SEMIALGEBRAIC SPLINES

MICHAEL DIPASQUALE, FRANK SOTTILE, AND LANYIN SUN

Abstract. We study bivariate splines over partitions defined by arcs of irreducible algebraic curves, which we call semialgebraic splines. Such splines were first considered by Wang, Chui, and Stiller. We compute the dimension of the space of semialgebraic splines in two extreme cases when the cell decomposition has a single interior vertex. If the forms defining the edges span a two-dimensional space of forms of degree n, then we compute the dimension of the spline space in every degree. In the other extreme, the curves have distinct slopes at the central vertex and do not simultaneously vanish at any other point. In this case we give a formula for the dimension of the spline space in large degree and bound how large the degree must be for the formula to be correct. We also study the dimension of the spline space in the case of a single interior vertex in some examples where the curves do not satisfy either extreme. The results are derived using commutative and homological algebra.

1. Introduction

A multivariate spline is a function on a domain in $\mathbb{R}^n$ that is piecewise a polynomial with respect to a cell decomposition Δ of the domain. A fundamental question is to describe the vector space of splines on Δ that have a given smoothness and whose polynomial constituents have at most a fixed degree. Traditionally, Δ is a simplicial [23] or polyhedral [21] complex. However, even in finite elements and isogeometric analysis it is useful to consider splines on more general cell decompositions, where some line segments are replaced by arcs [8]. This occurs when modeling nonlinear features such as circles and cylinders [14]. We consider the case when Δ is a planar complex whose cells are bounded by arcs of algebraic curves. We will call splines on Δ *semialgebraic splines*, as the cells are semialgebraic sets.

Semialgebraic splines were first studied by Wang and Chui [6, 24, 25], who observed that smoothness is equivalent to the usual existence of smoothing cofactors across 'edges' (arcs of algebraic curves) satisfying conformality conditions at each vertex. Building on this observation, Stiller [22] used tools from algebraic geometry to determine the dimensions of spline spaces in some cases when Δ has a single interior vertex. When Δ is a polyhedral complex, classical spline spaces were recast in terms of graded modules and homological algebra by Billera [1], who developed this with Rose [2, 3]. Further foundational work by

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Schenck and Stillman may be found in [19, 20]. We translate this homological machinery to semialgebraic splines and give the first step in a general treatment of the dimensions of bivariate semialgebraic spline spaces. That is, we study semialgebraic splines when Δ has a single interior vertex. This was also considered by Stiller [22], but we invoke hypotheses that are complementary to and less restrictive than his (see Remark 4.6). We have found no systematic study of semi-algebraic spline spaces since Stiller's paper. Wang and Zhu have studied the zero-locus of semi-algebraic splines [26], these 'piecewise semi-algebraic varieties' are simply semi-algebraic sets.

Our main contributions in the case of a single interior vertex are two-fold. Suppose that the forms defining the interior algebraic curves all have degree n (where n is a positive integer), and they span only a two-dimensional subspace of the forms of degree n. In this case we compute the dimension of the spline space in any degree (Corollary 3.6). Now consider the 'generic' case when the forms defining the interior algebraic curves have any prescribed degree, but they have no other common zeros in the complex projective plane, $\mathbb{P}^2(\mathbb{C})$. In this case we compute a polynomial of degree two in the variable d (d being the degree of a spline), the *Hilbert polynomial*, which gives the dimension of the spline space for $d \gg 0$ (Corollary 4.2). We also give a bound on how large d must be for this polynomial to give the dimension of the spline space (Corollary 5.10). Since the general bound may be far from optimal, we give a much tighter bound in the case of three interior curves (Proposition 5.12).

In Section 2, we fix our notation, give background on spline modules, and present some examples. In Section 3, the forms defining the curves all have degree n and span a two-dimensional vector space, so that they intersect in n^2 points in $\mathbb{P}^2(\mathbb{C})$ (counting multiplicities). In Section 4, the curves are smooth at the unique interior vertex v of Δ and their only common zero in $P^2(\mathbb{C})$ is v. In Section 5 we address the question of how large the degree must be for the formula of Corollary 4.2 to hold using the notion of Castelnuovo-Mumford regularity. We also expand on work of Stiller [22], showing how results from the algebraic theory of linkage can be used to evaluate the dimension of the spline space in low degree in some instances. We close with Section 6 where we give examples that suggest extensions of this work when Δ has a single interior vertex.

2. Spline Modules

Billera [1] introduced methods from homological algebra into the study of splines. This was refined by Billera and Rose [2, 3] and by Schenck and Stillman [19, 20], who viewed spaces of splines as homogeneous summands of graded modules over the polynomial ring, so that the dimension of spline spaces is given by the Hilbert function of the module. We fix our notation and make the straightforward observation that this homological approach carries over to semialgebraic splines, in the same spirit as Wang's observation that smoothing cofactors and conformality conditions for polyhedral splines carry over to semialgebraic splines [24, 25]. For more complete background, we recommend § 8.3 of [7]. Background concerning free resolutions and modules may be found in [7] or [11].

Let Δ be a finite cell complex in the plane $\mathbb{R}^2$, whose 1-cells are arcs of irreducible real algebraic curves. We call the 2-cells of Δ, *faces*, the 1-cells, *edges*, and 0-cells, *vertices*.

We assume that each vertex and edge of Δ lies in the boundary of some face (it is *pure*), that it is connected, and that it is *hereditary*: for any faces σ, σ' sharing a vertex υ, there is a sequence $\sigma = \sigma_0, \sigma_1 \dots, \sigma_n = \sigma'$ of faces containing υ such that each pair σ_{i-1}, σ_i for $i = 1, \dots, n$ shares an edge. Write $|\Delta| \subset \mathbb{R}^2$ for the support of Δ. We assume that $|\Delta|$ is contractible and require that each connected component of the intersection of two cells of Δ is a cell of Δ. Write Δ_i° for the set of i-cells of Δ that lie in the interior of $|\Delta|$. Every face σ of Δ inherits the orientation of $\mathbb{R}^2$ and we fix an orientation of each edge $\tau \in \Delta_1^\circ$.

Fig. 1 shows a cell complex with one interior vertex, three interior edges (oriented inwards) and three faces. Placing that vertex at the origin, $|\Delta|$ is the unit disc, and its

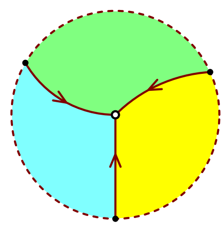


FIGURE 1. Cell complex with one interior vertex.

edges (in clockwise order) lie along the negative y-axis, the circle of radius 1 centered at $(0, 1)$, and the circle of radius $\sqrt{2}$ centered at $(1, -1)$.

Let R be a ring. A *chain complex* $\mathcal{C}$ is a sequence $C_0, C_1, \dots, C_n$ of R-modules with R-module maps $\partial_i \colon C_i \to C_{i-1}$, whose compositions vanish, $\partial_{i-1} \circ \partial_i = 0$, so that the kernel of ∂_{i-1} contains the image of ∂_i. (Here, $C_{-1} = C_{n+1} = 0$.) The *homology* of $\mathcal{C}$ is the sequence of R-modules $H_i(\mathcal{C}) := \operatorname{kernel}(\partial_{i-1}) / \operatorname{image}(\partial_i)$, for $i = 0, \dots, n$.

Let $R(\Delta)$ be the chain complex whose ith module has a basis given by the cells of Δ_i° and whose maps are induced by the boundary maps on the cells. For the cell complex Δ of Fig. 1, $R(\Delta)$ is $R^3 \to R^3 \to R$. Since the interior cells subdivide $|\Delta|$ with its boundary removed, the homology of the chain complex $R(\Delta)$ is the relative homology $H_i(|\Delta|, \partial|\Delta|; R)$. This always vanishes when $i = 0$. If $|\Delta|$ is connected and contractible, then we also have that $H_1(R(\Delta)) = 0$ and $H_2(R(\Delta)) = R$.

For integers $r, d \geq 0$, let $\widetilde{C}_d^r(\Delta)$ be the real vector space of functions f on $|\Delta|$ which have continuous rth order partial derivatives and whose restriction to each face σ of Δ is a polynomial f_σ of degree at most d. By [24] (see also [3, Cor. 1.3]), elements $f \in \widetilde{C}_d^r(\Delta)$ are lists $(f_\sigma \mid \sigma \in \Delta_2)$ of polynomials such that if $\tau \in \Delta_1^\circ$ is an interior edge with defining equation $g_\tau(x, y) = 0$ that borders the two-dimensional faces σ, σ', then g_τ^{r+1} divides the difference $f_\sigma - f_{\sigma'}$. (The quotient is the smoothing cofactor at τ.)

Fig. 2 displays the graphs of two splines on the complex Δ of Fig. 1. The spline on the left lies in $\widetilde{C}_3^0(\Delta)$ and that on the right lies in $\widetilde{C}_6^1(\Delta)$. These are splines on Δ of the lowest degree for the given smoothness that are not the restriction of a single polynomial.

Billera and Rose [2] observed that homogenizing spline spaces enables a global homological approach to computing them. Let $S := \mathbb{R}[x, y, z]$ be the homogeneous coordinate ring of $\mathbb{P}^2(\mathbb{R})$. Write $\langle G_1, \dots, G_t \rangle$ for the ideal of S generated by polynomials $G_1, \dots, G_t \in S$. Let $C_d^r(\Delta)$ be the vector space of lists $(F_\sigma \mid \sigma \in \Delta_2)$ of homogeneous forms in S of degree

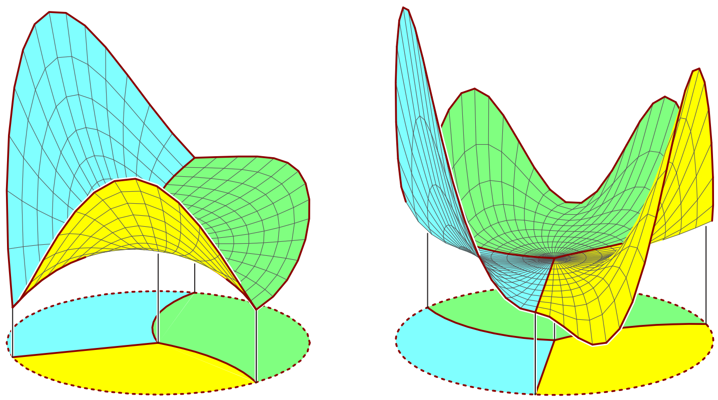


FIGURE 2. Graphs of splines.

d such that if $f_\sigma := F_\sigma(x,y,1)$ is the dehomogenization of F_σ, then $(f_\sigma \mid \sigma \in \Delta_2) \in \widetilde{C}^r_d(\Delta)$. Define $C^r(\Delta) := \bigoplus_d C^r_d(\Delta)$ to be the direct sum of these homogenized spline spaces. Call $C^r(\Delta)$ the *spline module*. It is a graded module of the graded ring S.

Lemma 2.1. *The spline module $C^r(\Delta)$ is finitely generated. It is the kernel of the map*

$$S^{\Delta_2} \simeq \bigoplus_{\sigma\in\Delta_2} S \xrightarrow{\partial} \bigoplus_{\tau\in\Delta_1^\circ} S/\langle G_\tau^{r+1}\rangle \ , \tag{1}$$

where G_τ is the homogeneous form defining the edge τ and if $F = (F_\sigma \mid \sigma \in \Delta_2) \in S^{\Delta_2}$ and $\tau \in \Delta_1^\circ$, then the τ-component of ∂F is the difference $F_\sigma - F_{\sigma'}$, where τ is a component of the intersection $\sigma \cap \sigma'$ and its the orientation agrees with that induced from σ, but is opposite to that induced from σ'.

Let $M = \bigoplus_d M_d$ be a finitely generated graded S-module. The *Hilbert function* of M records the dimensions of its graded pieces, $HF(M,d) := \dim_{\mathbb{R}} M_d$. There is an integer $d_0 \geq 0$ such that if $d > d_0$, then the Hilbert function is a polynomial, called the *Hilbert polynomial* of M, $HP(M,d)$ [11]. The *postulation number* of M is the minimal such d_0, the greatest integer at which the Hilbert function and Hilbert polynomial disagree. The reason for these definitions is that the problem of computing the dimensions $\dim C^r_d(\Delta)$ of the spline spaces is equivalent to computing the Hilbert function of the spline module $C^r(\Delta)$, which equals its Hilbert polynomial for $d > d_0$.

Table 1 gives the Hilbert function and Hilbert polynomial of $C^r(\Delta)$ for $r = 0, \ldots, 3$, where Δ is the cell complex of Fig. 1. The polynomials may be verified using Theorem 4.2. Its last row is the Hilbert function/polynomial of $\mathbb{R}[x,y,z]$, these are splines that are restrictions of polynomials on $\mathbb{R}^2$. The only splines in degrees less than $3r+3$ are such restrictions. The last column is the postulation number.

For $\tau \in \Delta_1^\circ$, define $J(\tau) := \langle G_\tau^{r+1}\rangle$, the principal ideal generated by G_τ^{r+1} and for $\upsilon \in \Delta_0^\circ$, define $J(\upsilon)$ to be the ideal generated by all $J(\tau)$ where τ is incident on υ. Let

TABLE 1. Hilbert function and polynomial of $C^r(\Delta)$.

$r\backslash d$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	Polynomial	d_0
0	1	3	6	13	23	36	52	71	93	118	146	177	211	248	$\frac{3}{2}d^2-\frac{1}{2}d+1$	1
1	1	3	6	10	15	21	30	44	61	81	104	130	159	191	$\frac{3}{2}d^2-\frac{11}{2}d+9$	5
2	1	3	6	10	15	21	28	36	45	57	73	94	118	145	$\frac{3}{2}d^2-\frac{21}{2}d+28$	9
3	1	3	6	10	15	21	28	36	45	55	66	78	93	111	$\frac{3}{2}d^2-\frac{31}{2}d+57$	13
$\binom{d+2}{2}$	1	3	6	10	15	21	28	36	45	55	66	78	91	105	$\frac{1}{2}d^2+\frac{1}{2}d$	0

$\mathcal{J}_1$ and $\mathcal{J}_0$ be the direct sums of these ideals,

$$\mathcal{J}_1 \ := \bigoplus_{\tau\in\Delta_1^\circ} J(\tau) \qquad \text{and} \qquad \mathcal{J}_0 \ := \bigoplus_{\upsilon\in\Delta_0^\circ} J(\upsilon)\,.$$

Then $\mathcal{J}\colon \mathcal{J}_1 \xrightarrow{\partial_1} \mathcal{J}_0$ is a complex of S-modules, with ∂_1 the obvious map. This is a subcomplex of the chain complex $\mathcal{S} := S(\Delta)$ that computes the homology of the pair $H_*(|\Delta|,\partial|\Delta|;S)$. We have the short exact sequence of complexes of S-modules,

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}/\mathcal{J} \longrightarrow 0\,, \tag{2}$$

where $\mathcal{S}/\mathcal{J}$ is the quotient complex,

$$0 \longrightarrow \bigoplus_{\sigma\in\Delta_2} S \xrightarrow{\partial_2} \bigoplus_{\tau\in\Delta_1^\circ} S/J(\tau) \xrightarrow{\partial_1} \bigoplus_{\upsilon\in\Delta_0^\circ} S/J(\upsilon) \longrightarrow 0.$$

Observe that $C^r(\Delta)$ is the kernel of ∂_2. That is, $C^r(\Delta) = H_2(\mathcal{S}/\mathcal{J})$. The short exact sequence (2) gives the long exact sequence in homology (note that $H_2(\mathcal{J})=0$).

$$\begin{aligned} 0 \to H_2(\mathcal{S}) \to H_2(\mathcal{S}/\mathcal{J}) \to H_1(\mathcal{J}) \to H_1(\mathcal{S}) & \\ \to H_1(\mathcal{S}/\mathcal{J}) \to H_0(\mathcal{J}) \to H_0(\mathcal{S}) \to H_0(\mathcal{S}/\mathcal{J}) \to 0\,. \end{aligned}$$

Proposition 2.2. *We have $H_0(\mathcal{S}/\mathcal{J})=0$. If the support $|\Delta|$ of Δ is contractible, then $H_1(\mathcal{S}/\mathcal{J}) \simeq H_0(\mathcal{J})$ and $C^r(\Delta) \simeq S \oplus H_1(\mathcal{J})$, with the factor of S the splines that are restrictions of polynomials.*

If there is a unique interior vertex υ, then $0 = H_1(\mathcal{S}/\mathcal{J}) = H_0(\mathcal{J})$ and $H_1(\mathcal{J})$ is the module of syzygies on the list of forms $(G_\tau^{r+1} \mid \tau\in\Delta_1^\circ)$.

Proof. Since $\mathcal{S}$ is the complex $S(\Delta)$, $H_0(\mathcal{S})=0$ so that $H_0(\mathcal{S}/\mathcal{J})=0$. If $|\Delta|$ is contractible, then $H_1(\mathcal{S})=0$ and $H_2(\mathcal{S})=S$. Thus the remaining long exact sequence splits into sequences of lengths 2 and 3. The first gives $H_1(\mathcal{S}/\mathcal{J}) \simeq H_0(\mathcal{J})$ and the second is

$$0 \longrightarrow S \longrightarrow C^r(\Delta) \longrightarrow H_1(\mathcal{J}) \longrightarrow 0\,,$$

giving the direct sum decomposition $C^r(\Delta) \simeq S \oplus H_1(\mathcal{J})$, as the first map has a splitting $(F_\sigma \mid \sigma\in\Delta_2) \mapsto F_{\sigma_0}$ given by any $\sigma_0\in\Delta_2$. The kernel $H_2(\mathcal{S})$ of the map ∂_2 of $\mathcal{S}$ is the submodule of splines that are restrictions of polynomials.

Lastly, if there is a unique interior vertex υ, then the forms $\{G_\tau^{r+1} \mid \tau\in\Delta_1^\circ\}$ generate $J(\upsilon)$. Thus $H_0(\mathcal{J})=0$ and the complex $\mathcal{J}$ is the first step in the resolution of the ideal $J(\upsilon)$ given the generators $(G_\tau^{r+1} \mid \tau \in \Delta_1^\circ)$. It follows that $H_1(\mathcal{J})$ is the module of syzygies (or relations) on the forms $(G_\tau^{r+1} \mid \tau \in \Delta_1^\circ)$. When $J(\upsilon)$ is minimally generated by $(G_\tau^{r+1} \mid \tau \in \Delta_1^\circ)$, then $H_1(\mathcal{J}) \simeq \operatorname{syz}(J(\upsilon))$. □

Write ϕ_2 for the number of faces of Δ, ϕ_1 for the number of interior edges, and ϕ_0 for the number of interior vertices, and for an interior edge $\tau \in \Delta_1^\circ$, let n_τ be the degree of the form G_τ defining τ.

Corollary 2.3. *Suppose that the support $|\Delta|$ of Δ is contractible. Then for r and d,*

$$\dim C_d^r(\Delta) \;=\; (\phi_2-\phi_1)\tbinom{d+2}{2}+\sum_{\tau\in\Delta_1^\circ}\tbinom{d-(r+1)n_\tau+2}{2}+\sum_{\upsilon\in\Delta_0^\circ}\dim(S/J(\upsilon))_d+\dim H_0(\mathcal{J})_d\,. \tag{3}$$

When Δ has a unique interior vertex υ, we have

$$\dim C_d^r(\Delta) \;=\; \sum_{\tau\in\Delta_1^\circ}\tbinom{d-(r+1)n_\tau+2}{2} + \dim(S/J(\upsilon))_d\,. \tag{4}$$

For $d \gg 0$, $\dim(S/J(\upsilon))_d$ is the degree of the scheme defined by $J(\upsilon)$.

Remark 2.4. If $I \subset S$ is a homogeneous ideal so that $HP(S/I,d) = \mu$ for some $\mu \in \mathbb{Z}_{>0}$ (equivalently $\dim(S/I)_d = \mu$ for $d \gg 0$), then μ is called the *multiplicity* of I, or the *degree* of the scheme defined by I.

Formula (4) is [22, Cor. 3.2], which is for mixed splines (see Remark 2.5). Recall that $S(-a)$ is the free S-module with one generator of degree a.

Proof. From the complex $\mathcal{S}/\mathcal{J}$, we have

$$\begin{aligned} &HF(H_2(\mathcal{S}/\mathcal{J}),d) \;-\; HF(H_1(\mathcal{S}/\mathcal{J}),d) \;+\; HF(H_0(\mathcal{S}/\mathcal{J}),d) \;= \\ &\qquad HF((\mathcal{S}/\mathcal{J})_2,d) \;-\; HF((\mathcal{S}/\mathcal{J})_1,d) \;+\; HF((\mathcal{S}/\mathcal{J})_0,d)\,. \end{aligned}$$

As $|\Delta|$ is contractible, $H_0(\mathcal{S}/\mathcal{J}) = 0$ and $H_1(\mathcal{S}/\mathcal{J}) \simeq H_0(\mathcal{J})$. Since $(\mathcal{S}/\mathcal{J})_2 \simeq S^{\Delta_2}$, its Hilbert function is $\phi_2\binom{d+2}{2}$. From the sum of short exact sequences defining $(\mathcal{S}/\mathcal{J})_1$,

$$\bigoplus_{\tau\in\Delta_1^\circ}\Big(S(-(r+1)n_\tau) \xrightarrow{\;\cdot G_\tau^{r+1}\;} S \longrightarrow S/\langle G_\tau^{r+1}\rangle \;=\; S/J(\tau)\Big)\,,$$

we have that

$$HF((\mathcal{S}/\mathcal{J})_1,d) \;=\; \phi_1\tbinom{d+2}{2} \;-\; \sum_{\tau\in\Delta_1^\circ}\tbinom{d-(r+1)n_\tau+2}{2}\,.$$

As $C^r(\Delta) = H_2(\mathcal{S}/\mathcal{J})$, and $(\mathcal{S}/\mathcal{J})_0 = \bigoplus_{\upsilon\in\Delta_0} S/J(\upsilon)$, this implies formula (3).

When Δ has a unique interior vertex υ, $H_0(\mathcal{J}) = 0$ and $\phi_2 = \phi_1$, giving (4). □

Remark 2.5. This formalism extends to the case of mixed splines as studied in [9, 10, 15, 22]. For each edge $\tau \in \Delta_1^0$ let $\alpha(\tau)$ be a nonnegative integer. Then $C^\alpha(\Delta)$ denotes the splines $(F_\sigma \mid \sigma \in \Delta_2)$ on Δ where if τ is an edge common to both σ and σ', then $G_\tau^{\alpha(\tau)+1}$ divides the difference $F_\sigma - F_{\sigma'}$. This is the kernel of the map of graded modules

$$\bigoplus_{\sigma\in\Delta_2} S \;\xrightarrow{\;\partial_2\;}\; \bigoplus_{\tau\in\Delta_1^\circ} S/\langle G_\tau^{\alpha(\tau)+1}\rangle\,.$$

This formalism extends as well to splines over cell complexes Δ of any dimension whose cells are semialgebraic sets. We leave the corresponding statements to the reader.

3. Semialgebraic splines with a single vertex I

We consider the first nontrivial case of semialgebraic splines—when the complex Δ has a single interior vertex υ and the forms defining the edges incident on υ form a pencil. That is, the forms G_τ span a two-dimensional subspace in the space of all forms of degree n vanishing at υ. This is always the case when the edges are line segments with at least two distinct slopes (so that the polynomials G_τ are linear forms). We determine the Hilbert polynomial of the spline module, showing that the multiplicity (see Remark 2.4) of the scheme $S/J(\upsilon)$ is n^2 times the multiplicity of the scheme S/I, where I is an ideal generated by powers of linear forms vanishing at υ. This has a simple form, which we give in Corollary 3.4.

This shows that the Hilbert polynomial of the spline module does not depend upon the real (as in real-number) geometry of the curves underlying the edges τ—it is independent of whether or not the curves are singular at υ or at any other point, and whether or not the other points at which they meet are real, complex, or at infinity.

Suppose that $L_1, \ldots, L_s$ are linear forms in $\mathbb{R}[x, y]$ defining distinct lines through the origin, so that they are pairwise coprime, and let I be the ideal generated by the powers $L_1^{r+1}, \ldots, L_s^{r+1}$. Observe that any $t \leq r+2$ of these powers are linearly independent ($r+2$ is the dimension of the space of forms of degree $r+1$). Recall that S/I has a unique (up to change of basis) *minimal free resolution* of the form

$$F_\bullet \,:\, 0 \longrightarrow F_\delta \xrightarrow{\psi_\delta} F_{\delta-1} \xrightarrow{\psi_{\delta-1}} \cdots \xrightarrow{\psi_1} S$$

with coker $\psi_1 = S/I$ and where the free module F_i equals $\bigoplus_j S(-a_{ij})$. The index δ of the last nonzero free module is the projective dimension of S/I. By the Hilbert Syzygy Theorem [11, Cor. 19.7], the projective dimension of an ideal in a polynomial ring is bounded above by the number of variables (in our case, three). The *Castelnuovo-Mumford regularity* (henceforth *regularity*) of S/I is the number $\max_{i,j}\{a_{ij} - i\}$.

We use the following results of Schenck and Stillman [19], describing the minimal free resolution and regularity of an ideal of powers of linear forms in two variables.

Proposition 3.1 ([19], Thm. 3.1)**.** *Let $I = \langle L_1^{r+1}, \ldots, L_t^{r+1}\rangle$ be an ideal minimally generated by the given powers of linear forms $L_1, \ldots, L_t \in R := \mathbb{R}[x, y]$ with $t > 1$. A minimal free resolution of R/I is given by*

$$R(-r-1-a)^{s_1} \oplus R(-r-2-a)^{s_2} \longrightarrow R(-r-1)^t \longrightarrow R\,, \tag{5}$$

where we have $s_1 := (t-1)a+t-r-2$ and $s_2 := r+1-(t-1)a$ with $a := \lfloor \frac{r+1}{t-1} \rfloor \geq 1$.

If m is the remainder of $r+1$ divided by $t-1$, then $s_1 = t-1-m > 0$ and $s_2 = m \geq 0$. Hence I always has syzygies of degree $r+1+a$.

Corollary 3.2 ([19], Cor. 3.4)**.** *The regularity of R/I is $r+\lceil \frac{r+1}{t-1} \rceil - 1$.*

Remark 3.3. As R/I is a finite-length module, the highest degree of a nonzero element in R/I equals the regularity of R/I [12, Cor. 4.4]. It follows from Corollary 3.2 that I contains all monomials of degree at least $r+\lceil\frac{r+1}{t-1}\rceil$, and thus the ideal $IS\subset S$ contains all monomials of S where the degree in x,y is at least $r+\lceil\frac{r+1}{t-1}\rceil$.

Tensoring the minimal free resolution (5) of R/I with S gives a minimal free resolution of IS (as S is a flat R-module). Taking Euler-Poincaré characteristic gives a formula for the multiplicity of the scheme defined by IS, which is the Hilbert polynomial of S/IS,

$$s_1\binom{d-(r+1+a)+2}{2}+s_2\binom{d-(r+2+a)+2}{2}\ -\ t\binom{d-(r+1)+2}{2}\ +\ \binom{d+2}{2}\ .$$

This simplifies nicely.

Corollary 3.4. *The multiplicity of the scheme defined by I is $\binom{a+r+2}{2}-t\binom{a+1}{2}$.*

3.1. **Curves in a pencil.** Now we suppose that $G_1,\dots,G_N$ are forms of degree n that underlie the edges of Δ, all of which are incident on the point $v=[0:0:1]$. Suppose that these forms define s distinct algebraic curves, that G_1 and G_2 are relatively prime, and each form G_i lies in the linear span of G_1 and G_2, so the curves lie in a pencil.

Proposition 3.5. *Set $t:=\min\{s,r+2\}$, and suppose that $G_1,\dots,G_t$ define distinct curves. Then the ideal $J:=\langle G_i^{r+1}\mid i=1,\dots,N\rangle$ is minimally generated by $G_1^{r+1},\dots,G_t^{r+1}$. Set $a:=\lfloor\frac{r+1}{t-1}\rfloor$. A minimal free resolution of S/J is given by*

$$S((-r-1-a)n)^{s_1}\oplus S((-r-2-a)n)^{s_2}\ \longrightarrow\ S((-r-1)n)^t\ \longrightarrow\ S\,, \tag{6}$$

where $s_1:=(t-1)a+t-r-2$ and $s_2:=r+1-(t-1)a$.

Proof. Since the forms G_1 and G_2 are relatively prime, they form a regular sequence. In this situation, Hartshorne [17] showed that the map $\varphi\colon T:=\mathbb{R}[u_1,u_2]\to S$ defined by $u_i\mapsto G_i$ for $i=1,2$ is an injection, and that S is flat as a T-module.

Let $L_1,\dots,L_N$ be the linear forms in T such that $\varphi(L_i)=G_i$ for $i=1,\dots,N$ and let $I:=\varphi^{-1}(J)$, which is the ideal $\langle L_1^{r+1},\dots,L_N^{r+1}\rangle$. As s of the G_i define distinct curves, the corresponding s linear forms are pairwise relatively prime, and their powers generate I. Then $t=\min\{s,r+2\}$ of these powers are linearly independent and thus are minimal generators of the ideal I. As $G_1,\dots,G_t$ are distinct, the powers $L_1^{r+1},\dots,L_t^{r+1}$ minimally generate I. Since φ is injective and $\varphi(T)=\mathbb{R}[G_1,G_2]$ contains the generators of J, we conclude that J is minimally generated by $G_1^{r+1},\dots,G_t^{r+1}$.

Applying φ to the exact sequence (5) and extending scalars to S gives the sequence (6) of free S-modules. The degrees change, as φ is a map of graded rings only if $\deg(u_i)=\deg(G_i)=n$. The sequence remains exact, as S is flat over $\varphi(T)$, and so it is a resolution of S/J. It remains minimal, as no map has a component of degree zero. ∎

Corollary 3.6. *Let J,n,N,a,t,s_1,s_2 be as in Proposition 3.5. The spline module $C^r(\Delta)$ is free as an S-module. More precisely,*

$$C^r(\Delta)\simeq S\oplus S(-(r+1)n)^{N-t}\oplus S(-(r+1+a)n)^{s_1}\oplus S(-(r+2+a)n)^{s_2}.$$

Its Hilbert function is

$$\dim C^r_d(\Delta)=\binom{d+2}{2}+(N-t)\binom{d-(r+1)n+2}{2}+s_1\binom{d-(r+1+a)n+2}{2}+s_2\binom{d-(r+2+a)n+2}{2}.$$

The multiplicity of the scheme defined by J equals $n^2\left(\binom{a+r+2}{2} - t\binom{a+1}{2}\right)$. The Hilbert polynomial for the spline module is

$$N\binom{d-(r+1)n+2}{2} \ + \ n^2\left(\binom{a+r+2}{2} - t\binom{a+1}{2}\right),$$

where we consider these binomial coefficients as polynomials in d. The postulation number is $(r+1+\lceil\frac{r+1}{t-1}\rceil)n-3$.

Proof. By Proposition 2.2, $C^r(\Delta) \simeq S \oplus H_1(\mathcal{J})$ and $H_1(\mathcal{J})$ is the module of syzygies on $\{G_1^{r+1}, \ldots, G_N^{r+1}\}$. Let these be ordered so that $\{G_1^{r+1}, \ldots, G_t^{r+1}\}$ minimally generate J, while each G_{t+i}^{r+1} for $i = 1, \ldots, N-t$ is a linear combination of $\{G_1^{r+1}, \ldots, G_t^{r+1}\}$. Then

$$H_1(\mathcal{J}) \ \simeq \ S(-(r{+}1)n)^{N-t} \oplus \operatorname{syz}(J(\upsilon)),$$

with a copy of $S(-(r{+}1)n)$ encoding the expression of G_{t+i} in terms of the minimal generators of J. The module $\operatorname{syz}(J(\upsilon))$ is the leftmost module in the minimal free resolution of $S/J(\upsilon)$ given in Proposition 3.5. It is free because $J(\upsilon)$ has projective dimension two. The structure of $C^r(\Delta)$ as a free S-module follows. We deduce the Hilbert function and polynomial from this. The postulation number d_0 is the largest integer which is less than at least one of the roots of the polynomials appearing as numerators in the binomial coefficients in the expression defining the Hilbert function, hence

$$d_0 = \begin{cases} (r+1+a)n-3 & \text{if } s_2 = 0 \\ (r+2+a)n-3 & \text{otherwise} \end{cases},$$

which is the same as $(r+1+\lceil\frac{r+1}{t-1}\rceil)n-3$. □

Observe that the multiplicity of the scheme defined by J is the product of the multiplicity, n^2 of the scheme defined by $\langle G_1, \ldots, G_N\rangle = \langle G_1, G_2\rangle$ and the multiplicity of the scheme defined by powers of linear forms as in Corollary 3.4.

Remark 3.7. The Hilbert function of the spline module $C^r(\Delta)$ when the forms underlying the edges lie in a pencil depends only on the numerical invariants N, s, r, n and *not* on the geometry in $\mathbb{R}^2$ of the curves underlying the edges. We illustrate this remark by considering several cases when $n = 2$ so that the edges are conics that lie in a pencil.

Let $G_1, G_2, G_3 \in \mathbb{R}[x, y, z]$ be nonproportional quadratic forms with $G_3 \in J := \langle G_1, G_2\rangle$, so that the three lie in a pencil, and suppose also that they vanish at $\upsilon = [0:0:1]$. Then J defines a zero-dimensional subscheme of $\mathbb{CP}^2$ of multiplicity four. The G_i are real, so there are several possibilities for the scheme defined by J in $\mathbb{R}^2$ (where $z \neq 0$). Fig. 3 shows four cell complexes Δ with $|\Delta|$ the unit disc having three faces and three edges defined by the quadratic forms G_1, G_2, and G_3 in $\mathbb{R}^2$. Their spline modules all have the same Hilbert function and polynomial, which is displayed in Table 2.

Starting from the upper left and moving clockwise in Fig. 3, we first have $G_1 := x^2 - 6xy + y^2 - 2xz + 6yz$, $G_2 = x^2 + 6xy + y^2 - 2xz - 6yz$, and $G_3 = 5G_1 + 4G_2$. These vanish at the four real points $[0:0:1]$, $[0:2:1]$, and $[1:\pm1:1]$. Next, we have $G_1 = x^2 + xy + y^2 - 2xz$, $G_2 = x^2 + xy - 2xz + 2yz$, and $G_3 = 3G_1 + 2G_2$. These vanish at the two real points $[0:0:1]$, $[2:0:1]$, and the two complex points $[2:\pm2\sqrt{-1}:1]$. For the third, we have $G_1 = 2x^2 + xy - 2y^2 - 4xz + 3yz$, $G_2 = x^2 + 4xy - y^2 - 2xz - 2yz$, and $G_3 = 6G_1 - 5G_2$. These vanish at the points $[0:0:1]$, $[2:0:1]$, and $[1:1:1]$, sharing

TABLE 2. Hilbert function and polynomial of three conics in a pencil.

$r\backslash d$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	Polynomial
0	1	3	7	13	22	34	49	67	88	112	139	169	202	238	$\frac{3}{2}d^2 - \frac{3}{2}d + 4$
1	1	3	6	10	15	21	30	42	57	75	96	120	147	177	$\frac{3}{2}d^2 - \frac{15}{2}d + 21$
2	1	3	6	10	15	21	28	36	46	58	73	91	112	136	$\frac{3}{2}d^2 - \frac{27}{2}d + 58$
3	1	3	6	10	15	21	28	36	45	55	66	78	93	111	$\frac{3}{2}d^2 - \frac{39}{2}d + 111$
4	1	3	6	10	15	21	28	36	45	55	66	78	91	105	$\frac{3}{2}d^2 - \frac{51}{2}d + 184$

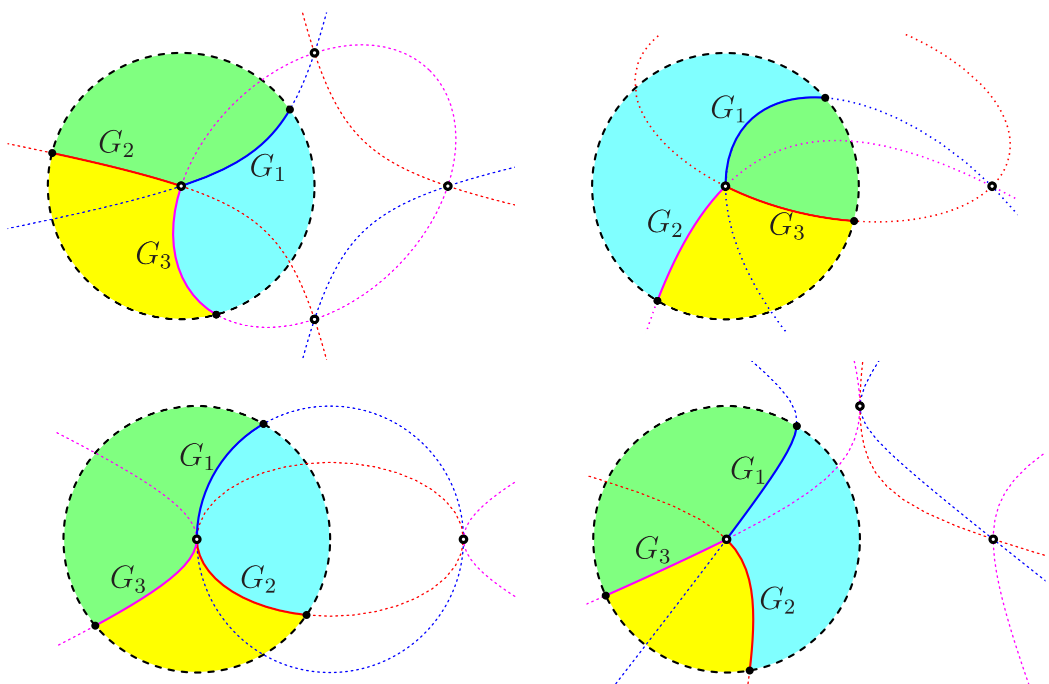


FIGURE 3. (clockwise from upper left) Four real points, two real and two complex points, two real and one double point, and two real double points.

a common vertical tangent at the third point, which has multiplicity two. For our last pencil, let $G_1 = x^2 + y^2 - 2xz$, $G_2 = x^2 + 3y^2 - 2xz$, and $G_3 = 4G_1 - 3G_2$. These vanish only at the points $[0 : 0 : 1]$ and $[2 : 0 : 1]$, and share common vertical tangents at those points, each of which has multiplicity two.

4. SEMIALGEBRAIC SPLINES WITH A SINGLE VERTEX II

Suppose that the complex Δ has a single interior vertex v, but the forms defining the edges incident on v are far from lying in a pencil in that they have no other common zeroes in $\mathbb{P}^2(\mathbb{C})$. We further suppose that the edge forms are smooth at v with distinct tangent directions. Under these assumptions, we determine the Hilbert polynomial of the spline module by showing that the multiplicities of the schemes $S/J(\upsilon)$ and S/I are equal, where I is generated by powers of the forms defining the tangents at υ.

Suppose that $\upsilon = [0 : 0 : 1] \in \mathbb{P}^2(\mathbb{R})$ and there are N interior edges incident on υ, defined by forms $G_1, \ldots, G_N$, of degrees $n_1, \ldots, n_N$ with $[0 : 0 : 1]$ their only common zero. Expand each form G_i as a polynomial in z,

$$G_i \;=\; \sum_{k=1}^{n_i} z^{n_i-k} G_{i,k}\,,$$

where $G_{i,k} \in \mathbb{R}[x, y]$ has degree k. Write $L_i := G_{i,1}$ for the coefficient of z^{n_i-1} in G_i, which is nonzero as G_i is smooth at υ. For an integer $r \geq 0$, let $J := J(\upsilon)$ be the ideal generated by $G_1^{r+1}, \ldots, G_N^{r+1}$ and I be the ideal generated by $L_1^{r+1}, \ldots, L_N^{r+1}$.

Theorem 4.1. *When $L_1, \ldots, L_N$ are distinct, the schemes S/J and S/I have the same Hilbert polynomial and degree.*

We prove this in two steps. In Subsection 4.1 we show that when r is small, these schemes coincide. In Subsection 4.2 we use toric degenerations to show that when r is large, the Hilbert polynomials are equal.

Corollary 4.2. *For $d \gg 0$,*

$$\dim C^r_d(\Delta) = \sum_{i=1}^{N} \tbinom{d-(r+1)n_i+2}{2} \;+\; \tbinom{r+a+2}{2} - t\tbinom{a+1}{2}\,,$$

where $t := \min\{N, r + 2\}$ and $a := \lfloor \frac{r+1}{t-1} \rfloor$.

Proof. It is equivalent to show that the Hilbert polynomial of $C^r(\Delta)$ is $\sum_{i=1}^{N} \binom{d-(r+1)n_i+2}{2} + \binom{r+a+2}{2} - t\binom{a+1}{2}$. This expression for the Hilbert polynomial follows from Corollary 2.3. $\square$

Remark 4.3. We address how large d must be in the statement of Corollary 4.2 in Section 5.

4.1. **Low powers.** Let $G_1, \ldots, G_N$, $L_1, \ldots, L_N$, I, and J be as above. Suppose that I is minimally generated by t of the powers L_i^{r+1}. We show that S/J and S/I define the same scheme when $2t \geq r + 3$. Let $\mathfrak{m} := \langle x, y, z\rangle$ be the irrelevant ideal. Recall that the saturation $(J : \mathfrak{m}^\infty)$ of the ideal J at $\mathfrak{m}$ is $\{f \mid \exists k \text{ with } \mathfrak{m}^k f \subset J\}$. This defines the same projective scheme as does J.

Lemma 4.4. *If $2t \geq r+3$, then $I = (J : \mathfrak{m}^\infty)$.*

Corollary 4.5. *If $2t \geq r+3$, then S/I and S/J define the same scheme.*

Proof of Lemma 4.4. We first show that $J \subset I$. Recall that $\deg(G_i) = n_i$. If we expand the form G_i^{r+1} of degree $n_i(r+1)$ as a polynomial in decreasing powers of z, we obtain

$$G_i^{r+1} \;=\; z^{(n_i-1)(r+1)} L_i^{r+1} \;+\; \sum_{k=1}^{(n_i-1)(r+1)} z^{(n_i-1)(r+1)-k} K_{i,r+1+k}\,,$$

where $K_{i,a} \in \mathbb{R}[x,y]$ is homogeneous of degree a. Since this degree is at least $r{+}2$ and our hypothesis implies that $2 \geq (r{+}1)/(t{-}1)$, this degree is at least $r{+}\lceil\frac{r+1}{t-1}\rceil$. By Remark 3.3, these polynomials $K_{i,a}$ lie in I. Since $L_i^{r+1} \in I$, we have that $G_i^{r+1} \in I$, and so $J \subset I$.

As υ is the only zero of J, to show that $I = (J\colon \mathfrak{m}^\infty)$, we only need to show that the localizations at υ of S/I and S/J are equal. We assume that the forms have been ordered so that $L_1^{r+1}, \dots, L_t^{r+1}$ are minimal generators of I. Let $J' \subset J$ be the ideal generated by $G_1^{r+1}, \dots, G_t^{r+1}$. It suffices to show that S/J' and S/I have the same localization at υ. Since $J' \subset J \subset I$, there are forms $A_{i,j} \in S = \mathbb{R}[x,y,z]$ such that

$$G_i^{r+1} \;=\; \sum_{j=1}^{t} A_{i,j} L_j^{r+1}\,, \tag{7}$$

for each $i = 1, \dots, t$. To show that the localizations agree, we show that the matrix A is invertible in the localization $S_{\langle x,y\rangle}$ of S at υ, as $\langle x,y\rangle$ defines the point υ.

Each form $A_{i,j}$ has degree $(n_i{-}1)(r{+}1)$. Let $A_{i,j}^{(n_i-1)(r+1)-k}$ denote the coefficient of z^k in the expansion of $A_{i,j}$ as a polynomial in z. The highest power of z appearing in (7) is $(n_i{-}1)(r{+}1)$. If we equate the coefficients of $z^{(n_i-1)(r+1)}$ in (7) (recalling that $G_i = z^{n_i-1}L_i + \cdots$), we obtain

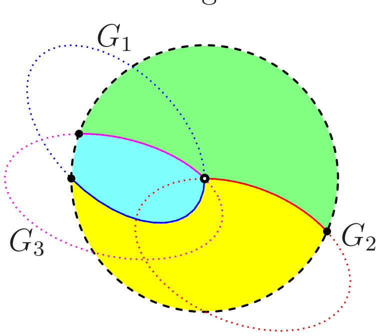
$$L_i^{r+1} \;=\; \sum_{j=1}^{t} A^0_{i,j} L_j^{r+1}\,.$$

As these powers $L_1^{r+1}, \dots, L_t^{r+1}$ are linearly independent, the matrix $A^0_{i,j}$ is the identity.

In particular, the entries of the matrix A that have a pure power of z are exactly the diagonal entries. Thus its determinant has the form $z^{(-t+\sum_i n_i)(r+1)} + g$, where $g \in \langle x,y\rangle$, which implies that A is invertible in the local ring $S_{\langle x,y\rangle}$. $\square$

Remark 4.6. Lemma 4.4 indicates how our results are complementary to Stiller's results in [22]. His most general results in [22, § 4] require that the minimal generators of $J(\upsilon)$, which have degree $n_i(r{+}1)$, are also minimal generators of the saturation of $J(\upsilon)$ (denoted $\mathscr{I}_X$ in [22]). This assumption is also made in [18]. Our assumptions that the edge forms are smooth at υ and that $J(\upsilon)$ is supported only at υ will imply that, to the contrary, the saturation of $J(\upsilon)$ is generated in degrees close to $r{+}1$.

Remark 4.7. The complex Δ on the left below has edges defined by the three homogeneous quadrics on the right.



$$\begin{aligned} G_1 &= xz \;+\; x^2 + xy + y^2 \\ G_2 &= 2yz \;+\; x^2 + xy + 2y^2 \\ G_3 &= \tfrac{3}{2}(x+y)z \;+\; x^2 + xy + 3y^2 \end{aligned}$$

Here $L_1 = x$, $L_2 = y$, and $L_3 = x + y$ and the hypotheses of Lemma 4.4 hold for $r \leq 3$, hence

$$\left(\langle G_1^{r+1}, G_2^{r+1}, G_3^{r+1}\rangle \ :\ \mathfrak{m}^\infty\right) \ =\ \langle L_1^{r+1}, L_2^{r+1}, L_3^{r+1}\rangle\,,$$

for $r = 0, \ldots, 3$. However, for $r = 4$, we do not have the containment

$$\langle G_1^5, G_2^5, G_3^5\rangle \ \subset\ \langle L_1^5, L_2^5, L_3^5\rangle\,.$$

If we set $J := \langle G_1^5, G_2^5, G_3^5\rangle$, then

$$(J\ :\ \mathfrak{m}^\infty) \ =\ \langle 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 - y^5,\ x^5 - y^5,\ y^6,\ xy^5,\ 5x^2y^4 + y^5z\rangle\,.$$

Each generator of $(J : \mathfrak{m}^\infty)$ is in $\langle L_1^5, L_2^5, L_3^5\rangle$ except for the polynomial $5x^2y^4 + y^5z$.

4.2. **Distinct tangents.** The results of Subsection 4.1 imply Theorem 4.1 when r is small relative to N. By Remark 4.7, we cannot have $(J : \mathfrak{m}^\infty) = I$ in general, so other arguments are needed. We use toric degenerations to show that the schemes S/J and S/I have the same Hilbert polynomial. We start with the following simple lemma.

Lemma 4.8. *Suppose that I is an ideal of S defining a scheme supported at $[0 : 0 : 1]$. Then $(I : \mathfrak{m}^\infty) = (I : z^\infty)$.*

Proof. We always have $(I : \mathfrak{m}^\infty) \subset (I : z^\infty)$. Since the only zero of I is $[0 : 0 : 1]$, there is a $k > 0$ such that $\langle x, y\rangle^k \subset I$. Let $f \in (I : z^\infty)$ so that there is some $\ell > 0$ with $fz^\ell \in I$. Let $x^ay^bz^c$ be a monomial of degree at least $k + \ell$. Then either $a + b \geq k$ or $c \geq \ell$. In either case, $fx^ay^bz^c \in I$, and so $f \in (I : \mathfrak{m}^k) \subset (I : \mathfrak{m}^\infty)$, which completes the proof. □

We recall the notion of initial degeneration of an ideal, which is explained in [11, § 15]. Given an integer vector $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{Z}^3$, a monomial $x^ay^bz^c$ of S has a weight $a\omega_1 + b\omega_2 + c\omega_3$. For a polynomial $F \in S$, let $\mathrm{in}_\omega F$ be the sum of terms of F whose monomials have the largest weight with respect to ω among all terms of F. The initial ideal of an ideal I of S with respect to ω is

$$\mathrm{in}_\omega I \ :=\ \langle \mathrm{in}_\omega F \mid F \in I\rangle\,.$$

The utility of this definition is that there is a flat degeneration of the scheme defined by I into the scheme defined by $\mathrm{in}_\omega I$. Consequently, S/I and $S/\mathrm{in}_\omega I$ have the same Hilbert function. This flat degeneration is induced by the torus action on S where $\tau \in \mathbb{C}^\times$ acts on a monomial $x^ay^bz^c$ by $\tau^{-(a\omega_1+b\omega_2+c\omega_3)}x^ay^bz^c$, and is called a toric degeneration.

We now fix the weight vector $\omega := (0, 0, 1)$, so that $\mathrm{in}_\omega F$ consists of the terms of F with the highest power of z. Let $G_1, \ldots, G_N$ be forms in S whose only common zero is $[0 : 0 : 1]$, so that the radical of $J := \langle G_1, \ldots, G_N\rangle$ is $\langle x, y\rangle$. For each i, let c_i be the highest power of z occurring in G_i and define $F_i \in \mathbb{R}[x, y]$ to be the coefficient of z^{c_i} in G_i, so that $z^{c_i}F_i = \mathrm{in}_\omega G_i$. Set $I := \langle F_1, \ldots, F_N\rangle$.

Lemma 4.9. *If $\mathrm{in}_\omega J \subset I$, then S/J and S/I have the same Hilbert polynomial.*

Proof. We first observe that $\langle x, y\rangle$ is the radical of $\mathrm{in}_\omega J$. Since $\sqrt{J} = \langle x, y\rangle$, there is some k with $\langle x, y\rangle^k \subset J$. As $\langle x, y\rangle^k$ is a monomial ideal, we have that $\mathrm{in}_\omega\langle x, y\rangle^k = \langle x, y\rangle^k$ and hence $\langle x, y\rangle^k \subset \mathrm{in}_\omega J$, which shows that $\sqrt{\mathrm{in}_\omega J} = \langle x, y\rangle$.

Since $\mathrm{in}_\omega G_i = z^{c_1}F_i$, we have $I \subset (\mathrm{in}_\omega J\colon z^\infty)$. By Lemma 4.8, $(\mathrm{in}_\omega J : z^\infty) = (\mathrm{in}_\omega J\colon \mathfrak{m}^\infty)$, so we have that $I \subset (\mathrm{in}_\omega J\colon \mathfrak{m}^\infty)$.

As I is defined by polynomials in x and y, if $z^c f \in I$, then $f \in I$, so that I is saturated with respect to $\mathfrak{m} = \langle x, y, z\rangle$. Saturating the inclusion $\mathrm{in}_\omega J \subset I$ gives $(\mathrm{in}_\omega J \colon \mathfrak{m}^\infty) \subset I$. Thus $(\mathrm{in}_\omega J \colon \mathfrak{m}^\infty) = I$ and S/I and $S/\mathrm{in}_\omega J$ have the same Hilbert polynomial. The lemma follows as $S/\mathrm{in}_\omega J$ and S/J have the same Hilbert polynomial, by flatness. □

We need to have that $\mathrm{in}_\omega J \subset I$ to apply Lemma 4.9. By construction, the initial forms $\mathrm{in}_\omega G_i = z^{c_i} F_i$ of the generators of J lie in I. To show that $\mathrm{in}_\omega J \subset I$, we must understand what happens when there is cancellation of these initial forms, which may be accomplished by understanding the syzygies of I

To that end, suppose that $F_1, \dots, F_N$ are minimal generators for I. Write a_{1i} for the degree of F_i. An ideal $I \subset S$ is Cohen-Macaulay if the codimension of S/I is equal to its projective dimension. The ring $\mathbb{R}[x, y]/I$ has finite length (since $\sqrt{I} = \langle x, y\rangle$), so I has projective dimension two, which does not change if I is considered as an ideal of S. Since I has codimension two, it is Cohen-Macaulay and a structure theorem due to Hilbert-Burch [11, Thm. 20.15] says that S/I has a minimal free resolution of the form,

$$0 \longrightarrow \bigoplus_{i=1}^{N-1} S(-a_{2j}) \longrightarrow \bigoplus_{i=1}^{N} S(-a_{1j}) \longrightarrow S \longrightarrow 0\,. \tag{8}$$

Lemma 4.10. *Given I, J and $F_1, \dots, F_N$ as above, if $\max_{s,t} |a_{2s} - a_{2t}| \le 2$, then $\mathrm{in}_\omega J \subset I$.*

Example 4.11. The condition on the second syzygies is necessary. Indeed, suppose that

$$J \;:=\; \langle\, y^6 + x^5 z\,,\, 2x^2y^4 + x^4yz\,,\, x^6 + y^5 z \rangle\,,$$

so $I = \langle x^5, x^4y, y^5\rangle$ and $\sqrt{J} = \langle x, y\rangle$. The minimal free resolution of S/I has the form

$$S(-6) \oplus S(-9) \;\longrightarrow\; S(-5)^3 \;\longrightarrow\; S \;\longrightarrow\; S/I\,,$$

so the condition on the second syzygies of Lemma 4.10 does not hold. Notice that

$$2x^3y^4 - y^7 \;=\; x(2x^2y^4 + x^4yz) - y(y^6 + x^5 z)$$

is in the ideal $\mathrm{in}_\omega J$ but not in the ideal I. Using Macaulay2 [16], we compute that the multiplicity of S/J is 20, while that of the scheme S/I is 21.

Proof of Lemma 4.10. Let $F \in J$ be a homogeneous form of degree d. We will show that $\mathrm{in}_\omega F \in I$. Write F in terms of the generators of J,

$$F \;=\; \sum_{i=1}^{N} H_i G_i\,,$$

where $H_1, \dots, H_N \in S$.

Suppose that $\deg G_i = n_i$. Expanding G_i as a polynomial in z gives

$$G_i \;=\; \sum_{k=0}^{c_i} z^k g_{i, n_i - k}\,,$$

where $g_{i,n_i-k} \in \mathbb{R}[x, y]$ has degree $n_i - k$ and c_i is the highest power of z occurring in G_i. Note that $F_i = g_{i,n_i-c_i}$ and that $a_{1i} = n_i - c_i$ in the Hilbert-Burch resolution (8).

Let γ_i be the highest power of z that occurs in H_i and note that the degree η_i of H_i is $d{-}n_i$. Expand H_i as a polynomial in z,

$$H_i \;=\; \sum_{k=0}^{\gamma_i} z^k h_{i,\eta_i-k}\,,$$

where $h_{i,\eta_i-k} \in \mathbb{R}[x,y]$ has degree $\eta_i - k$.

If we expand F as a polynomial in z, we have

$$F \;:=\; \sum_{k=0}^{\mu} z^k f_{d-k}\,,$$

where $f_{d-k} \in \mathbb{R}[x,y]$ has degree $d-k$ and μ is the maximum of $c_i+\gamma_i$. Then

$$\mathrm{in}_\omega F \;=\; z^m f_{d-m}\,, \qquad \text{where } m \;:=\; \max\{k \mid f_{d-k} \neq 0\} \;\leq\; \mu\,.$$

Suppose that the forms are numbered so that for $i=1,\dots,p$ we have $\mu = c_i+\gamma_i$, but if $i>p$, then $\mu > c_i+\gamma_i$. Then the coefficient $f_{d-\mu}$ of z^μ in F is

$$f_{d-\mu} \;=\; \sum_{i=1}^{p} h_{i,\eta_i-\gamma_i} g_{i,n_i-c_i}\,. \tag{9}$$

Since $g_{i,n_i-c_i} = F_i$ lies in I, if we have $f_{d-\mu} \neq 0$, then $\mathrm{in}_\omega F \in I$ as desired.

Suppose on the contrary that $f_{d-\mu}=0$. Since I is minimally generated by $F_1,\dots,F_N$, the sum in (9) is a syzygy of the ideal I. Then the degree $d{-}\mu$ of the sum (9) is at least one of the degrees a_{2s} in the Hilbert-Burch resolution (8). Let a_2^m be the minimum of the a_{2s} and a_2^M be the maximum. Then $a_2^m \leq d{-}\mu$, and so every term f_{d-k} for $k<\mu$ in the expansion of F with respect to z has degree at least a_2^m+1 (recall that $f_{d-\mu}=0$).

However, the regularity of S/I is a_2^M-2, so that I contains every monomial in x,y of degree at least a_2^M-1. As $|a_{2s}-a_{2t}| \leq 2$, we have $a_2^M \leq a_2^m+2$, and so $a_2^M-1 \leq a_2^m+1$, so that every term f_{d-k} with $k<\mu$ in the expansion of F lies in I, which implies that $F \in I$ and in particular $\mathrm{in}_\omega F \in I$, completing the proof. □

Let $G_1,\dots,G_N \in S$ be forms of the same degree n with $[0:0:1]$ their only common zero such that their linear terms $L_1,\dots,L_N$ at $[0:0:1]$ are distinct and nonzero.

Proof of Theorem 4.1. For any $r \geq 0$, let J be the ideal generated by $G_1^{r+1},\dots,G_N^{r+1}$ and let I be the ideal generated by the powers $L_1^{r+1},\dots,L_N^{r+1}$ of linear forms. These powers of linear forms are distinct and they are linearly independent if and only if $N \leq r+2$.

Suppose first that $N > r+2$. Then I is generated by $t=r+2$ of these powers. In this case, $2t = 2r+4 > r+3$ and the theorem follows by Corollary 4.5.

If instead $N \leq r{+}2$, then I is minimally generated by these powers. By Proposition 3.1 the second syzygies of I differ by at most one, so the hypotheses of Lemma 4.10 hold and $\mathrm{in}_\omega J \subset I$. But then Lemma 4.9 implies the statement of the theorem. □

Remark 4.12. Extensions of Theorem 4.1 and Corollary 4.2 to mixed smoothness (where varying orders of continuity are imposed across interior edges) require a minimal free resolution for an ideal generated by arbitrary powers of linear forms in two variables. This is provided by Geramita and Schenck in [15]. Note that the condition on second syzygies needed in Lemma 4.10 is satisfied for ideals generated by arbitrary powers of linear forms. We leave the details as an exercise for the interested reader.

Remark 4.13. When the hypotheses of Lemma 4.10 hold, we may determine the multiplicity of $S/J(\upsilon)$. We cannot relax the condition on distinct tangents (see Example 6.1), however the forms may have controlled singularities at υ. For instance, if each form has at worst a cusp singularity at υ, then the ideal I defining the tangent cone is generated by (possibly different) powers of linear forms, similar to Remark 4.12. As long as the underlying linear forms are distinct, Lemma 4.10 computes the multiplicity of $S/J(\upsilon)$.

5. Hilbert Function and Regularity

Suppose that the cell complex Δ has a single interior vertex, υ, and that the forms defining its edges are smooth at υ with distinct tangents as in Section 4. The formula of Corollary 4.2 for the Hilbert polynomial of the spline module $C^r(\Delta)$ only gives the dimension of $C^r_d(\Delta)$ when d exceeds the postulation number of the spline module. By Corollary 2.3 (4), the dimension of $C^r_d(\Delta)$ differs from an explicit polynomial by the dimension of $(S/J(\upsilon))_d$, which is the Hilbert function of $S/J(\upsilon)$. We study the entire Hilbert function in some cases and give bounds on the postulation number of $S/J(\upsilon)$ using (Castelnuovo-Mumford) regularity.

5.1. **Hilbert Function.** Stiller used Max Noether's '$AF + BG$ Theorem' to compute the Hilbert function of $S/J(\upsilon)$ in a special case [22, Thm. 4.9] (see Example 6.2). As explained by Eisenbud, Green, and Harris, [13], one generalization of Max Noether's theorem leads to linkage. We use linkage to study the Hilbert function of $S/J(\upsilon)$ in some cases. We will only consider the case when Δ has three edges defined by pairwise coprime forms G_1, G_2, G_3 of degrees n_1, n_2, n_3 as in Fig. 1. Set $J(\upsilon) = \langle G_1^{r+1}, G_2^{r+1}, G_3^{r+1}\rangle$, $K := \langle G_1^{r+1}, G_2^{r+1}\rangle$, $K' := (K : G_3^{r+1})$, and assume $G_3^{r+1} \notin K$ so that $K' \neq S$.

Lemma 5.1. *The ideal K' is Cohen-Macaulay of codimension two.*

Proof. We use that an ideal $I \subset S$ is Cohen-Macaulay of codimension k if and only if $Ext^i(S/I, S) = 0$ for $i \neq k$ (see [11, Pro. 18.4,Co. 19.15]). Now, since $K \subset K'$ and K has codimension two, K' has codimension at least two, so $Ext^i(S/K', S) = 0$ for $i < 2$ [11, Prop. 18.4]. The short exact sequence

$$0 \longrightarrow S/K' \xrightarrow{\cdot G_3} S/K \longrightarrow S/J(\upsilon) \longrightarrow 0\,,$$

yields a long exact sequence in Ext, by which we see the map $Ext^3(S/K, S) \to Ext^3(S/K', S)$ is surjective. Since $Ext^3(S/K, S) = 0$ (as S/K is Cohen-Macaulay of codimension two), $Ext^3(S/K', S) = 0$ as well. It follows that $Ext^2(S/K', S)$ is the only nonvanishing Ext term for S/K', so K' must be Cohen-Macaulay of codimension two. $\square$

Set $K'' := (K : K')$. Since K' is codimension two and Cohen-Macaulay, so is K'' [11, Thm. 21.23], and $(K : K'') = K'$. The ideals K', K'' are said to be *linked*. There is a particularly nice relationship between the Hilbert functions of K' and K''.

Proposition 5.2. [13, Thm. CB7] *Let K, K', K'' be as above and set $s = n_1(r+1) + n_2(r+1) - 3$. Then*

$$\dim\left(K'/K\right)_d \;=\; \operatorname{mult}\left(S/K''\right) \;-\; \dim\left(S/K''\right)_{s-d}\,,$$

where $\dim(S/K'')_{s-d}$ *is zero for* $d > s$.

We show how this proposition may be used when $r = 0$. In what follows, we use the convention that $\binom{A}{B} = 0$ when $A < B$.

Proposition 5.3. *Suppose that $K = \langle G_1, G_2\rangle$ defines a scheme Γ of n_1n_2 distinct points in $\mathbb{P}^2(\mathbb{C})$ and that G_3 vanishes at υ but not any other point of Γ. Then*

$$\dim\left(S/J(\upsilon)\right)_d \;=\; \begin{cases} \dim(S/K)_d - \dim(S/K)_{d-n_3} & d \le n_1 + n_2 + n_3 - 3 \\ 1 & d \ge n_1 + n_2 + n_3 - 2. \end{cases}$$

Also, $\dim\left(S/K\right)_d - \dim\left(S/K\right)_{d-n_3}$ *is equal to*

$$\binom{d+2}{2} \;-\; \left(\sum_{i=1}^{3}\binom{d+2-n_i}{2}\right) \;+\; \left(\sum_{1\le i<j\le 3}\binom{d+2-n_i-n_j}{2}\right) \;-\; \binom{d+2-n_1-n_2-n_3}{2}.$$

Proof. As the points of Γ are distinct, K is the (radical) ideal of all polynomials vanishing on Γ. Since G_3 only vanishes at υ, the ideal $K' := (K : G_3)$ is the ideal of $\Gamma \smallsetminus \{\upsilon\}$. Thus $K'' = (K : K')$ is the ideal of υ, so that $K'' = \langle x, y\rangle$. By Proposition 5.2,

$$\dim(K'/K)_d \;=\; 1 - \dim(S/K'')_{n_1+n_2-3-d} \;=\; \begin{cases} 0 & d \le n_1 + n_2 - 3 \\ 1 & d > n_1 + n_2 - 3 \end{cases}.$$

The ideal K' is related to $J(\upsilon) = \langle G_1, G_2, G_3\rangle$ via the multiplication sequence

$$0 \;\to\; S(-n_3)/K' \;\xrightarrow{\cdot G_3}\; S/K \;\to\; S/J(\upsilon) \;\to\; 0\,. \tag{10}$$

Using (10), the tautological short exact sequence

$$0 \;\to\; K'/K \;\to\; S/K \;\to\; S/K' \;\to\; 0\,,$$

and taking Euler-Poincaré characteristic yields

$$\dim\left(S/J(\upsilon)\right)_d \;=\; \dim\left(S/K\right)_d \;-\; \dim\left(S/K\right)_{d-n_3} \;+\; \dim\left(K'/K\right)_{d-n_3}\,.$$

Observing that $\dim(S/K)_d = n_1n_2$ for $d \ge n_1+n_2-2$ yields the first claim. The expression for $\dim(S/K)_d - \dim(S/K)_{d-n_3}$ follows from

$$\dim(S/K)_d \;=\; \binom{d+2}{2} \;-\; \binom{d+2-n_1}{2} \;-\; \binom{d+2-n_2}{2} \;+\; \binom{d+2-n_1-n_2}{2}\,. \qquad \square$$

Remark 5.4. The hypotheses of Proposition 5.3 can be weakened to assume that G_1, G_2 define a complete intersection scheme $\Gamma \subset \mathbb{P}^2(\mathbb{C})$ in which υ is a reduced point and G_3 vanishes only at $\upsilon \in \Gamma$. This does not change the conclusion.

Corollary 5.5. *Suppose that Δ consists of three arcs of degrees n_1, n_2, n_3 defined by three irreducible polynomials G_1, G_2, G_3 meeting at the single interior point υ. Suppose* *furthermore that* $K = \langle G_1, G_2 \rangle$ *defines a set* Γ *of* $n_1 n_2$ *distinct points in* $\mathbb{P}^2(\mathbb{C})$ *and that* G_3 *vanishes at* υ *but not any other point of* Γ*. Then*

$$\dim C^0_d(\Delta) \;=\; \begin{cases} \binom{d+2}{2} + \displaystyle\sum_{1\le i<j\le 3} \binom{d+2-n_i-n_j}{2} & d \le n_1+n_2+n_3-3 \\ 1 + \displaystyle\sum_{i=1}^{3} \binom{d+2-n_i}{2} & d \ge n_1+n_2+n_3-2. \end{cases}$$

Proof. This follows from Proposition 5.3 and Corollary 2.3. The absence of the term $\binom{d+2-n_1-n_2-n_3}{2}$ is because it vanishes when $d \le n_1+n_2+n_3-3$. $\square$

Example 5.6. Suppose Δ has three interior edges defined by irreducible cubics G_1, G_2, G_3, so that G_1, G_2 vanish simultaneously at a set X of nine distinct points and G_3 vanishes at the central vertex υ but no other point of X. From Corollary 5.5,

$$\dim C^0_d(\Delta) = \begin{cases} \binom{d+2}{2} + 3\binom{d-4}{2} & d \le 6 \\ 1 + 3\binom{d-1}{2} & d \ge 7. \end{cases}$$

These dimensions are recorded in the following table.

d	0	1	2	3	4	5	≥ 6
$\dim C^0_d(\Delta)$	1	3	6	10	15	21	$\frac{3}{2}d^2 - \frac{9}{2}d + 4$

.

Remark 5.7. The computation $\dim(S/J(\upsilon))_6 = 1$ has a hidden application of the classical Cayley-Bacharach theorem; namely that $\dim(K'/K)_3 = 0$. This statement says that any cubic vanishing on eight of the nine points defined by K must also vanish on the ninth point. Proposition 5.2 generalizes this classical result.

5.2. **Regularity.** As determining the Hilbert function of $S/J(\upsilon)$ is difficult, we turn now to bounding its postulation number. This is controlled by the regularity of $S/J(\upsilon)$.

Proposition 5.8 ([12], Thm. 4.2)**.** *The Hilbert function* $\dim(S/J(\upsilon))_d$ *agrees with the Hilbert polynomial for* $d \ge \mathrm{reg}(S/J(\upsilon)) + 1$*. Thus, the postulation number of* $S/J(\upsilon)$ *is at most the regularity of* $S/J(\upsilon)$*.*

The regularity of quotients S/I for some ideal I has been studied intensively. One of the tightest general bounds applicable to our situation is due to Chardin and Fall [5].

Proposition 5.9. [5, Cor. 0.2] *Let* S *be a polynomial ring in three variables and* I *an ideal generated in degree at most* n *satisfying* $\dim(S/I) \le 1$*. Then* $\mathrm{reg}(S/I) \le 3(n-1)$*.*

Corollary 5.10. *Suppose* Δ *has a single interior vertex* υ *and* N *edges defined by forms* $G_1, \dots, G_N$ *of degrees* $n_1 \le \cdots \le n_N = n$*, meeting smoothly at* υ *with distinct tangents. Set* $t = \min\{N, r+2\}$ *and* $a = \lfloor \frac{r+1}{t-1} \rfloor$*. Then*

$$\dim C^r_d(\Delta) \;=\; \sum_{i=1}^{N} \binom{d-n_i(r+1)+2}{2} + \binom{r+a+2}{2} - t\binom{a+1}{2}$$

for $d \ge 3n(r+1) - 2$*.*

Proof. This follows from Corollary 4.2, Proposition 5.8, and Proposition 5.9. □

The bound in Corollary 5.10 is not optimal (see Table 3). We derive a tighter bound when Δ has three edges defined by forms G_1, G_2, G_3 of degrees n_1, n_2, n_3 meeting at a single interior vertex v, as in Fig. 1. We take v to be the point $[0 : 0 : 1]$ with ideal $\langle x, y\rangle$. Then $J(v) := \langle G_1^{r+1}, G_2^{r+1}, G_3^{r+1}\rangle$. Even in this simple case determining the Hilbert function is difficult. (See also [18], where three-generated ideals in $\mathbb{C}[x, y, z]$ are studied in the context of plane Cremona maps.)

Our regularity bound is a translation of [18, Thm. 1.2]. We use *local cohomology*. Let $\mathfrak{m} = \langle x, y, z\rangle$. The zeroth local cohomology of an S-module M is

$$H^0_{\mathfrak{m}}(M) \ := \ \{m \in M \mid \mathfrak{m}^k m = 0 \text{ for some } k \geq 0\}.$$

If I is an ideal of S, then $H^0_{\mathfrak{m}}(S/I) = (I : \mathfrak{m}^\infty)/I$.

For $i > 0$, the local cohomology functors $H^i_{\mathfrak{m}}(\ )$ for $i > 0$ are the *right derived functors* of $H^0_{\mathfrak{m}}(\ )$. If $M \to \mathcal{I}$ is an injective resolution of M then $H^i_{\mathfrak{m}}(M) := H^i(H^0_{\mathfrak{m}}(\mathcal{I}))$, the i^{th} cohomology of the complex $H^0_{\mathfrak{m}}(\mathcal{I})$. For more on local cohomology, see [12, App. 1]. If M is graded and finitely generated, then $H^i_{\mathfrak{m}}(M)$ is graded and *Artinian* in that $H^i_{\mathfrak{m}}(M)_d = 0$ for $d \gg 0$. Also $H^i_{\mathfrak{m}}(M) \neq 0$ only for the range $\operatorname{depth}(M) \leq i \leq \dim(M)$ [12, Prop. A1.16]. These are important since $\operatorname{reg}(M)$ may be identified using local cohomology.

Proposition 5.11. [12, Thm 4.3] *If M is a finitely generated graded S-module, then* $\operatorname{reg}(M)$ *is the smallest integer d satisfying:*

(1) $H^0_{\mathfrak{m}}(M)_d \neq 0$, *and*
(2) $H^i_{\mathfrak{m}}(M)_{i+d-1} = 0$ *for all $i > 0$.*

Let $(S/I)^*$ be the graded dual of S/I, in degree $-d$ it is the dual vector space to $(S/I)_d$. Given a graded S-module M of finite length, $\alpha(M)$ is the lowest degree of a nonzero homogeneous component of M and $\Omega(M)$ is the highest degree.

Proposition 5.12. *Let G_1, G_2, G_3 be forms of degrees $1 \leq n_1 \leq n_2 \leq n_3$, with $n_3 \geq 2$ whose only common zero in $\mathbb{P}^2$ is v. Then*

$$\operatorname{reg}(S/J(v)) \ \leq \ (n_1{+}n_2{+}n_3{-}1)(r{+}1) - 3.$$

If $2t \geq r{+}3$ and $n_1 > 1$, with t as in the statement of Lemma 4.4, then equality holds.

Proof. Let $J = J(v)$. We show first that $\Omega(H^0_{\mathfrak{m}}(S/J)) \leq (n_1{+}n_2{+}n_3{-}1)(r{+}1) - 3$, with equality if $2t \geq r + 3$. This bound will follow from [4, Lem. 5.8].

Specializing the second part of [4, Lem. 5.8] to the case $i = 0$ gives

$$H^0_{\mathfrak{m}}(S/J)((n_1{+}n_2{+}n_3)(r{+}1){-}3) \ \cong \ H^0_{\mathfrak{m}}(S/J)^*\,,$$

where the $(n_1{+}n_2{+}n_3)(r{+}1){-}3$ in parentheses denotes a graded shift of $H^0_{\mathfrak{m}}(S/J)$.

Using the identification $H^0_{\mathfrak{m}}(S/J) = (J : \mathfrak{m}^\infty)/J$, this yields

$$\big((J : \mathfrak{m}^\infty)/J\big)((n_1{+}n_2{+}n_3)(r{+}1){-}3) \ \cong \ ((J : \mathfrak{m}^\infty)/J)^*\,.$$

This implies that

$$\alpha((J : \mathfrak{m}^\infty)/J) \ - \ (n_1{+}n_2{+}n_3)(r{+}1){+}3 \ = \ -\Omega((J : \mathfrak{m}^\infty)/J)\,,$$

so

$$\alpha((J:\mathfrak{m}^\infty)/J) \ + \ \Omega((J:\mathfrak{m}^\infty)/J) \ = \ (n_1{+}n_2{+}n_3)(r{+}1){-}3\,.$$

Compare this to the first statement of [18, Thm. 1.2]. As J is $\langle x,y\rangle$-primary, $(J:\mathfrak{m}^\infty) = (J:z^\infty)$. Since no pure power of z appears in any of the forms G_1, G_2, G_3 (they all vanish at $[0:0:1]$), the maximum power of z in G_i^{r+1} is $(n_i{-}1)(r{+}1)$. Hence $\alpha((J:z^\infty)/J) \geq r+1$, so

$$\begin{aligned}\Omega\left((J:\mathfrak{m}^\infty)/J\right) \ &= \ (n_1{+}n_2{+}n_3)(r{+}1){-}3 \ - \ \alpha\left(J:\mathfrak{m}^\infty/J\right)\\ &\leq \ (n_1{+}n_2{+}n_3)(r{+}1){-}3 \ - \ (r{+}1)\\ &= \ (n_1{+}n_2{+}n_3{-}1)(r{+}1){-}3\,.\end{aligned}$$

Hence $\Omega(H^0_\mathfrak{m}(S/J)) \leq (n_1+n_2+n_3-1)(r+1)-3$, as desired. If $2t \geq r{+}3$, then Lemma 4.4 shows that $(J:\mathfrak{m}^\infty) = \langle L_1^{r+1}, L_2^{r+1}, L_3^{r+1}\rangle$, hence $\alpha((J:\mathfrak{m}^\infty)/J) = r+1$ and $\Omega(H^0_\mathfrak{m}(S/J)) = (n_1{+}n_2{+}n_3{-}1)(r{+}1){-}3$.

Now we show that $\Omega(H^1_\mathfrak{m}(S/J)) \leq (n_1+n_2)(r+1)-4$. Compare this statement to the second part of [18, Thm. 1.2]. By local duality [12, Thm. A1.9],

$$H^1_\mathfrak{m}(S/J) \ \cong \ \mathit{Ext}^2(S/J, S(-3))^*\,.$$

Hence $\Omega(H^1_\mathfrak{m}(S/J)) = -\alpha(\mathit{Ext}^2(S/J,S(-3)))$. Now let $I = \langle G_1^{r+1}, G_2^{r+1}\rangle \subset J$. Then I is a complete intersection, so S/I has a minimal free resolution of the form

$$0 \ \longrightarrow \ S(-a-b) \ \longrightarrow \ S(-a)\oplus S(-b) \ \longrightarrow \ S,$$

where $a = \deg(G_1^{r+1}) = n_1(r+1)$ and $b = \deg(G_2^{r+1}) = n_2(r+1)$. In particular, $\mathit{Ext}^2(S/I,S) \cong S(a+b)/I$. Let $\gamma = G_3^{r+1}$ and set $c = \deg(\gamma) = n_3(r+1)$. We have a short exact sequence

$$0 \ \longrightarrow \ S(-c)/(I:\gamma) \ \xrightarrow{\cdot\gamma} \ S/I \ \longrightarrow \ S/J \ \longrightarrow \ 0\,.$$

Since $\operatorname{codim}(S/(I:\gamma)) \geq 2$, the long exact sequence in *Ext* yields

$$0 \ \longrightarrow \ \mathit{Ext}^2(S/J,S) \ \longrightarrow \ \mathit{Ext}^2(S/I,S) \ \xrightarrow{\cdot\gamma} \ \mathit{Ext}^2(S/(I:\gamma),S) \ \longrightarrow \ \cdots\,,$$

hence $\mathit{Ext}^2(S/J,S)$ is the kernel of the map given by multiplication by γ on $\mathit{Ext}^2(S/I,S)$. Since $\mathit{Ext}^2(S/I,S) \cong S(a+b)/I$, we have

$$\mathit{Ext}^2(S/J,S) \ \cong \ \big((I:\gamma)/I\big)(a+b)\,,$$

where the $a+b$ in parentheses represents a degree shift. Since $\gamma \notin I$, it follows that $I:\gamma$ is generated in degrees ≥ 1, hence $\alpha(\mathit{Ext}^2(S/J,S)) \geq -a-b+1$. It follows that $\Omega(H^1_\mathfrak{m}(S/J)) = -\alpha(\mathit{Ext}^2(S/J,S(-3))) \leq a+b-4$.

Now, by Proposition 5.11,

$$\begin{aligned}\operatorname{reg}(S/I) \ &= \ \max\{\Omega(H^0_\mathfrak{m}(S/I)), \Omega(H^1_\mathfrak{m}(S/I))+1\}\\ &\leq \ \max\{(n_1+n_2+n_3-1)(r+1)-3, (n_1+n_2)(r+1)-3\}\\ &= \ (n_1+n_2+n_3-1)(r+1)-3\,,\end{aligned}$$

as we assume n_1, n_2, n_3 are all at least one. Equality holds if $2t \geq r+3$. $\square$

Corollary 5.13. *If Δ has three edges defined by forms G_1, G_2, G_3 of degrees n_1, n_2, n_3 meeting smoothly with distinct tangents at a single interior vertex, then*

$$\dim C^r_d(\Delta) \;=\; \sum_{i=1}^{3} \tbinom{d-n_i(r+1)+2}{2} + \tbinom{r+a+2}{2} - t\tbinom{a+1}{2}$$

for $d \geq (n_1+n_2+n_3-1)(r+1)-2$, where $t = \min\{3, r+2\}$ and $a = \lfloor \frac{r+1}{t-1} \rfloor$.

Proof. This follows from Corollary 4.2, Proposition 5.12, and Proposition 5.8. □

Example 5.14. For the cell complex of Fig. 1, whose Hilbert function, polynomial, and postulation number are shown in Table 1, Table 3 shows how the bounds on the postulation number in Corollary 5.10 and Corollary 5.13 compare with the actual postulation number, d_0. This indicates that for r small, we should expect the bound in Corollary 5.13 to be close to exact, while the bound in Corollary 5.10 may be quite far off.

Table 3. Comparing bounds with the postulation number in Example 5.14.

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
d_0	1	5	9	13	17	20	24	28	32	35	39	43	47	50	54
Cor. 5.13	1	5	9	13	17	21	25	29	33	37	41	45	49	53	57
Cor. 5.10	4	10	16	22	28	34	40	46	52	58	64	70	76	82	88

6. Examples

We illustrate some limitations and possible extensions of our results for cell complexes Δ with a single interior vertex υ. In Section 3, we determined the Hilbert function of the spline module when the curves defining the edges of Δ lie in a pencil. As noted in Remark 3.7, this Hilbert function does not depend upon the geometry of the curves in that pencil, only on their number and degree. In Section 4, we determined the Hilbert polynomial of the spline module in nearly the opposite case—when the curves vanish simultaneously only at the vertex υ and they have distinct tangents at υ.

Our first example is from [7]—three curves of different degrees, but only two tangents at υ. In the remaining examples, the curves are three conics defining schemes of multiplicity three and two (intermediate between the cases of Sections 3 and 4). We show that the Hilbert polynomial of the spline module depends upon the geometry of the curves.

Example 6.1. This example appears in [7, § 8.3, Exer. 13]. Let Δ consist of portions of the three curves $G_1 = yz - x^2$, $G_2 = xz + y^2$, and $G_3 = yz^2 - x^3$ in the unit disc in $\mathbb{R}^2$ where $z \neq 0$ meeting at the origin as in Fig. 4. We have $J(\upsilon) = \langle G_1^{r+1}, G_2^{r+1}, G_3^{r+1}\rangle$ and $C^r(\Delta) \cong S \oplus \mathrm{syz}(J(\upsilon))$ by Proposition 2.2.

The tangents of G_1, G_2, G_3 at $(0,0)$ are $L_1 = y$, $L_2 = x$, and $L_3 = y$. Let $I = \langle L_1^{r+1}, L_2^{r+1}, L_3^{r+1}\rangle = \langle x^{r+1}, y^{r+1}\rangle$. Since the tangents are not distinct, we cannot use Theorem 4.1 to compute the multiplicity of the scheme $S/J(\upsilon)$. However, if $r \leq 1$,

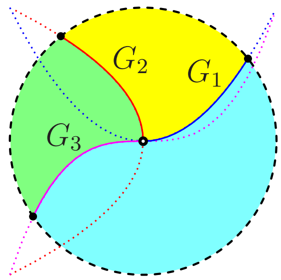


FIGURE 4. Complex Δ in Example 6.1.

then the schemes S/I and $S/J(\upsilon)$ have the same multiplicity by Corollary 4.5. Using Corollary 2.3,

$$HP(C^r(\Delta), d) \;=\; 2\binom{d-2(r+1)}{2} + \binom{d-3(r+1)}{2} + (r+1)^2\,,$$

if $r \leq 1$. For $r \geq 2$, we replace G_3^{r+1} by $G_3^{r+1} - z^{r+1}G_1^{r+1}$, which has leading term $x^2y^rz^{2r+1}$ in z. Set $I' := \langle x^{r+1}, x^2y^r, y^{r+1}\rangle$. The minimal free resolution of S/I' has the form

$$0 \longrightarrow S(-2r-1) \oplus S(-r-3) \longrightarrow S^3 \longrightarrow S\,. \tag{11}$$

The ideal I' is generated by the leading forms of G_1^{r+1} G_2^{r+1}, and $G_3^{r+1} - z^{r+1}G_1^{r+1}$, which generate $J(\upsilon)$. By Lemmas 4.9 and 4.10, $S/J(\upsilon)$ has the same Hilbert polynomial as S/I' when $2 \leq r \leq 4$. Using Corollary 2.3,

$$\begin{aligned} HP(C^r(\Delta), d) &= 2\binom{d-2(r+1)}{2} + \binom{d-3(r+1)}{2} + HP(S/J(\upsilon), d) \\ &= 2\binom{d-2(r+1)}{2} + \binom{d-3(r+1)}{2} + HP(S/I', d) \\ &= 2\binom{d-2(r+1)}{2} + \binom{d-3(r+1)}{2} + (2 + r + r^2)\,, \end{aligned}$$

where the final equality follows from the minimal free resolution of I' (11).

If $r > 4$, the techniques of this paper will not suffice to compute $HP(C^r(\Delta), d)$. Computations in Macaulay2 show that the saturation of $\mathrm{in}_\omega(J(\upsilon))$ is $I' = \langle x^{r+1}, x^2y^r, y^{r+1}\rangle$ for $r = 5$, where $\omega = (0, 0, 1)$. This cannot be concluded from Lemma 4.10 since the condition on second syzygies fails. Further computations in Macaulay2 show

$$\left(\mathrm{in}_\omega(J(\upsilon)) : \mathfrak{m}^\infty\right) \;=\; \langle x^{r+1}, y^{r+1}, x^2y^r, x^6y^{r-1}\rangle$$

for $r = 6, 7, 8, 9$. For $r = 10$,

$$\left(\mathrm{in}_\omega(J(\upsilon)) : \mathfrak{m}^\infty\right) \;=\; \langle x^{r+1}, y^{r+1}, x^2y^r, x^6y^{r-1}, x^{10}y^{r-2}\rangle\,,$$

indicating a growth in the number of generators of the saturation $(\mathrm{in}_\omega(J(\upsilon)) : \mathfrak{m}^\infty)$. See Table 4. For $r \geq 5$ different techniques will be needed to compute $HP(C^r(\Delta), d)$. The column headed d_0 gives the postulation number (computed only through $r = 5$). The final column is the regularity bound from Corollary 5.13.

Example 6.2. Suppose that G_1, G_2, and G_3 are conics underlying the edges of a cell complex Δ with a single vertex υ that do not lie in a pencil, but simultaneously vanish in at least another point. By Corollary 2.3 (4), the Hilbert function of $C^r(\Delta)$ is

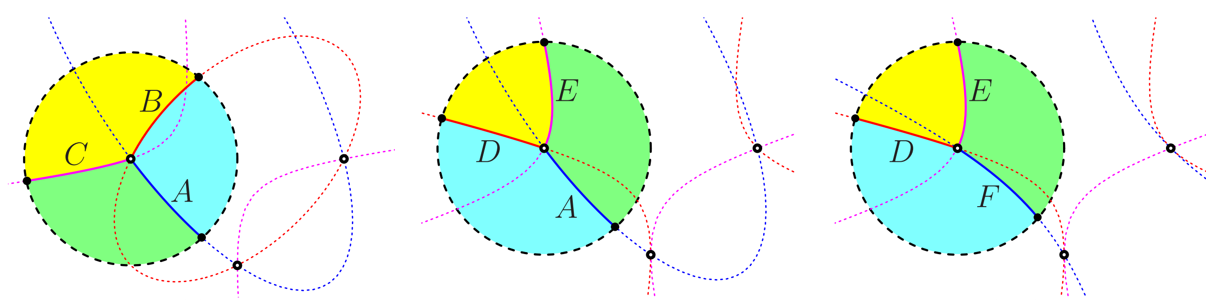
$$HF(C^r(\Delta), d) \;=\; 3\binom{d-2r}{2} + \dim(S/J(\upsilon))_d\,.$$

TABLE 4. Table for cell complex Δ in Example 6.1.

r	$\mathrm{sat}(\mathrm{in}_\omega(J(\upsilon)))$	$HP(C^r(\Delta), d)$	$HP(S/J(\upsilon), d)$	d_0	$6(r+1)-3$
0	$\langle x, y\rangle$	$\frac{3}{2}d^2 - \frac{5}{2}d + 2$	1	3	3
1	$\langle x^2, y^2\rangle$	$\frac{3}{2}d^2 - \frac{19}{2}d + 20$	4	9	9
2	$\langle x^3, y^3, x^2y^2\rangle$	$\frac{3}{2}d^2 - \frac{33}{2}d + 56$	8	15	15
3	$\langle x^4, y^4, x^2y^3\rangle$	$\frac{3}{2}d^2 - \frac{47}{2}d + 111$	14	21	21
4	$\langle x^5, y^5, x^2y^4\rangle$	$\frac{3}{2}d^2 - \frac{61}{2}d + 185$	22	27	27
5	$\langle x^6, y^6, x^2y^5\rangle$	$\frac{3}{2}d^2 - \frac{75}{2}d + 278$	32	32	33
6	$\langle x^7, y^7, x^2y^6, x^6y^5\rangle$	$\frac{3}{2}d^2 - \frac{89}{2}d + 389$	43		39
7	$\langle x^8, y^8, x^2y^7, x^6y^6\rangle$	$\frac{3}{2}d^2 - \frac{103}{2}d + 519$	56		45
8	$\langle x^9, y^9, x^2y^8, x^6y^7\rangle$	$\frac{3}{2}d^2 - \frac{117}{2}d + 668$	71		51
9	$\langle x^{10}, y^{10}, x^2y^9, x^6y^8\rangle$	$\frac{3}{2}d^2 - \frac{131}{2}d + 836$	88		57
10	$\langle x^{11}, y^{11}, x^2y^{10}, x^6y^9, x^{10}y^8\rangle$	$\frac{3}{2}d^2 - \frac{145}{2}d + 1022$	106		63

We compute the Hilbert functions of $S/J(\upsilon)$ for different choices of three conics. Stiller [22, Thm. 4.9] did this when $r = 0$ and when the conics define 1, 2, or 3 simple points.

We first consider three cases where the conics define a scheme of multiplicity three, consisting of the three points $\upsilon = [0 : 0 : 1]$, $[2 : 0 : 1]$, and $[1 : -1 : 1]$. The first triple is $A := 2x^2+2xy+y^2-4xz-3y$, $B := x^2-xy+y^2-2xz+yz$, and $C := x^2-8xy-y^2-2xz+6yz$. Their curves have distinct tangents at each of three points. The next triple is A, $D := x^2 + 4xy - y^2 - 2xz - 6yz$, and $E := x^2 - 3xy - y^2 - 2xz + yz$. The curves of D and E are tangent at $[1 : -1 : 1]$. The third triple is D, E, and $F := 2x^2 + 5xy + y^2 - 4xz - 6yz$. The curves of F and D are also tangent at the point $[2 : 0 : 1]$. We display the resulting cell complexes in the affine plane $\mathbb{R}^2$ with $z \neq 0$ in Fig. 5.



(I) (II) (III)

FIGURE 5. Three conics defining three points.

Table 5 gives the Hilbert functions of $S/J(\upsilon)$ for $d \leq 18$ and $r \leq 4$ for each of these triples. While the Hilbert functions agree for $r = 0$ (as shown by Stiller [22, Thm. 4.9]), they differ for all larger r in both the postulation number and Hilbert polynomial.

Table 5. Hilbert functions of $S/J(v)$ for cell complexes of Fig. 5.

I	d																		
r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
1	1	3	6	10	12	12	10	9	9	9	9	9	9	9	9	9	9	9	9
2	1	3	6	10	15	21	25	27	27	25	21	21	21	21	21	21	21	21	21
3	1	3	6	10	15	21	28	36	42	46	48	48	46	42	36	36	36	36	36
4	1	3	6	10	15	21	28	36	45	55	63	69	73	75	75	73	69	63	57

II	d																		
r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
1	1	3	6	10	12	12	10	10	10	10	10	10	10	10	10	10	10	10	10
2	1	3	6	10	15	21	25	27	27	25	22	22	22	22	22	22	22	22	22
3	1	3	6	10	15	21	28	36	42	46	48	48	46	42	38	38	38	38	38
4	1	3	6	10	15	21	28	36	45	55	63	69	73	75	75	73	69	63	60

III	d																		
r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
1	1	3	6	10	12	12	11	11	11	11	11	11	11	11	11	11	11	11	11
2	1	3	6	10	15	21	25	27	27	25	23	23	23	23	23	23	23	23	23
3	1	3	6	10	15	21	28	36	42	46	48	48	46	42	40	40	40	40	40
4	1	3	6	10	15	21	28	36	45	55	63	69	73	75	75	73	69	63	63

We find similar behavior when the three quadrics define a scheme of multiplicity two, for us the points $[0:0:1]$ and $[2:0:1]$. Let $A := x^2+xy+y^2-2xz$, $B := 2x^2+xy+2y^2-4xz-2yz$, $C := x^2+xy+2y^2-2xz+6yz$, and $D := x^2-xy-2y^2-2xz+2yz$. Then $\langle A,B,C\rangle$ and $\langle A,B,D\rangle$ both define the same scheme consisting of those two reduced points. They have distinct tangents at $[0:0:1]$, and A, B, and C have distinct tangents at $[2:0:1]$, but B and D are tangent at $[2:0:1]$. Fig. 6 shows the resulting cell

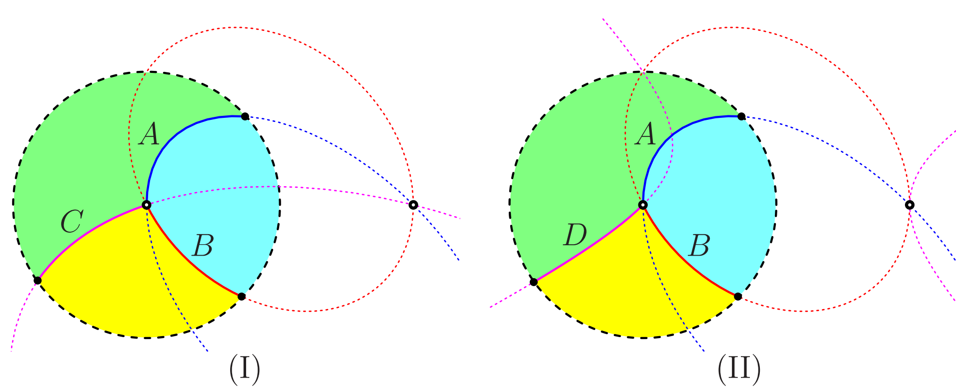


Figure 6. Three conics defining two points.

complexes and the underlying curves. Table 6 shows the Hilbert functions of $S/J(\upsilon)$.

TABLE 6. Hilbert functions of $S/J(\upsilon)$ for three quadrics defining two points.

I	d																	
r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	1	3	6	10	12	12	10	7	6	6	6	6	6	6	6	6	6	6
2	1	3	6	10	15	21	25	27	27	25	21	16	14	14	14	14	14	14
3	1	3	6	10	15	21	28	36	42	46	48	48	46	42	36	29	25	24

II	d																	
r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	1	3	6	10	12	12	10	7	7	7	7	7	7	7	7	7	7	7
2	1	3	6	10	15	21	25	27	27	25	21	16	15	15	15	15	15	15
3	1	3	6	10	15	21	28	36	42	46	48	48	46	42	36	29	26	26

Remark 6.3. The multiplicity of a zero-dimensional scheme is the sum of its local multiplicities at each point of its support. For $S/J(\upsilon)$, this is

$$\operatorname{mult}(S/J(\upsilon)) \;=\; \sum_{\nu \in \operatorname{supp}(S/J(\upsilon))} \operatorname{mult}_\nu(S/J(\upsilon)),$$

where $\operatorname{mult}_\nu(S/J(\upsilon))$ is the vector space dimension of the local ring $(S/J(\upsilon))_{\mathfrak{m}_\nu}$ with $\mathfrak{m}_\nu$ the maximal ideal of the point ν. This is the multiplicity of the tangent cone of $S/J(\upsilon)$ at ν (see [11, § 5.4]). Thus we should expect that we can read off the multiplicity of the schemes in Example 6.2 as sums of local multiplicities which depend only on the geometry of the tangent cones at points in the support of $S/J(\upsilon)$. This is indeed the case; to see this, we write the multiplicites of Tables 5 and 6 as sums of the multiplicity in Corollary 3.4 and the multiplicity in Table 4. Call the multiplicity in Corollary 3.4 the *generic multiplicity*; by Theorem 4.1 this is the multiplicity of $S/J(\upsilon)$ when tangents are distinct.

In Table 5, note that if the tangents of the edge forms at all points in the support of $S/J(\upsilon)$ are distinct, then the multiplicity of $S/J(\upsilon)$ is thrice the generic multiplicity. If the tangents of edge forms are distinct at two points of support but two tangents coincide at the third point, then the geometry at the third point is the same as in Example 6.1. The multiplicity of $S/J(\upsilon)$ is twice the generic multiplicity plus the multiplicity given in Table 4. If tangents of edge forms are distinct at one point but two tangents coincide at both other points, then the multiplicity of $S/J(\upsilon)$ is the generic multiplicity plus twice the multiplicity given in Table 4. The same observations can be made in Table 6.

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Michael DiPasquale, Department of Mathematics, Oklahoma State University, Stillwater, OK 74078-1058, USA
E-mail address: midipasq@gmail.com
URL: http://math.okstate.edu/people/mdipasq/

Frank Sottile, Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
E-mail address: sottile@math.tamu.edu
URL: http://www.math.tamu.edu/~sottile

Lanyin Sun, School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023 China
E-mail address: lanyinsun@mail.dlut.edu.cn