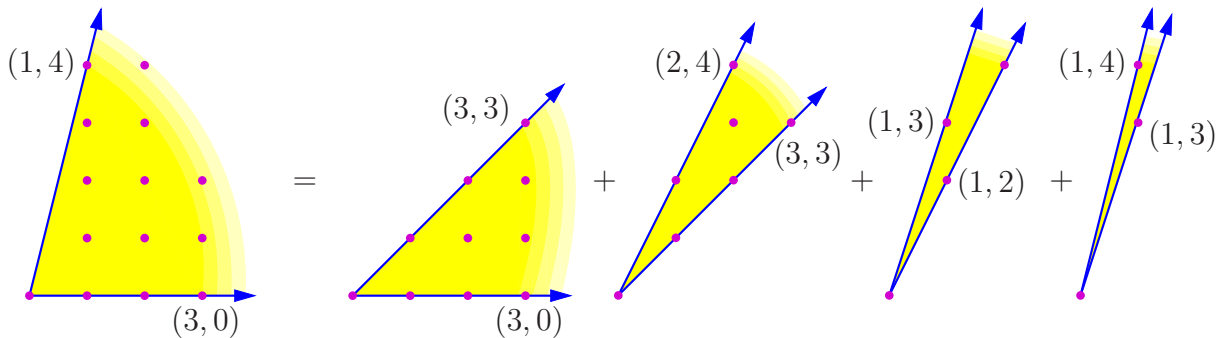
Here *polynomial time* refers to the input data of $\mathcal{K}$, that is, the algorithm runs in time polynomial in the input length of, say, the halfspace description of $\mathcal{K}$.

Brion's Formula implies that an identical complexity statement can be made about the generating function $\sigma_{\mathcal{P}}(x)$ for any rational polytope $\mathcal{P}$. From here it is a short step (which nevertheless needs some justification) to see that one can *count* integer points in a rational polytope in polynomial time.

We illustrate Barvinok's short signed decomposition for the cone $\mathcal{K} := (0,0) + \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(1,4)$, ignoring cones of smaller dimension.
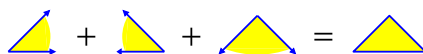

While $\mathcal{K}$ is the difference of two unimodular cones, it has a unique decomposition as a sum of *four* unimodular cones.

In general the cone $(0,0) + \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(1,n)$ is the difference of two unimodular cones, but it has a unique decomposition into n unimodular cones.

Arguably the most famous consequence of Barvinok's Theorem applies to *Ehrhart quasi-polynomials*—the counting functions $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$ in the positve-integer variable t for a given rational polytope $\mathcal{P}$. One can show that the generating function $\sum_{t\geq 1} L_{\mathcal{P}}(t)\, x^t$ is a rational function, and Barvinok's Theorem implies that this rational function can be computed in polynomial time. Barvinok's algorithm has been implemented in the software packages `barvinok` [16] and `LattE` [7]. The method of irrational decomposition has also been implemented in `LattE`, vastly improving its performance [12].

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