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DENSE FEWNOMIALS

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ABSTRACT. We derive new bounds of fewnomial type for the number of real solutions to systems of polynomials that have structure intermediate between fewnomials and generic (dense) polynomials. This uses a modified version of Gale duality for polynomial systems. We also use stratified Morse theory to bound the total Betti number of a hypersurface defined by such a dense fewnomial. These bounds contain and generalize previous bounds for ordinary fewnomials obtained by Bates, Bertrand, Bihan, and Sottile. As with their results, our bounds hold for polynomials with real exponents.

1. INTRODUCTION

The classical theorem of Bézout [4] bounds the number of solutions to a system of polynomials by the product of their degrees. While this Bézout bound is sharp for generic systems of polynomial equations, that is no longer the case when the equations possess additional structure. For example, Kushnirenko [2] showed that if the polynomials all have the same Newton polytope, then the number of nondegenerate solutions to such a system is at most the volume of the Newton polytope, suitably normalized.

Bounds for the number of nondegenerate real solutions are governed by Kushnirenko's "fewnomial principle": roughly, few monomials implies few solutions or restricted topology [11]. This principle was established by Khovanskii in his fundamental work on fewnomials [10] in which he showed that a system of n polynomials in n variables where the polynomials have $1+k+n$ distinct monomials has fewer than

$$2^{\binom{k+n}{2}}\,(n+1)^{k+n} \tag{1.1}$$

nondegenerate positive solutions. This bound is remarkable as it is independent of the degrees and Newton polytopes of the polynomials, which control the number of complex solutions. Few of the complex solutions to a fewnomial system can be real.

Khovanskii's bound was lowered by Bihan and Sottile [5] to

$$\frac{e^2+3}{4}\,2^{\binom{k}{2}}\,n^k\,. \tag{1.2}$$

For this, they transformed the original polynomial system into an equivalent Gale dual [6] system of rational functions, whose number of solutions they bounded. An essential step for the bound on Gale systems uses Khovanskii's generalization of the classical Rolle Theorem.

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The bound (1.2) is smaller than Khovanskii's bound (1.1) because Khovanskii's bound is a specialization to polynomials of a bound for more general functions and the proof of (1.2) takes advantage of some special geometry enjoyed by polynomials.

We derive bounds of fewnomial type for polynomial systems with structure intermediate between that of fewnomials and general polynomials. These bounds can be dramatically smaller than the fewnomial bound (1.2), and like that bound do not depend upon the degrees of the polynomials involved. A collection $\mathcal{A} \subset \mathbb{Z}^n$ of exponent vectors is *(d,ℓ)-dense* if there are integers d,ℓ such that $\mathcal{A}$ admits a decomposition of the form

$$\mathcal{A} \;=\; \psi(d\Delta^\ell \cap \mathbb{Z}^\ell) \;\bigcup\; \mathcal{W}\,, \tag{1.3}$$

where $\mathcal{W}$ consists of n affinely independent vectors, $\psi\colon \mathbb{Z}^\ell \to \mathbb{Z}^n$ is an affine-linear map, and Δ^ℓ is the unit simplex in $\mathbb{R}^\ell$.

A Laurent polynomial whose support $\mathcal{A}$ is (d,ℓ)-dense (1.3) is a *(d,ℓ)-dense fewnomial*. We show that a system of (d,ℓ)-dense fewnomials in n variables has fewer than

$$\frac{e^2+3}{4}\, 2^{\binom{\ell}{2}}\, n^\ell \cdot d^\ell \tag{1.4}$$

nondegenerate positive solutions. To compare this to (1.2), note that a (d,ℓ)-dense set $\mathcal{A}$ may contain $n+\binom{d+\ell}{\ell}$ points, and so the parameter k in (1.2) could be $\binom{d+\ell}{\ell}-1$ for a (d,ℓ)-dense set. Thus the systems in (1.2) are the special case $(d,\ell)=(1,k)$, and when $d>1$, the bound (1.4) is dramatically smaller than (1.2).

In [1], the methods of [5] were extended to establish the strict upper bound

$$\frac{e^4+3}{4}\, 2^{\binom{k}{2}}\, n^k\,, \tag{1.5}$$

for the number of nonzero real solutions to a fewnomial system—not just positive solutions—when the exponent vectors span a sublattice of $\mathbb{Z}^n$ of odd index. The same arguments show that if the exponent vectors $\mathcal{A}$ span a sublattice of $\mathbb{Z}^n$ of odd index, then the number of nondegenerate real solutions to a system of (d,ℓ)-dense fewnomials is at most

$$\frac{e^4+3}{4}\, 2^{\binom{\ell}{2}}\, n^\ell \cdot d^\ell\,. \tag{1.6}$$

Khovanskii also gave a bound for the sum $b_*(X)$ of the Betti numbers of a smooth hypersurface in the positive orthant $\mathbb{R}^n_>$ defined by a polynomial with $1+k+n$ monomial terms [10] (Corollary 4, p. 91),

$$b_*(X) \;\leq\; (2n^2-n+1)^{k+n}\,(2n)^{n-1}\,2^{\binom{k+n}{2}}\,. \tag{1.7}$$

Bihan and Sottile [7] used the fewnomial bound and stratified Morse theory for a manifold with corners [9] to lower this to

$$b_*(X) \;<\; \frac{e^2+3}{4}\, 2^{\binom{k}{2}} \cdot \sum_{i=0}^{n} \binom{n}{i}\, i^k\,. \tag{1.8}$$

The same arguments show that when X is defined by a (d,ℓ)-dense fewnomial, we have

$$b_*(X) \;<\; \frac{e^2+3}{4}\, 2^{\binom{\ell}{2}} \cdot d^\ell \cdot \sum_{i=0}^{n} \binom{n}{i}\, i^\ell\,. \tag{1.9}$$

An important step in these arguments is a version of Gale duality for dense fewnomials, which generalizes Gale duality for polynomial systems as established in [6].

These bounds (1.4), (1.6), and (1.9) simultaneously generalize the results of [5], [1], and [7], which are the cases when $(d, \ell) = (1, k)$. The case $\ell = 1$ of the bound (1.5) was established in [3], where a $(d, 1)$-dense fewnomial was called a near circuit.

This paper is structured as follows. We begin in Section 2 with definitions and examples of dense fewnomials, give the precise statements of our main theorems, and study an example when $n = 2$. Section 3 is devoted to establishing the variant of Gale duality appropriate for dense fewnomials, and in Section 4, which is the heart of this paper, we establish the bounds (1.4) and (1.6). We develop the necessary tools and give the proof of our bound (1.9) for the sum of Betti numbers in Section 5.

We thank Maurice Rojas, who suggested looking for extensions of the fewnomial bound as a class project for Rusek and Shakalli.

2. Dense Fewnomials

An integer vector $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ is the exponent of a Laurent monomial,

$$x^\alpha \ := \ x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \, .$$

A polynomial f with *support* $\mathcal{A} \subset \mathbb{Z}^n$ is one whose exponent vectors lie in $\mathcal{A}$,

$$f \ = \ \sum_{\alpha \in \mathcal{A}} a_\alpha x^\alpha \qquad (a_\alpha \in \mathbb{R}) \, .$$

We are interested in systems of n polynomials, each with support $\mathcal{A}$,

$$f_1(x_1, \ldots, x_n) \ = \ f_2(x_1, \ldots, x_n) \ = \ \cdots \ = \ f_n(x_1, \ldots, x_n) \ = \ 0 \, .$$

A solution x to this system is *nondegenerate* if the differentials of the polynomials are linearly independent at x. Nondegenerate solutions occur with algebraic multiplicity 1, and their number is preserved under perturbations of the f_i. We obtain novel bounds on the number of real solutions to a system of polynomials when the set of exponent vectors has structure that is intermediate between fewnomials and dense polynomials.

Kushnirenko [2] showed that a general system of polynomials, all with support $\mathcal{A}$, will have $n! \operatorname{vol}(\operatorname{conv}(\mathcal{A}))$ complex solutions, the normalized volume of the convex hull of the exponent vectors. While this is also a bound for the number of real solutions, there is another bound which depends only upon the number of exponents. Specifically, a *fewnomial system* is one in which the support $\mathcal{A}$ consists of $1+k+n$ monomials, but is otherwise unstructured, and such a system has the bound (1.2) on its number of positive solutions [5]. When the exponents $\mathcal{A}$ affinely span a sublattice of odd index in $\mathbb{Z}^n$, a fewnomial system has the bound (1.5) for its number of nondegenerate nonzero real solutions [1].

A *dense fewnomial* is a polynomial whose support $\mathcal{A}$ is intermediate between fewnomials and general polynomials in the following way. Let d, ℓ, n be positive integers, $\triangle^\ell \subset \mathbb{R}^\ell$ be the standard unit simplex, the convex hull of the original and the ℓ unit basis vectors, and $\psi \colon \mathbb{Z}^\ell \to \mathbb{Z}^n$ be an affine-linear map. A *(d, ℓ)-dense fewnomial* is a Laurent polynomial f whose support $\mathcal{A} \subset \mathbb{Z}^n$ admits a decomposition

$$\mathcal{A} \ = \ \psi(d\triangle^\ell \cap \mathbb{Z}^\ell) \ \bigcup \ \mathcal{W} \, , \tag{2.1}$$

where $\mathcal{W} = \{w_1, \ldots, w_n\}$ consists of n affinely independent vectors. Such a set $\mathcal{A}$ (2.1) is *(d, ℓ)-dense*.

We give some examples of (d, ℓ)-dense sets of exponent vectors.

(1) Any collection $\mathcal{A}$ of $1{+}\ell{+}n$ exponent vectors that affinely spans $\mathbb{R}^n$ is $(1, \ell)$-dense. To see this, let $\mathcal{W} \subset \mathcal{A}$ be n affinely independent vectors. Writing $\mathcal{A} - \mathcal{W} = \{v_0, v_1, \ldots, v_\ell\}$, these vectors are the image of the integer points in $\triangle^\ell$ under the affine map ψ that takes the ith unit vector to v_i and the origin to v_0.

Thus, ordinary fewnomials with $1{+}k{+}n$ monomials are $(1, k)$-dense fewnomials.

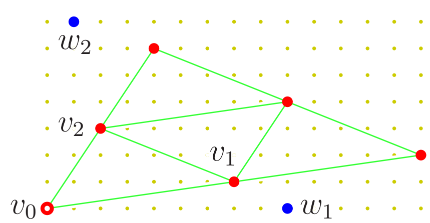
(2) When $\ell = 1$, the exponent vectors $\mathcal{A}$ of a dense fewnomial form a *near circuit* in the terminology of [3]. There, it was shown that if $\mathcal{A}$ spans $\mathbb{Z}^n$, then a system with support $\mathcal{A}$ has at most $2dn{+}1$ nonzero real solutions.

(3) A general (d, ℓ)-dense set $\mathcal{A}$ has the following form,

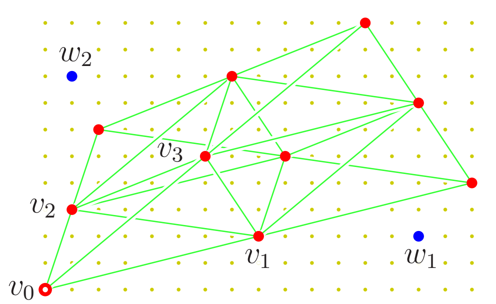
$$\mathcal{A} \ := \ \Big\{v_0 \ + \ \sum_{m=1}^{\ell} \lambda_m v_m \mid 0 \leq \lambda_m \,, \ \sum_i \lambda_i \leq d\Big\} \bigcup \mathcal{W}\,, \tag{2.2}$$

where $\mathcal{W} = \{w_1, \ldots, w_n\} \subset \mathbb{Z}^n$ is affinely independent and $v_0, v_1, \ldots, v_\ell$ are integer vectors. Below is an example of a $(2, 2)$-dense set $\mathcal{A}$ in $\mathbb{Z}^2$, where $\mathcal{W} = \{(9, 0), (1, 7)\}$, $v_0 = (0, 0)$ is the open circle, $v_1 = (7, 1)$, and $v_2 = (2, 3)$.

(2.3)



Here is a $(2, 3)$-dense set in $\mathbb{Z}^2$, where $\mathcal{W} = \{(14, 2), (1, 8)\}$, $v_0 = (0, 0)$ is the open circle, $v_1 = (8, 2)$, $v_2 = (1, 3)$, and $v_3 = (6, 5)$.



Theorem 2.4. *Suppose that $\mathcal{A} \subset \mathbb{Z}^n$ is (d, ℓ)-dense. Then a system*

$$f_1(x_1, \ldots, x_n) \ = \ f_2(x_1, \ldots, x_n) \ = \ \cdots \ = \ f_n(x_1, \ldots, x_n) \ = \ 0$$

of real polynomials with support $\mathcal{A}$ has fewer than

$$\frac{e^2{+}3}{4}\, 2^{\binom{\ell}{2}}\, n^\ell \cdot d^\ell$$

nondegenerate positive solutions. If the affine span of $\mathcal{A}$ is a sublattice of $\mathbb{Z}^n$ with odd index, then the number of nondegenerate real solutions is less than

$$\frac{e^4+3}{4}\, 2^{\binom{\ell}{2}}\, n^\ell \cdot d^\ell\,.$$

The bounds in Theorem 2.4 hold if the support of the system is only a subset of $\mathcal{A}$. Indeed, the number of nondegenerate solutions does not decrease if we perturd the polynomials by any function, and so we may perturb the coefficients of the system to obtain one whose support is exactly $\mathcal{A}$ without decreasing the number of nondegenerate solutions.

When $n = d = \ell = 2$, this bound for positive solutions is 83. Suppose that we replace the vectors v_1 and v_2 in (2.3) by $(7r, r)$ and $(2s, 3s)$, respectively where $r, s \geq 1$. Then the Kushnirenko bound for the resulting $(2, 2)$-dense configuration is $76rs + 18r + 22s$, so a general system with this support will have this number of complex solutions, at most 83 of which can be positive. This bound of 83 is also significantly smaller than the bound of Bihan and Sottile (1.2) in this case, which is $\frac{e^2+3}{4}2^{\binom{5}{2}}2^5 = 85107.15$.

A generic system of (d, ℓ)-dense fewnomials (as in (2.1)),

$$f_1(x) \ =\ f_2(x) \ =\ \cdots \ =\ f_n(x) \ =\ 0\,, \tag{2.5}$$

will have an invertible matrix of coefficients of the monomials $\{x^{w_i} \mid i = 1, \ldots, n\}$, and so we may solve (2.5) for these monomials to get the equivalent system

$$x^{w_i} \ =\ \sum_{p\in d\Delta^\ell\cap\mathbb{Z}^\ell} a_{i,p}x^{\psi(p)}\,, \qquad \text{for} \quad i = 1, \ldots, n, . \tag{2.6}$$

For each $i = 1, \ldots, n$, define the degree d polynomial in variables $y \in \mathbb{R}^\ell$,

$$h_i(y) \ :=\ \sum_{p\in d\Delta^\ell\cap\mathbb{Z}^\ell} a_{i,p}y^p\,.$$

Following the notation in (2.2), we translate the set $\mathcal{A}$ by $-v_0$, which amounts to multiplying the equations (2.5) and (2.6) by x^{-v_0}, and does not change their solutions in the algebraic torus. Thus, we may assume that $v_0 = 0$ so that ψ is a linear map, and then let $\mathcal{V} := \{v_1, \ldots, v_\ell\} \subset \mathbb{Z}^n$ be the images of the standard basis vectors under ψ. Then we have $x^{w_i} = h_i(x^{v_1}, \ldots, x^{v_\ell})$. A linear relation among the vectors in $\mathcal{V}$ and $\mathcal{W}$,

$$\sum_{m=1}^{\ell} b_m v_m \ +\ \sum_{i=1}^{n} c_i w_i \ =\ 0\,, \tag{2.7}$$

implies the multiplicative relation among the monomials

$$\prod_{m=1}^{\ell} (x^{v_m})^{b_m} \ \cdot\ \prod_{i=1}^{n} (x^{w_i})^{c_i} \ =\ 1\,.$$

If we use (2.6) to first substitute $h_i(x^{v_1}, \ldots, x^{v_\ell})$ for x^{w_i} for $i = 1, \ldots, n$ in this expression, and then substitute y_m for x^{v_m}, for $m = 1, \ldots, \ell$, we obtain

$$\prod_{m=1}^{\ell} (y_m)^{b_m} \ \cdot\ \prod_{i=1}^{n} (h_i(y))^{c_i} \ =\ 1\,.$$

Write $\beta = (b_1, \ldots, b_\ell)$ for the vector of the coefficients of $\mathcal{V}$ in (2.7) and $\gamma = (c_1, \ldots, c_n)$ for the vector of the coefficients of $\mathcal{W}$. Then we may write the left hand side of this last expression compactly as $y^\beta \cdot h(y)^\gamma$.

Now suppose that $(\beta_j, \gamma_j) \in \mathbb{Z}^\ell \oplus \mathbb{Z}^n$ for $j = 1, \ldots, \ell$ is a basis for the $\mathbb{Z}$-module of linear relations among the vectors in $\mathcal{V} \bigcup \mathcal{W}$. Then the system

$$y^{\beta_j} \ \cdot\ h(y)^{\gamma_j} \ =\ 1\,, \qquad j = 1, \ldots, \ell \tag{2.8}$$

is a *(d, ℓ)-dense Gale system* dual to the original system (2.5) of polynomials.

Theorem 2.9. *Let* (2.5) *be a system of* (d, ℓ)*-dense fewnomials and* (2.8) *be its corresponding dual* (d, ℓ)*-dense Gale system. Then the number of nondegenerate positive solutions to* (2.5) *is equal to the number of nondegenerate positive solutions to* (2.8) *where* $h_i(y) > 0$ *for each* $i = 1, \ldots, n$.

If the exponents $\mathcal{A}$ *affinely span a sublattice of* $\mathbb{Z}^n$ *of odd index and the relations* $(\beta^j, \gamma^j) \in \mathbb{Z}^\ell \oplus \mathbb{Z}^n$ *for* $j = 1, \ldots, \ell$ *span a sub* $\mathbb{Z}$*-module of odd index in the module of all linear relations, then the number of nondegenerate real solutions to* (2.5) *is equal to the number of nondegenerate real solutions to* (2.8) *in* $(\mathbb{R}^\times)^\ell$ *where no* $h_i(y)$ *vanishes.*

This follows from Theorem 3.7 on Gale duality for (d, ℓ)-dense fewnomials, which has three parts, for $\mathbb{C}$, for $\mathbb{R}$, and for $\mathbb{R}_>$. The parts of Theorem 3.7 for $\mathbb{R}$ and for $\mathbb{R}_>$ are a stronger reformulation of Theorem 2.9.

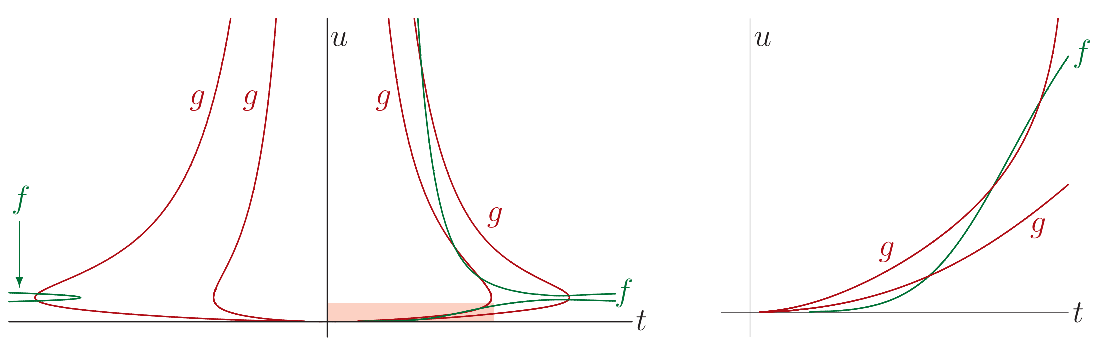
Thus we may prove Theorem 2.4 by establishing bounds for (d, ℓ)-dense Gale systems.

Let us consider an example of this duality. The system of Laurent polynomials,

$$\begin{aligned} f &:= 27t^{-5} + 31 - 16t^2u - 16t^2u^{-1} - 16t^4u^2 + 40t^4 - 16t^4u^{-2} = 0\,, \\ g &:= 12t + 40 - 32t^2u - 32t^2u^{-1} + 5t^4u^2 + 6t^4 + 5t^4u^{-2} = 0\,, \end{aligned} \tag{2.10}$$

has 36 complex solutions with nonzero coordinates, ten of which are real and eight of which lie in the positive quadrant. We show the curves $f = 0$ and $g = 0$ defined by the polynomials (2.10). In the picture on the left, the horizontal scale has been exaggerated. Its shaded region is shown on the right, where now the vertical scale is exaggerated.

(2.11)

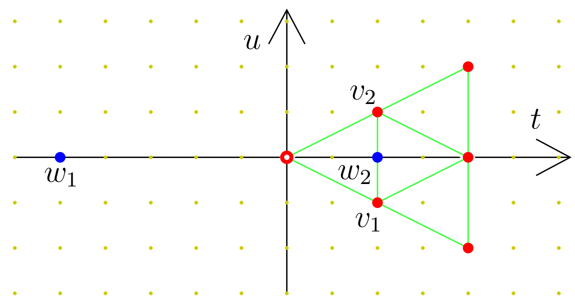


Here are numerical approximations to the ten real solutions.

$$\begin{aligned} &(0.619, 0.093),\ (0.839, 0.326),\ (1.003, 0.543),\ (1.591, 0.911),\ (-1.911, 0.864), \\ &(0.619, 10.71),\ (0.839, 3.101),\ (1.003, 1.843),\ (1.591, 1.097),\ (-1.911, 1.158)\,. \end{aligned}$$

(The repetition in the t-coordinates is explained by the symmetry $u \mapsto u^{-1}$ of (2.10).)

The system (2.10) is a system of $(2,2)$-dense fewnomials, as we may see from its support.



If we solve (2.10) for the monomials t^{-5} and t, we obtain

$$\begin{aligned} t^{-5} &= \tfrac{1}{27}(-\,31 + 16t^2u + 16t^2u^{-1} + 16t^4u^2 - 40t^4 + 16t^4u^{-2}) \\ t &= \tfrac{1}{12}(-\,40 + 32t^2u + 32t^2u^{-1} - \ 5t^4u^2 - \ 6t^4 - \ 5t^4u^{-2})\,. \end{aligned} \tag{2.12}$$

Write $h_1(t^2u, t^2u^{-1})$ and $h_2(t^2u, t^2u^{-1})$ for the polynomials on the right-hand side of (2.12), which becomes $t^5 = h_1(t^2u, t^2u^{-1})$ and $t = h_2(t^2u, t^2u^{-1})$. We convert this into a (d,ℓ)-dense Gale system dual to (2.10). First observe that

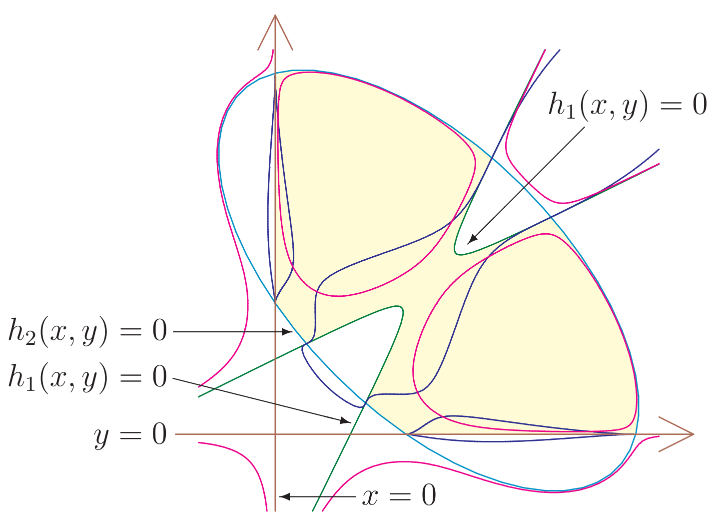
$$\begin{aligned} (t^2u)^{1}\,(t^2u^{-1})^{1}\,(t^{-5})^{1}\,(t)^{1} &= 1 \\ (t^2u)^{2}\,(t^2u^{-1})^{2}\,(t^{-5})^{1}\,(t)^{-3} &= 1\,. \end{aligned} \tag{2.13}$$

We use the equations (2.12) to replace the monomials t^{-5} and t in (2.13) and then apply the substitutions $x := t^2u$ and $y := t^2u^{-1}$ (so that $x^2 = t^4u^2$, $xy = t^4$, and $y^2 = t^4u^{-2}$). Then, after clearing denominators and rearranging, we have

$$\begin{aligned} x^{1}y^{1}h_1(x,y)^{1}h_2(x,y)^{1} \ - \ 1 &= 0 \\ x^{2}y^{2}h_1(x,y)^{1} \ - \ h_2(x,y)^{3} &= 0\,. \end{aligned} \tag{2.14}$$

This system has 36 complex solutions, ten of which are real and eight of which lie in the shaded region in the picture below where $x, y, h_1(x,y)$, and $h_2(x,y)$ are all positive.

(2.15)



The numbers of complex, real, and suitably positive solutions to the two systems (2.10) and (2.14) is a consequence of Theorem 3.7 on structured Gale duality. Here are numerical approximations to the ten real solutions of (2.14)

$$\begin{gathered}(4.229, 3.154),\ (4.098, 0.036),\ (2.777, 2.306),\ (2.184, 0.227),\ (1.853, 0.546),\\ (3.154, 4.229),\ (0.036, 4.098),\ (2.306, 2.777),\ (0.227, 2.184),\ (0.546, 1.853)\,.\end{gathered}$$

We remark that there is no relation between the two pairs of curves in (2.11) and (2.15). Gale duality only asserts a scheme-theoretic equality between the points of intersection of each pair of curves.

3. Gale Duality for (d,ℓ)-dense fewnomials

Gale duality [6] asserts that a system of n polynomials in n variables involving a total of $1+k+n$ distinct monomials is equivalent to a system of k rational functions of a particular form in the complement of an arrangement of $k+n$ hyperplanes in $\mathbb{R}^k$. A modification of Gale duality asserts that a system of (d,ℓ)-dense fewnomials is equivalent to a system ℓ $(\leq k)$ rational functions in the complement $\mathcal{M}(\mathbb{R})$ of the coordinate axes of $\mathbb{R}^\ell$ and of n degree d hypersurfaces. We will call such a system a (d,ℓ)-dense Gale system. Write $\mathbb{T}$ for the non-zero complex numbers, $\mathbb{C}^\times$.

3.1. **(d,ℓ)-dense polynomials.** Suppose that $\mathcal{A} = \psi(d\triangle^\ell \cap \mathbb{Z}^\ell) \bigcup \mathcal{W}$ is (d,ℓ)-dense and that it affinely spans $\mathbb{Z}^n$. Translating $\mathcal{A}$ by $\psi(0)$ if necessary, we may assume that ψ is linear. Write $\mathcal{V} = \{v_1, \dots, v_\ell\} \subset \mathbb{Z}^n$ for the images under ψ of the standard basis vectors of $\mathbb{Z}^\ell$ and list the elements of $\mathcal{W}$ as $\{w_1, \dots, w_n\}$. Consider the map

$$\begin{array}{rccl}\varphi : & \mathbb{T}^n & \longrightarrow & \mathbb{T}^\ell \times \mathbb{T}^n \\ & x & \longmapsto & (x^{v_1}, \dots, x^{v_\ell},\, x^{w_1}, \dots, x^{w_n})\,.\end{array}$$

Write $y = (y_1, \dots, y_\ell)$ for the coordinates of the first factor $\mathbb{T}^\ell$ and $z = (z_1, \dots, z_n)$ for the coordinates of the second factor $\mathbb{T}^n$. A polynomial with support $\mathcal{A}$ has the form

$$\begin{aligned} f \ &= \sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_\ell)\\ |\lambda|=d}} a_\lambda x^{\lambda_1 v_1} \cdots x^{\lambda_\ell v_\ell} \ + \ \sum_{i=1}^n a_i x^{w_i} \\ &= \varphi^*\Bigl(\sum_{|\lambda|=d} a_\lambda y_1^{\lambda_1} \cdots y_\ell^{\lambda_\ell} \ + \ \sum_{i=1}^n a_i z_i\Bigr) \ = \ \varphi^*(h(y) + \Lambda(z))\,, \end{aligned} \tag{3.1}$$

the pullback along φ of a polynomial h of degree d and a linear form Λ.

3.2. **Gale duality for (d,ℓ)-dense fewnomials.** Let $\mathcal{A}$ and $\mathcal{W}$ be as in the previous subsection. Suppose that $f_1(x) = \cdots = f_n(x) = 0$ is a system of (d,ℓ)-dense fewnomials with support $\mathcal{A}$. By (3.1), there exist polynomials $h_1(y), \dots, h_n(y)$ of degree d in the variables $y = (y_1, \dots, y_\ell)$ and linear forms $\Lambda_1(z), \dots, \Lambda_n(z)$ in variables $z = (z_1, \dots, z_n)$ such that

$$f_i(x) \ = \ \varphi^*(h_i(y) + \Lambda_i(z)) \qquad i = 1, \dots, n\,.$$

Since we wish to enumerate non-degenerate solutions, we may assume that the polynomials $h_i(y), \Lambda_i(z)$ are generic, for perturbing the coefficients of the f_i can only increase their number of non-degenerate solutions.

Thus we may assume that $\Lambda_1(z), \ldots, \Lambda_n(z)$ are linearly independent. Replacing the polynomials $f_1, \ldots, f_n$ by appropriate linear combinations, we may assume that $\Lambda_i(z) = -z_i$ for each i. Then our system becomes

$$\varphi^*(h_1(y) - z_1) \;=\; \varphi^*(h_2(y) - z_2) \;=\; \cdots \;=\; \varphi^*(h_n(y) - z_n) \;=\; 0\,. \tag{3.2}$$

If we define $H \subset \mathbb{C}^\ell \times \mathbb{C}^n$ by the equations

$$H \;=\; \{(y, z) \in \mathbb{C}^\ell \times \mathbb{C}^n \mid z_1 = h_1(y), \ldots, z_n = h_n(y)\}\,,$$

then our system (3.2) has the alternative geometric description as $\varphi^*(H)$. Since $\mathbb{Z}\mathcal{A} = \mathbb{Z}^n$, φ is an isomorphism onto its image, and we deduce the following lemma.

Lemma 3.3. *The system* (3.2) *is isomorphic to the intersection* $\varphi(\mathbb{T}^n) \cap H$ *in* $\mathbb{C}^\ell \times \mathbb{C}^n$*.*

This is the first step in Gale duality for (d, ℓ)-dense fewnomials. For the second step observe that H is isomorphic to $\mathbb{C}^\ell$, as it is the graph of the function $\mathbb{C}^\ell \to \mathbb{C}^n$ given by $y \mapsto (h_1(y), \ldots, h_n(y))$. Let $\Psi\colon \mathbb{C}^\ell \to H$ be the isomorphism between $\mathbb{C}^\ell$ and this graph. Then the system (3.2) is equivalent to

$$\varphi(\mathbb{T}^n) \cap \Psi(\mathbb{C}^\ell)\,.$$

We determine the equations in $\mathbb{T}^\ell \times \mathbb{T}^n$ that define $\varphi(\mathbb{T}^n)$.

For $\beta = (b_1, \ldots, b_\ell) \in \mathbb{Z}^\ell$ and $\gamma = (c_1, \ldots, c_n) \in \mathbb{Z}^n$, let

$$y^\beta \cdot z^\gamma \;=\; y_1^{\beta_1} \cdots y_\ell^{\beta_\ell} \cdot z_1^{c_1} \cdots z_n^{c_n}\,.$$

We similarly write $h(y)^\gamma$ for $h_1(y)^{c_1} \cdots h_n(y)^{c_n}$.

Suppose that $\mathcal{B} \subset \mathbb{Z}^\ell \oplus \mathbb{Z}^n$ is a basis for the $\mathbb{Z}$-linear relations among the exponent vectors $\mathcal{V} \bigcup \mathcal{W}$. As $\mathcal{A}$ spans $\mathbb{R}^n$, so does $\mathcal{V} \bigcup \mathcal{W}$, and so $\mathcal{B}$ consists of ℓ vectors, $\{(\beta_1, \gamma_1), \ldots, (\beta_\ell, \gamma_\ell)\}$. Then the image $\varphi(\mathbb{T}^n) \subset \mathbb{T}^\ell \times \mathbb{T}^n$ is the subtorus defined by

$$y^{\beta_j} \cdot z^{\gamma_j} \;=\; 1 \qquad \text{for } j = 1, \ldots, \ell\,. \tag{3.4}$$

Proposition 3.5. *The pullback of* $\varphi(\mathbb{T}^n) \cap H$ *along the map* Ψ *is the system*

$$y^{\beta_j} \cdot h(y)^{\gamma_j} \;=\; 1 \qquad \textit{for } j = 1, \ldots, \ell\,. \tag{3.6}$$

This is well-defined in $\mathbb{C}^\ell$ *in the complement* $\mathcal{M}(\mathbb{C})$ *of the coordinate planes and the hypersurfaces* $h_i(y) = 0$ *for* $i = 1, \ldots, n$*.*

We may now state our main theorem on structured Gale duality, which contains a stronger version of Theorem 2.9. The saturation of a submodule $\mathcal{B} \subset \mathbb{Z}^\ell \oplus \mathbb{Z}^n$ is the set $(\mathcal{B} \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^\ell \oplus \mathbb{Z}^n$ of integer points in its linear span.

Theorem 3.7. *Suppose that* $\mathcal{A}, \mathcal{W}, \psi,$ *and* $\mathcal{V}$ *are as above and that* $\mathcal{A}$ *spans* $\mathbb{Z}^n$*. Then the solution set to* (3.2) *in* $\mathbb{T}^n$ *is scheme-theoretically isomorphic to the solution set of the system of rational functions* (3.6) *defined in* $\mathbb{T}^\ell$ *in the complement of the hypersurfaces* $h_i(y) = 0$*, for* $i = 1, \ldots, n$*.*

If the coefficients of the polynomials f_i *are real, then so are those of* $h_i(y)$*. If the span of* $\mathcal{A}$ *has odd index in* $\mathbb{Z}^n$ *and the integer span of the exponents* $\mathcal{B}$ *has odd index in its saturation, then the analytic subscheme defined in* $(\mathbb{R}^\times)^n$ *by* (3.2) *is isomorphic to the analytic subscheme defined by* (3.6) *in the complement of the hypersurfaces* $h_i(y) = 0$ *in* $(\mathbb{R}^\times)^\ell$*.*

If now the exponents $\mathcal{A}$ only span a full rank sublattice of $\mathbb{Z}^n$ and the exponents $\mathcal{B}$ only span a full rank sublattice of the module of linear relations among $\mathcal{V}\bigcup\mathcal{W}$, then the analytic subscheme of $\mathbb{R}^n_{>0}$ defined by (3.2) *is isomorphic to the analytic subscheme defined by* (3.6) *in the subset of $\mathbb{R}^\ell_{>0}$ defined by $h_i(y)>0$ for $i=1,\ldots,n$.*

Proof. The first statement concerning complex solutions is immediate from Proposition 3.5 and the observation that the system (3.2) is the pullback of the intersection $\varphi(\mathbb{T}^n)\cap H$ along the map φ, if we know that the map φ is injective. Since $\mathcal{A}$ spans $\mathbb{Z}^n$, so does $\mathcal{V}\bigcup\mathcal{W}$, and the map φ is injective.

As the affine span of $\mathcal{A}$ has odd index in $\mathbb{Z}^n$, the map φ is injective on $(\mathbb{R}^\times)^n$. As $\mathbb{Z}\mathcal{B}$ has odd index in its saturation, the equations $y^\beta\cdot z^\gamma=1$ for $(\beta,\gamma)\in\mathcal{B}$ define the image $\varphi((\mathbb{R}^\times)^n)$ in the real torus $(\mathbb{R}^\times)^\ell\times(\mathbb{R}^\times)^n$. These facts in turn imply the second statement.

Similarly, the hypotheses of the third statement imply the same facts about the positive part of the real torus, $\mathbb{R}^n_{>0}$. Observing that the subset of $\mathbb{R}^\ell_{>0}$ defined by $h_i(y)>0$ for $i=1,\ldots,n$ is the pullback of $\mathbb{R}^\ell_{>0}\times\mathbb{R}^n_{>0}$ under the map Ψ completes the proof. $\square$

4. Bounds for (d,ℓ)-dense Gale systems

By Theorem 2.9, Theorem 2.4 follows from bounds for (d,ℓ)-dense Gale systems, which we give and prove below.

Let d,ℓ,n be positive integers and $y_1,\ldots,y_\ell$ be indeterminates. Suppose that $h_i(y)$ for $i=1,\ldots,n$ are generic degree d polynomials. Define

$$\begin{aligned}\Delta\ &:=\ \{y\in\mathbb{R}^\ell_{>0}\mid h_i(y)>0\text{ for }i=1,\ldots,n\}\qquad\text{and}\\ \mathcal{M}(\mathbb{R})\ &:=\ \{y\in(\mathbb{R}^\times)^\ell\mid h_i(y)\neq0\text{ for }i=1,\ldots,n\}\,,\end{aligned}$$

and write $\mathcal{M}(\mathbb{C})$ for the complexification of $\mathcal{M}(\mathbb{R})$.

Theorem 4.1. *With the preceding definitions, suppose that $\beta_1,\ldots,\beta_\ell\in\mathbb{Z}^\ell$ and $\gamma_1,\ldots,\gamma_\ell\in\mathbb{Z}^n$ are vectors such that $\mathcal{B}:=\{(\beta_1,\gamma_1),\ldots,(\beta_\ell,\gamma_\ell)\}\subset\mathbb{Z}^\ell\oplus\mathbb{Z}^n$ are linearly independent. Then the number of solutions to*

$$y^{\beta_j}\cdot h(y)^{\gamma_j}\ =\ 1\,,\qquad \textit{for}\quad j\ =\ 1,\ldots,\ell\,, \tag{4.2}$$

in the positive region Δ is less than

$$\frac{e^2+3}{4}\,2^{\binom{\ell}{2}}\,n^\ell\cdot d^\ell\,.$$

If the integer span of $\mathcal{B}$ has odd index in its saturation, then the number of solutions in $\mathcal{M}(\mathbb{R})$ is less than

$$\frac{e^4+3}{4}\,2^{\binom{\ell}{2}}\,n^\ell\cdot d^\ell\,.$$

We will deduce these bounds from several lemmata which we now formulate. Their proofs are given in subsequent subsections.

For a vector α, let $\alpha^\pm$ be the coordinatewise maximum of $\pm\alpha$ and 0 so that $\alpha^\pm$ is nonnegative, and $\alpha=\alpha^+-\alpha^-$. Hence $(1,-2)^+=(1,0)$ and $(1,-2)^-=(0,2)$. Set

$$g_k(y)\ :=\ y^{2\beta_k^+}h(y)^{2\gamma_k^+}-y^{2\beta_k^-}h(y)^{2\gamma_k^-}\,. \tag{4.3}$$

Then $g_k(y) = 0$ for $y \in \mathcal{M}(\mathbb{C})$ if and only if

$$\prod_{i=1}^{\ell}(y_i^2)^{\beta_{k,i}} \cdot \prod_{i=1}^{n}(h_i(y)^2)^{\gamma_{k,i}} \;=\; 1 \tag{4.4}$$

Notice that the system

$$g_1(y) \;=\; g_2(y) \;=\; \cdots \;=\; g_\ell(y) \;=\; 0\,,$$

is equivalent in Δ to the system (4.2) and in $\mathcal{M}(\mathbb{R})$, it contains the system (4.2) as a subsystem. We will bound the number of solutions to this expanded system in Δ and in $\mathcal{M}(\mathbb{R})$ to obtain our bounds for the system (4.2) in Theorem 4.1.

We state two important reductions.

Reduction 4.5. *It suffices to prove Theorem* 4.1 *under the following additional assumptions.*

(1) *For each $j = 1, \ldots, \ell$, the set $\mu_j \subset \mathcal{M}(\mathbb{C})$ defined by the equations*

$$y^{2\beta_k} \cdot h(y)^{2\gamma_k} \;:=\; \prod_{i=1}^{\ell}(y_i^2)^{\beta_{k,i}} \cdot \prod_{i=1}^{n}(h_i(y)^2)^{\gamma_{k,i}} \;=\; 1 \qquad k = 1, \ldots, j\,,$$

is smooth and has codimension j. This condition holds for all sufficiently generic polynomials $h_i(y)$ of degree d.

(2) *For each $k = 1, \ldots, \ell$ define $b_k := \beta_{k,1} + \cdots + \beta_{k,\ell} + d(\gamma_{k,1} + \cdots + \gamma_{k,n})$. Then every minor of the $\ell \times (1 + \ell + n)$ matrix whose kth row is $(-b_k, \beta_k, \gamma_k)$ is nonzero.*

We establish these reductions in Subsection 4.1.

Our bounds are based on an induction which comes from the Khovanskii-Rolle Theorem, or more precisely, the induction is based on a modified form which was used in [8], and which ensures that the hypotheses in subsequent lemmata hold. See [5] and [8] for more discussion. For D equal to either $\mathcal{M}(\mathbb{R})$ or its positive chamber Δ and C an algebraic curve in D, let $\mathrm{ubc}_D(C)$ be the number of noncompact connected components of C. Write $V_D(f_1, \ldots, f_\ell)$ for the common zeroes in D of functions $f_1, \ldots, f_\ell$.

Lemma 4.6 (Modified Khovanskii-Rolle Theorem)**.** *There exist polynomials $G_1, G_2, \ldots, G_\ell$ where, for each $j = 1, \ldots, \ell$, $G_{\ell-j}(y)$ is a generic polynomial with degree $2^j n \cdot d$ such that the following hold.*

(1) *For each $j = 0, \ldots, \ell - 1$, the system*

$$g_1(y) \;=\; \cdots \;=\; g_j(y) \;=\; G_{j+1}(y) \;=\; \cdots \;=\; G_\ell(y) \;=\; 0\,,$$

has only nondegenerate solutions in $\mathcal{M}(\mathbb{C})$ and the system

$$g_1(y) \;=\; \cdots \;=\; g_{j-1}(y) \;=\; G_{j+1}(y) \;=\; \cdots \;=\; G_\ell(y) \;=\; 0\,, \tag{4.7}$$

(g_j is omitted) defines a smooth curve in $\mathcal{M}(\mathbb{C})$.

(2) *For $j = 1, \ldots, \ell$, let the smooth real algebraic curve $C_j \subset \mathcal{M}(\mathbb{R})$ be defined by the solutions to* (4.7) *in $\mathcal{M}(\mathbb{R})$. For D equal to either $\mathcal{M}(\mathbb{R})$ or Δ, we have*

$$|V_D(g_1, \ldots, g_j, G_{j+1}, \ldots, G_\ell)| \;\leq\; |V_D(g_1, \ldots, g_{j-1}, G_j, \ldots, G_\ell)| + \mathrm{ubc}_D(C_j)\,.$$

This implies the following estimate. Let D be equal to either $\mathcal{M}(\mathbb{R})$ or Δ. Then we have

$$|V_D(g_1,\dots,g_\ell)| \;\le\; |V_D(G_1,\dots,G_\ell)| + \mathrm{ubc}_D(C_1) + \cdots + \mathrm{ubc}_D(C_\ell)\,. \tag{4.8}$$

Our next lemma estimates these quantities.

Lemma 4.9. *We have*

(1) $V_\Delta(G_1,\dots,G_\ell) \le V_{\mathcal{M}}(G_1,\dots,G_\ell) \le 2^{\binom{\ell}{2}} n^\ell \cdot d^\ell$,

(2) $\mathrm{ubc}_\Delta(C_j) \le \frac{1}{2} 2^{\binom{\ell-j}{2}} n^{\ell-j}\binom{1+\ell+n}{j} \cdot d^\ell$.

(3) $\mathrm{ubc}_{\mathcal{M}}(C_j) \le \frac{1}{2} 2^{\binom{\ell-j}{2}} 2^j n^{\ell-j}\binom{1+\ell+n}{j} \cdot d^\ell$.

(4) $2^{\binom{\ell-j}{2}} n^{\ell-j}\binom{1+\ell+n}{j} \le \frac{1}{2}\frac{2^j}{j!} \cdot 2^{\binom{\ell}{2}} n^\ell$.

Statement (1) follows from Lemma 4.6 by Bézout's Theorem, as $2^{\binom{\ell}{2}} n^\ell d^\ell$ is the product of the degrees of the polynomials $G_1,\dots,G_\ell$. Statement (4) follows from the proof of Lemma 3.5 of [5]. We prove the other statements of Lemma 4.9 in Subsection 4.3.

Proof of Theorem 4.1. Lemma 4.9 and the estimate (4.8) give us the estimate

$$|V_\Delta(g_1,\dots,g_\ell)| \;\le\; 2^{\binom{\ell}{2}} n^\ell \cdot d^\ell \;+\; \sum_{j=1}^{\ell} \frac{1}{4}\frac{2^j}{j!} \cdot 2^{\binom{\ell}{2}} n^\ell \cdot d^\ell \;\le\; \Bigl(1 \;+\; \frac{1}{4}\sum_{j=1}^{\ell}\frac{2^j}{j!}\Bigr) \cdot 2^{\binom{\ell}{2}} n^\ell \cdot d^\ell\,.$$

The sum is a partial sum of the power series for e^2-1, and so we obtain

$$|V_\Delta(g_1,\dots,g_\ell)| \;<\; \frac{e^2+3}{4} 2^{\binom{\ell}{2}} n^\ell \cdot d^\ell\,.$$

The estimation for $|V_{\mathcal{M}}(g_1,\dots,g_\ell)|$ is similar. Using Lemma 4.9(3) for $\mathrm{ubc}_{\mathcal{M}}(C_j)$, the corresponding sum is now a partial sum for e^4-1, and so we obtain

$$|V_{\mathcal{M}}(g_1,\dots,g_\ell)| \;<\; \frac{e^4+3}{4} 2^{\binom{\ell}{2}} n^\ell \cdot d^\ell\,,$$

which completes the proof of Theorem 4.1. □

4.1. **Proof of reductions.** The Reduction 4.5(1) will follow from Bertini's Theorem that a general linear section of a smooth quasi-projective variety is smooth and of the expected dimension. First, define $\mathbb{G}_j \subset \mathbb{T}^\ell \times \mathbb{T}^n$ to be the subtorus defined by the equations

$$y^{2\beta_k} z^{2\gamma_k} \;=\; 1 \qquad \text{for} \quad k = 1,\dots,j\,.$$

As in Section 3, let $\Psi\colon \mathbb{C}^\ell \to \mathbb{C}^\ell \times \mathbb{C}^n$ be the map defined by

$$\Psi \;:\; y \;\longmapsto\; (y,\, h_1(y),\dots,h_n(y))\,.$$

Then $\mu_j = \Psi^{-1}(\mathbb{G}_j \cap \Psi(\mathbb{C}^\ell))$. Since Ψ is an isomorphism onto its image and $\mathbb{G}_j$ has codimension j, it suffices to show that $\mathbb{G}_j \cap \Psi(\mathbb{C}^\ell)$ is transverse.

But this follows because $\Psi(\mathbb{C}^\ell)$ is the pullback of a linear subspace L along the map

$$\begin{array}{rcl} \mathbb{C}^\ell \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^{\binom{\ell+d}{d}} \times \mathbb{C}^n \\ (y,z) & \longmapsto & ((y^\lambda\colon |\lambda| = d),\, z)\,. \end{array}$$

The linear space L is defined by the coefficients of the polynomials in a system Gale dual to the system (4.2). Choosing L to be generic, we may apply Bertini's Theorem and deduce that μ_j is smooth and of codimension j. We also see that this may be accomplished by choosing the polynomials $h_i(y)$ to be sufficiently generic.

For the second reduction, observe that our equations (4.3) and (4.4) are equivalent to

$$f_k(y) \;:=\; \sum_{m=1}^{\ell} \beta_{k,m} \log|y_m| \;+\; \sum_{i=1}^{n} \gamma_{k,i} \log|h_i(y)| \;=\; 0\,, \qquad k = 1, \ldots, \ell\,, \tag{4.10}$$

in $\mathcal{M}(\mathbb{R})$. We may perturb them by changing the coefficients $\beta_{k,m}$ and $\gamma_{k,i}$ without increasing their numbers of nondegenerate solutions. Thus, we can satisfy Reduction 4.5(2) with real exponents. Since the rational numbers are dense in the real numbers, we may satisfy Reduction 4.5(2) with rational exponents. Finally, by clearing denominators, we may assume the exponents are integral.

4.2. **Proof of Lemma 4.6.** We will establish Lemma 4.6 by downward induction on j. The main step is provided by the Khovanskii-Rolle Theorem (see §3.4 in [10] or Theorem 3.3 in [5]), which we present in the simplified form in which we need it.

Theorem 4.11 (Khovanskii-Rolle)**.** *Let $f_1, \ldots, f_\ell$ be smooth functions defined on a domain $D \subset \mathbb{R}^\ell$ where*

$$f_1(y) \;=\; f_2(y) \;=\; \cdots \;=\; f_{\ell-1}(y) \;=\; 0\,,$$

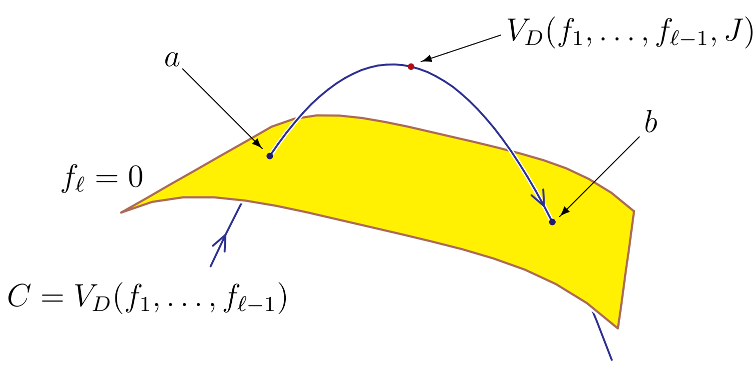
defines a smooth curve C in D. Let

$$J \;:=\; J(f_1, \ldots, f_\ell) \;:=\; \det\left(\frac{\partial f_i}{\partial y_j}\right)_{i,j=1,\ldots,\ell}$$

be the Jacobian determinants of $f_1, \ldots, f_\ell$. If $V_D(f_1, \ldots, f_{\ell-1}, J)$ is finite and if C has finitely many components in D, then $V_D(f_1, \ldots, f_\ell)$ is finite and we have

$$|V_D(f_1, \ldots, f_\ell)| \;\leq\; |V_D(f_1, \ldots, f_{\ell-1}, J)| + \mathrm{ubc}_D(C)\,. \tag{4.12}$$

Proof. Note that on C, the Jacobian J is proportional to the evaluation of the differential of f_ℓ on a tangent vector to C. Given two consecutive solutions a, b to $f_\ell = 0$ along an arc of C, The Jacobian will have different signs at a and at b, and therefore will vanish at least once between a and b.



The estimate (4.12) follows as compact components of C contain as many arcs connecting zeroes of f_ℓ as zeroes of f_ℓ, while noncompact components contain one arc fewer. ∎

To deduce Lemma 4.6, we will iterate the Khovanskii-Rolle Theorem, showing that the appropriate Jacobians have the claimed degrees and ensuring that its hypotheses are satisfied.

Observe that Lemma 4.6(1) with $j = \ell$ holds by the assumptions we make in Reduction 4.5. We prove Lemma 4.6 by downward induction on $j = \ell, \ldots, 1$. Specifically, we will assume that Statement (1) holds of some j and then construct a polynomial G_j such that (2) holds for j and (1) holds for $j{-}1$.

To construct the polynomials G_j, we replace the rational functions $g_k(y)$ for $k = 1, \ldots, \ell$ in Lemma 4.6(2) by the logarithmic functions $f_k(y)$ for $k = 1, \ldots, \ell$ defined in (4.10). We may do this, as if $y \in \mathcal{M}(\mathbb{R})$, then $f_k(y) = 0$ if and only if $g_k(y) = 0$.

First, we need to determine the degrees of the Jacobians.

Lemma 4.13. *Let $1 \leq j \leq \ell$ and suppose that $G_{j+1}, \ldots, G_\ell$ are polynomials, where G_i has degree $2^{\ell-i}n \cdot d$, but is otherwise general, for each $i = 1, \ldots, \ell$. Then*

$$\prod_{m=1}^{\ell} y_m \cdot \prod_{i=1}^{n} h_i(y) \cdot J(f_1, \ldots, f_j,\, G_{j+1}, \ldots, G_\ell) \tag{4.14}$$

is a polynomial with degree $2^{\ell-j}n \cdot d$.

We use this to deduce Lemma 4.6. Suppose that we have polynomials $G_{j+1}, \ldots, G_\ell$ where G_i is a generic polynomial with degree $2^{\ell-i}n \cdot d$, for each $i = j{+}1, \ldots, \ell$, and Lemma 4.6(1) holds for j.

Let $C_j \subset \mathcal{M}(\mathbb{R})$ be the smooth real algebraic curve defined by

$$f_1(y) \;=\; \cdots \;=\; f_{j-1}(y) \;=\; G_{j+1}(y) \;=\; \cdots \;=\; G_\ell(y) \;=\; 0\,.$$

Let $\overline{G}_j$ be the product of the Jacobian $J(f_1, \ldots, f_j,\, G_{j+1}, \ldots, G_\ell)$ with the polynomial

$$\Upsilon(y) \;:=\; \prod_{m=1}^{\ell} y_m \cdot \prod_{i=1}^{n} h_i(y)\,.$$

Then $\overline{G}_j$ is a polynomial with degree $2^{\ell-j}n \cdot d$, by Lemma 4.13. Since Υ does not vanish in $\mathcal{M}(\mathbb{R})$, the Jacobian and $\overline{G}_\ell$ have the same zero set in $\mathcal{M}(\mathbb{R})$. Then we have

$$\begin{aligned} |V_D(f_1, \ldots, f_j,\, G_{j+1}, \ldots, G_\ell)| \;\leq& \\ &|V_D(f_1, \ldots, f_{j-1},\, \overline{G}_j,\, G_{j+1}, \ldots, G_\ell)| \;+\; \mathrm{ubc}_D(C_j)\,, \end{aligned} \tag{4.15}$$

by the Khovanskii-Rolle Theorem.

Note however that we do not know if $\overline{G}_j$ is a generic polynomial with degree $2^{\ell-j}n \cdot d$, and in particular, we do not know if the hypotheses of Lemma 4.6(1) for $j{-}1$ hold. These hypotheses may be achieved by perturbing $\overline{G}_j$ to a nearby generic polynomial G_j with degree $2^{\ell-j}n \cdot d$. To ensure that this perturbation does not destroy the estimate (4.15), we only need to guarantee that the signs of G_j and $\overline{G}_j$ are the same at every point of $V_D(f_1, \ldots, f_j,\, G_{j+1}, \ldots, G_\ell)$, but this will hold for all sufficiently small perturbations, as there are only finitely many such points and $\overline{G}_j$ is nonzero at each. These conditions are equivalent to every point of $V_D(f_1, \ldots, f_j,\, G_{j+1}, \ldots, G_\ell)$ being nondegenerate, which is ensured by the genericity of $G_{j+1}, \ldots, G_\ell$. This completes the proof of Lemma 4.6.

Proof of Lemma 4.13. Write ∂_m for $\partial/\partial y_m$, and consider the expression (4.14), writing the Jacobian in block form, with j rows for $f_1,\ldots,f_j$ and $\ell-j$ rows for $G_{j+1},\ldots,G_\ell$.

$$\prod_{m=1}^{\ell} y_m\cdot\prod_{i=1}^{n} h_i(y)\cdot\det\left(\frac{\big(\partial_m f_k(y)\big)_{k=1,\ldots,j}^{m=1,\ldots,\ell}}{\big(\partial_m G_k(y)\big)_{k=j+1,\ldots,\ell}^{m=1,\ldots,\ell}}\right) = \prod_{i=1}^{n} h_i(y)\cdot\det\left(\frac{\big(y_m\partial_m f_k(y)\big)_{k=1,\ldots,j}^{m=1,\ldots,\ell}}{\big(y_m\partial_m G_k(y)\big)_{k=j+1,\ldots,\ell}^{m=1,\ldots,\ell}}\right). \tag{4.16}$$

Laplace expansion along the first j rows of the matrix on the right expresses its determinant as a sum of products of maximal minors of the two blocks. We will prove Lemma 4.13 by showing that each term in that sum is a polynomial with degree $2^{\ell-j}n\cdot d$.

First, the lower block $\big(y_m\partial_m G_k(y)\big)_{k=j+1,\ldots,\ell}^{m=1,\ldots,\ell}$ is a matrix of polynomials whose entries in row k are the toric derivatives $y_m\partial_m G_k(y)$ of G_k. Thus every entry in row k has degree $\deg(G_k)=2^{\ell-k}n\cdot d$, and therefore each minor has degree

$$2^{\ell-(j+1)}n\cdot d\ +\ \cdots\ +\ 2n\cdot d\ +\ n\cdot d\ =\ (2^{\ell-j}-1)n\cdot d\,. \tag{4.17}$$

For the upper block, note that $y_m\partial_m f_k(y)$ is

$$\beta_{k,m}\ +\ \sum_{i=1}^{k}\gamma_{k,i}y_m\partial_m\log|h_i(y)|\ =\ \beta_{k,m}\ +\ \sum_{i=1}^{n}\gamma_{k,i}\frac{y_m\partial_m h_i(y)}{h_i(y)}\,.$$

In particular, the upper block is a product of a $j\times(\ell+n)$ matrix and a $(\ell+n)\times\ell$ matrix,

$$\big(y_m\partial_m f_k(y)\big)_{k=1,\ldots,j}^{m=1,\ldots,\ell}\ =\ \big(\,\beta_{k,q}\ \mid\ \gamma_{k,i}\,\big)\cdot\begin{pmatrix} I_\ell\\ y_m\partial_m h_i(y)/h_i(y)\end{pmatrix}\ ,$$

where $I_\ell=(\delta_{q,m})$ is the $\ell\times\ell$ identity matrix. By the Cauchy-Binet formula, a $j\times j$ minor of $\big(y_m\partial_m f_k(y)\big)$ is a sum of products of $j\times j$ minors of the two matrices on the right. Consider now the product of $\prod_{i=1}^n h_i(y)$ with a term in this sum.

The first matrix $\big(\beta_{k,q}\mid\gamma_{k,i}\big)$ contains constants, and a $j\times j$ minor of the second involves no more than $p:=\min\{n,j\}$ rows from its lower $n\times\ell$ block $(y_m\partial_m h_i(y)/h_i(y))$. This minor is a sum of $j!$ terms, each one a product of a constant and p entries of the matrix of the form $y_m\partial_m h_i(y)/h_i(y)$, for different rows i. Multiplying this term by $\prod_{i=1}^n h_i(y)$ will clear all denominators and result in a product of p terms of the form $y_m\partial_m h_i(y)$ and complementary $n-p$ terms of the form $h_i(y)$. Each of these terms has degree d, so each term coming from the expansion of this $j\times j$ minor has degree $n\cdot d$, and therefore the product of $\prod_{i=1}^n h_i(y)$ by each $j\times j$ minor of the upper block of (4.16) will have degree $n\cdot d$.

Together with (4.17), this completes the proof. □

4.3. **Proof of Lemma 4.9.** We only need to prove Statements (2) and (3) of Lemma 4.9. By Reduction 4.5 and Lemma 4.6, we may assume that the polynomials $h_i(y)$ and $G_j(y)$ are generic given their degrees.

The complexification of the real curve C_j is defined in $\mathbb{C}^\ell\supset\mathcal{M}(\mathbb{C})$ by

$$g_1(y)\ =\ \cdots\ =\ g_{j-1}(y)\ =\ G_{j+1}(y)\ =\ \cdots\ =\ G_\ell(y)\ =\ 0\,, \tag{4.18}$$

and it lies on the codimension $j-1$ subvariety $\mu_{j-1} \subset \mathcal{M}(\mathbb{C})$ defined by

$$g_1(y) \;=\; \cdots \;=\; g_{j-1}(y) \;=\; 0\,.$$

We bound the number of unbounded components of C_j by first describing the points where $\mu_{j-1}(\mathbb{C})$ meets the boundary of $\mathcal{M}(\mathbb{C})$, then bound the number of real solutions to

$$G_{j+1}(y) \;=\; \cdots \;=\; G_\ell(y) \;=\; 0 \tag{4.19}$$

on these boundary points, and lastly by determining the number of components of C_j incident upon each such real solution.

To accomplish this, consider $\mathcal{M}(\mathbb{C}) \subset \mathbb{C}^\ell$ as a subset of projective space $\mathbb{P}^\ell$. Its boundary $\partial\mathcal{M}(\mathbb{C}) := \mathbb{P}^\ell \setminus \mathcal{M}(\mathbb{C})$ consists of the finite coordinate planes $y_m = 0$ for $m = 1, \ldots, \ell$, the coordinate plane at infinity $y_0 = 0$, and the degree d hypersurfaces $h_i(y) = 0$ for $i = 1, \ldots, n$. Strictly speaking, we must homogenize polynomials h_i, g_j, G_k with respect to the coordinate y_0 at infinity. When working on an affine patch where $y_m \neq 0$, we de-homogenize them by setting $y_m = 1$.

By our assumption that the polynomials h_i were general, this boundary $\partial\mathcal{M}(\mathbb{C})$ forms a divisor with normal crossings whose components are the coordinate planes and the hypersurfaces $h_i(y) = 0$. The common zeroes of any q of the polynomials $h_1, \ldots, h_n$ and $j-q$ of the coordinates $y_0, \ldots, y_\ell$ is a smooth subvariety of codimension j, called a *codimension-j stratum*. The union of these j-fold intersections of the components of the boundary divisor is called the *codimension-j skeleton* of $\partial\mathcal{M}(\mathbb{C})$.

Lemma 4.20. *The closure $\overline{\mu_{j-1}}$ meets $\partial\mathcal{M}(\mathbb{C})$ in a union of codimension-j strata and in the neighborhood of a real point of $\overline{\mu_{j-1}}$ lying in the relative interior of a codimension-j stratum, μ_{j-1} has one branch in each of the 2^j components of $\mathcal{M}(\mathbb{R})$ incident on that point.*

Proof. Since $\mu_{j-1} \subset \mathcal{M}(\mathbb{C})$ has codimension $j-1$ in $\mathbb{P}^\ell$, the intersection of its closure $\overline{\mu_{j-1}}$ with the boundary divisor $\partial\mathcal{M}(\mathbb{C})$ will have codimension j in $\mathbb{P}^\ell$. We prove the first part of the lemma by showing that this intersection consists of points lying within the codimension-j skeleton of $\partial\mathcal{M}(\mathbb{C})$, and therefore is a union of components of the codimension-j skeleton.

Let Y be a point of $\partial\mathcal{M}(\mathbb{C})$ that does not lie in the codimension-j skeleton. We show that $Y \notin \overline{\mu_{j-1}}$. Since $\partial\mathcal{M}(\mathbb{C})$ is a divisor with normal crossings whose components are defined by the coordinates y_i and forms h_i, at least one, but no more than $j-1$ of the coordinates $y_0, y_1, \ldots, y_\ell$ and forms $h_1, \ldots, h_n$ vanish at Y. Reordering the coordinates and forms if necessary, we may assume that the forms which vanish at Y are among $h_1, \ldots, h_q$ and the coordinates are among $y_{q+1}, \ldots, y_{j-1}$. Since the assertion about Y is local, we may restrict to the affine patch U where none of the remaining coordinates or forms vanish.

The equations

$$y^{2\beta_k} h(y)^{2\gamma_k} \;=\; 1 \qquad k = 1, \ldots, j-1\,, \tag{4.21}$$

define $\mu_{j-1} \subset \mathcal{M}(\mathbb{C})$. By Reduction 4.5(2) on the homogenized exponent vectors, there is an integer linear combination of the first $j-1$ rows of the matrix $(-b_k, \beta_k, \gamma_k)$ so that the columns corresponding to $h_1, \ldots, h_q$ and $y_{q+1}, \ldots, y_{j-1}$ become diagonal. These same linear combinations transform the equations (4.21) into equations of the form

$$\begin{aligned} h_i(y)^{a_i} \;&=\; y^{\alpha_i} \cdot h(y)^{\delta_i} \qquad i = 1, \ldots, q\,, \\ y_i^{a_i} \;&=\; y^{\alpha_i} \cdot h(y)^{\delta_i} \qquad i = q{+}1, \ldots, j{-}1\,, \end{aligned} \tag{4.22}$$

where $a_i>0$ and the components of α_i in positions $q{+}1,\dots,j{-}1$ vanish as do the components of δ_i in positions $1,\dots,q$. That is, $h_1,\dots,h_q$ and $y_{q+1},\dots,y_{j-1}$ do not appear on the right of these expressions.

Since the expressions in (4.22) are well-defined functions in U, the regular functions

$$\begin{array}{rcll} h_i(y)^{a_i} & - & y^{\alpha_i}\cdot h(y)^{\delta_i} & \text{for}\quad i=1,\dots,q\,,\quad\text{and}\\ y_i^{a_i} & - & y^{\alpha_i}\cdot h(y)^{\delta_i} & \text{for}\quad i=q{+}1,\dots,j{-}1 \end{array}$$

vanish on $\mu_{j-1}\cap U$, and hence on $\overline{\mu_{j-1}}\cap U$. However, these cannot all vanish at Y, for none of the functions $y^{\alpha_i}h^{\delta_i}$ for $i=1,\dots,j{-}1$ vanish at Y, but at least one of $h_1^{a_1},\dots,h_q^{a_q}$, $y_{q+1}^{a_{q+1}},\dots,y_{j-1}^{a_{j-1}}$ vanishes at Y.

To complete the proof, suppose that $Y\in\overline{\mu_{j-1}}(\mathbb{R})\cap\partial\mathcal{M}(\mathbb{R})$ is a real point lying on a codimension-j stratum of $\partial\mathcal{M}(\mathbb{C})$ but not on a stratum of larger codimension. Reordering the coordinates and functions if necessary and working locally, we may assume that the polynomials $h_1(y),\dots,h_q(y)$ and coordinates $y_{q+1},\dots,y_j$ vanish at Y. Thus in the affine neighborhood U of Y where none of the other polynomials or coordinates vanish, μ_{j-1} is defined by equations of the form

$$\begin{array}{rcll} h_i(y)^{a_i} & = & y_j^{c_i}\cdot y^{\alpha_i}\cdot h(y)^{\delta_i} & i=1,\dots,q\,,\\ y_i^{a_i} & = & y_j^{c_i}\cdot y^{\alpha_i}\cdot h(y)^{\delta_i} & i=q{+}1,\dots,j\,, \end{array}\tag{4.23}$$

where $a_i>0$ and none of $h_1,\dots,h_q$ and $y_{q+1},\dots,y_j$ appear in the expressions $y^{\alpha_i}\cdot h(y)^{\delta_i}$ for $i=1,\dots,j{-}1$. In fact, we must have $c_i>0$ as $Y\in\overline{\mu_{j-1}}$. In a neighborhood of Y in $\mathbb{RP}^\ell$, the complement $\mathcal{M}(\mathbb{R})$ has 2^j chambers given by the signs of the functions $h_1(y),\dots,h_q(y)$, $y_{q+1},\dots,y_j$. Since the exponents in (4.23) have every component even (this comes from the evenness of the exponents in (4.21)), there is a component of μ_{j-1} in each of these chambers. $\square$

We complete the proof of Statements (2) and (3) of Lemma 4.9. We estimate the number of unbounded components of the curve C_j by first bounding the number of points where its closure in $\mathbb{RP}^\ell$ meets the boundary divisor, and then bounding the number of components of C_j incident upon each point.

The estimate for the number of points in $\overline{C_j}\cap\partial\mathcal{M}(\mathbb{R})$ is simply the number of points in the codimension-j skeleton where

$$G_{j+1}(y)\;=\;G_{j+2}(y)\;=\;\cdots\;=\;G_\ell(y)\;=\;0\,.\tag{4.24}$$

Consider a stratum where q of the polynomials h_i vanish and $j{-}q$ of the coordinates y_m vanish. Since the polynomials G_i and h_i are general given their degrees, the number of points on this stratum will be at most the product of these degrees, which is

$$d^q\cdot 2^{\ell-j-1}nd\cdot 2^{\ell-j-2}nd\cdots 2nd\cdot nd\;=\;2^{\binom{\ell-j}{2}}\cdot n^{\ell-j}d^{\ell-j}d^q\,.$$

Since there are $\binom{\ell+1}{j-q}\cdot\binom{n}{q}$ such strata, the number of points where C_j meets the boundary is at most

$$2^{\binom{\ell-j}{2}}\cdot n^{\ell-j}d^{\ell-j}\cdot\;\sum_{q=0}^{j}\binom{\ell+1}{j-q}\binom{n}{q}\cdot d^q\;<\;2^{\binom{\ell-j}{2}}\cdot n^{\ell-j}\binom{1+\ell+n}{j}\cdot d^\ell\,.\tag{4.25}$$

As the polynomials G_i are general the variety defined by (4.24) is transverse to the codimension-j stratum, so there is at most one branch of C_j in each component of $\mathcal{M}(\mathbb{R})$ meeting such a point. Thus the number (4.25) bounds the number of ends of components of C_j in Δ, so it bounds twice the number of unbounded components of C_j in Δ. For the bound in $\mathcal{M}(\mathbb{R})$, we multiply this by 2^j, as there are 2^j components of $\mathcal{M}(\mathbb{R})$ meeting each such point, and each component of $\mathcal{M}(\mathbb{R})$ contains at most one branch of C_j.

This completes the proof of the Lemma 4.9.

5. Betti Number Bounds

Using the bound (1.4) and stratified Morse theory for a manifold with corners [9], we prove the following theorem.

Theorem 5.1. *Let X be a hypersurface in $\mathbb{R}^n_>$ defined by a (d,ℓ)-dense fewnomial. Then*

$$b_*(X) \ < \ \frac{e^2+3}{4}\, 2^{\binom{\ell}{2}} \cdot d^\ell \cdot \sum_{i=0}^{n} \tbinom{n}{i}\, i^\ell\,.$$

Let $f(x_1,\dots,x_n)$ be a real Laurent polynomial with (d,ℓ)-dense support $\mathcal{A}\subset\mathbb{Z}^n$ such that $X:=\mathcal{V}(f)\subset\mathbb{R}^n_>$ is a smooth hypersurface. By a logarithmic change of coordinates, $x_i=e^{z_i}$, we may work with exponential sums in $\mathbb{R}^n$ instead of polynomials in $\mathbb{R}^n_>$. Then the (d,ℓ)-dense fewnomial $f=\sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha$ becomes the exponential sum

$$\varphi \ := \ \sum_{\alpha\in\mathcal{A}} c_\alpha e^{z\cdot\alpha}.$$

In this way, the bounds (1.2) and (1.4) for positive solutions to fewnomial systems hold for real solutions to systems of exponential sums with the same exponents.

Let $Z:=\mathcal{V}(\varphi)\subset\mathbb{R}^n$ be the hypersurface defined by φ, which is homeomorphic to X. We will prove Theorem 5.1 in these logarithmic coordinates and with real exponents.

Theorem 5.2. *The sum of the Betti numbers of a hypersurface in $\mathbb{R}^n$ defined by an exponential sum whose exponent vectors are (d,ℓ)-dense $\mathcal{A}=\psi(d\triangle^\ell\cap\mathbb{Z}^\ell)\bigcup\mathcal{W}\subset\mathbb{R}^n$ is at most*

$$\frac{e^2+3}{4}\, 2^{\binom{\ell}{2}} \cdot d^\ell \cdot \sum_{i=0}^{n} \tbinom{n}{i}\, i^\ell\,.$$

An affine change of coordinates replaces $\mathcal{W}$ with another set of independent vectors and replaces ψ with another affine map but it does not change the (d,ℓ)-dense structure. We may thus assume that the vectors in $\mathcal{W}$ are the standard unit basis vectors in $\mathbb{R}^n$, and so φ includes the coordinate exponentials e^{z_i} for $i=1,\dots,n$. Let $M\colon=(M_0,M_1,\dots,M_n)$ be a list of positive numbers and set

$$\Delta(M) \ := \ \{z\in\mathbb{R}^n \mid z_i\geq -M_i,\, i=1,\dots,n \quad \text{and} \quad \sum_i z_i\leq M_0\},$$

which is a nonempty simplex. We will use stratified Morse theory to bound the Betti numbers of $Y:=Z\cap\Delta_M$ when M is general.

Theorem 5.3. *For M general, the sum of the Betti numbers of Y is at most*

$$\frac{e^2+3}{4}\, 2^{\binom{\ell}{2}} \cdot d^\ell \cdot \sum_{i=0}^{n} \tbinom{n}{i}\, i^\ell\,.$$

Theorem 5.2 is a consequence of Theorem 5.3. See Theorem 1′ in [7] for a detailed proof.

Proof of Theorem 5.3. Given positive numbers $M = (M_0, M_1, \ldots, M_n)$, define affine hyperplanes in $\mathbb{R}^n$ by

$$H_0 \ := \ \{z \mid \sum_i z_i = M_0\} \quad \text{ and } \quad H_i \ := \ \{z \mid z_i = -M_i\}, \qquad \text{for } i = 1, \ldots, n\,.$$

For each proper subset $S \subset \{0, \ldots, n\}$, define an affine linear subspace

$$H_S \ := \ \bigcap_{i \in S} H_i\,.$$

Since each $M_i > 0$, this has dimension $n - |S|$, and these subspaces are the affine linear subspaces supporting the faces of the simplex Δ_M.

Choose M generic so that for all S the subspace H_S meets Z transversally. For each subset S, set $Z_S := Z \cap H_S$. This is a smooth hypersurface in H_S and therefore has dimension $n - |S| - 1$. The boundary stratum Y_S of $Y = Z \cap \Delta_M$ lying in the relative interior of the face supported by H_S is an open subset of Z_S.

For a nonzero vector $u \in \mathbb{R}^n$, the directional derivative $D_u\varphi$ is

$$\sum_{\alpha \in \mathcal{A}} (u \cdot \alpha) c_\alpha e^{z \cdot \alpha}$$

which is an exponential sum having the same exponents as φ. Let L_u be the linear function on $\mathbb{R}^n$ defined by $z \mapsto u \cdot z$.

The critical points of the function L_u restricted to Z are the zeroes of the system

$$\varphi(z) = 0 \quad \text{ and } \quad D_v\varphi(z) = 0 \quad \text{ for } v \in u^\perp\,.$$

When u is general and we choose a basis for $u^\perp$, this becomes a system of n exponential sums in n variables having the same support as the original polynomial. Therefore, the whole system has (d, ℓ)-dense support, $\mathcal{A} = \psi(d\triangle^\ell \cap \mathbb{Z}^\ell) \bigcup \{e_1, \ldots, e_n\}$. By Theorem 2.4, the number of solutions is at most

$$\frac{e^2+3}{4}\, 2^{\binom{\ell}{2}}\, n^\ell \cdot d^\ell\,.$$

We use this to estimate the number of critical points of the function L_u restricted to Z_S. The restriction φ_S of φ to H_S defines Z_S as a hypersurface in H_S. We determine this restriction. Suppose first that $0 \notin S$. If $i \in S$, then we may use the equation $z_i = -M_i$ to eliminate the variable z_i and the exponential e^{z_i} from φ. The effect of these substitutions for $i \in S$ on the exponents is the projection π sending $e_i \mapsto 0$ for $i \in S$. Then $\pi \circ \psi$ is still affine and so φ_S is still a (d, ℓ)-dense fewnomial but with n replaced by $n - |S|$, and thus the number of critical points of $L_u|_{H_S}$ on Z_S is bounded by

$$\frac{e^2+3}{4}\, 2^{\binom{\ell}{2}}\, (n - |S|)^\ell \cdot d^\ell\,.$$

If $0 \in S$, then we could use fewnomial theory to estimate the number of critical points of $L_u|_{H_S}$ on Z_S, but will not need that estimate.

Let u be a general vector in $\mathbb{R}^n$ such that L_u is a Morse function for the stratified space Y. By Proposition 2 in [7], the sum of the Betti numbers of Y is bounded by the number of critical points p of L_u for which L_u achieves its minimum on the normal slice $N(p)$ at p. Since the strata Y_S of Y are open subsets of the manifolds Z_S, this number is bounded above by the number of such critical points of L_u on the manifolds Z_S. Just as in [7], we can alter u so that no critical point in any Z_S with $0 \in S$ contributes. Therefore, the sum of the Betti numbers of Y is bounded above by

$$\frac{e^2+3}{4}\, 2^{\binom{\ell}{2}} \cdot d^\ell \cdot \sum_{S \subset \{1,\dots,n\}} (n-|S|)^\ell \;=\; \frac{e^2+3}{4}\, 2^{\binom{\ell}{2}} \cdot d^\ell \cdot \sum_{i=0}^{n} \binom{n}{i} (n-i)^\ell \,. \qquad \square$$

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