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HOPF STRUCTURES ON THE MULTIPLIHEDRA

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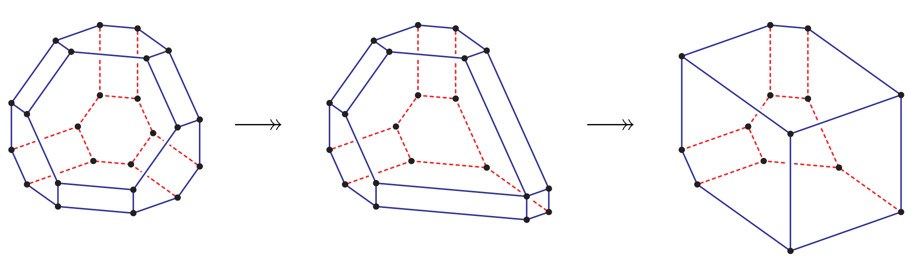
Abstract. We investigate algebraic structures that can be placed on vertices of the multiplihedra, a family of polytopes originating in the study of higher categories and homotopy theory. Most compelling among these are two distinct structures of a Hopf module over the Loday–Ronco Hopf algebra.

Introduction

The permutahedra $\mathfrak{S}_\bullet$ form a family of highly symmetric polytopes that have been of interest since their introduction by Schoute in 1911 [23]. The associahedra $\mathcal{Y}_\bullet$ are another family of polytopes that were introduced by Stasheff as cell complexes in 1963 [25], and with the permutahedra were studied from the perspective of monoidal categories and H-spaces [17] in the 1960s. Only later were associahedra shown to be polytopes [11, 13, 18]. Interest in these objects was heightened in the 1990s, when Hopf algebra structures were placed on them in work of Malvenuto, Reutenauer, Loday, Ronco, Chapoton, and others [6, 14, 16]. More recently, the associahedra were shown to arise in Lie theory through work of Fomin and Zelevinsky on cluster algebras [7].

We investigate Hopf structures on another family of polyhedra, the multiplihedra, $\mathcal{M}_\bullet$. Stasheff introduced them in the context of maps preserving higher homotopy associativity [26] and described their 1-skeleta. Boardman and Vogt [5], and then Iwase and Mimura [12] described the multiplihedra as cell complexes, and only recently were they shown to be convex polytopes [8]. These three families of polytopes are closely related. For each integer $n \geq 1$, the permutahedron $\mathfrak{S}_n$, multiplihedron $\mathcal{M}_n$, and associahedron $\mathcal{Y}_n$ are polytopes of dimension $n-1$ with natural cellular surjections $\mathfrak{S}_n \twoheadrightarrow \mathcal{M}_n \twoheadrightarrow \mathcal{Y}_n$, which we illustrate when $n = 4$.



The faces of these polytopes are represented by different flavors of planar trees; permutahedra by ordered trees (set compositions), multiplihedra by bi-leveled trees (Section 2.1), and associahedra by planar trees. The maps between them forget

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the additional structure on the trees. These maps induce surjective maps of graded vector spaces spanned by the vertices, which are binary trees. The span $\mathfrak{S}Sym$ of ordered trees forms the Malvenuto-Reutenauer Hopf algebra [16] and the span $\mathcal{Y}Sym$ of planar binary trees forms the Loday-Ronco Hopf algebra [14]. The algebraic structures of multiplication and comultiplication on $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ are described in terms of geometric operations on trees and the composed surjection $\tau\colon \mathfrak{S}_\bullet \twoheadrightarrow \mathcal{Y}_\bullet$ gives a surjective morphism $\boldsymbol{\tau}\colon \mathfrak{S}Sym \twoheadrightarrow \mathcal{Y}Sym$ of Hopf algebras.

We define $\mathcal{M}Sym$ to be the vector space spanned by the vertices of all multiplihedra. The factorization of τ induced by the maps of polytopes, $\mathfrak{S}Sym \twoheadrightarrow \mathcal{M}Sym \twoheadrightarrow \mathcal{Y}Sym$, does not endow $\mathcal{M}Sym$ with the structure of a Hopf algebra. Nevertheless, some algebraic structure does survive the factorization. We show in Section 3 that $\mathcal{M}Sym$ is an algebra, which is simultaneously a $\mathfrak{S}Sym$-module and a $\mathcal{Y}Sym$–Hopf module algebra, and the maps preserve these structures.

We perform a change of basis in $\mathcal{M}Sym$ using Möbius inversion that illuminates its comodule structure. Such changes of basis helped to understand the coalgebra structure of $\mathfrak{S}Sym$ [1] and of $\mathcal{Y}Sym$ [2]. Section 4 discusses a second $\mathcal{Y}Sym$ Hopf module structure that may be placed on the positive part $\mathcal{M}Sym_+$ of $\mathcal{M}Sym$. This structure also arises from polytope maps between $\mathfrak{S}_\bullet$ and $\mathcal{Y}_\bullet$, but not directly from the algebra structure of $\mathfrak{S}Sym$. Möbius inversion again reveals an explicit basis of $\mathcal{Y}Sym$ coinvariants in this alternate setting.

1. Basic Combinatorial Data

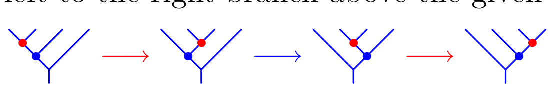
The structures of the Malvenuto-Reutenauer and Loday-Ronco algebras are related to the weak order on ordered trees and the Tamari order on planar trees. There are natural maps between the weak and Tamari orders which induce a morphism of Hopf algebras. We first recall these partial orders and then the basic structure of these Hopf algebras. In Section 1.3 we establish a formula involving the Möbius functions of two posets related by an interval retract. This is a strictly weaker notion than that of a Galois correspondence, which was used to study the structure of the Loday-Ronco Hopf algebra.

1.1. $\mathfrak{S}_\bullet$ and $\mathcal{Y}_\bullet$.

The 1-skeleta of the families of polytopes $\mathfrak{S}_\bullet, \mathcal{M}_\bullet$, and $\mathcal{Y}_\bullet$ are Hasse diagrams of posets whose structures are intertwined with the algebra structures we study. We use the same notation for a polytope and its poset of vertices. Similarly, we use the same notation for a cellular surjection of polytopes and the poset map formed by restricting that surjection to vertices.

For the permutahedron $\mathfrak{S}_n$, the corresponding poset is the (left) *weak order*, which we describe in terms of permutations. A cover in the weak order has the form $w \lessdot (k, k{+}1)w$, where k preceeds $k{+}1$ among the values of w. Figure 1 displays the weak order on $\mathfrak{S}_4$. We let $\mathfrak{S}_0 = \{\emptyset\}$, where $\emptyset$ is the empty permutation of $\emptyset$.

Let $\mathcal{Y}_n$ be the set of rooted, planar binary trees with n nodes. The cover relations in the *Tamari order* on $\mathcal{Y}_n$ are obtained by moving a child node directly above a given node from the left to the right branch above the given node. Thus



is an increasing chain in $\mathcal{Y}_3$ (the moving vertices are marked with dots). Figure 1 shows the Tamari order on $\mathcal{Y}_4$.

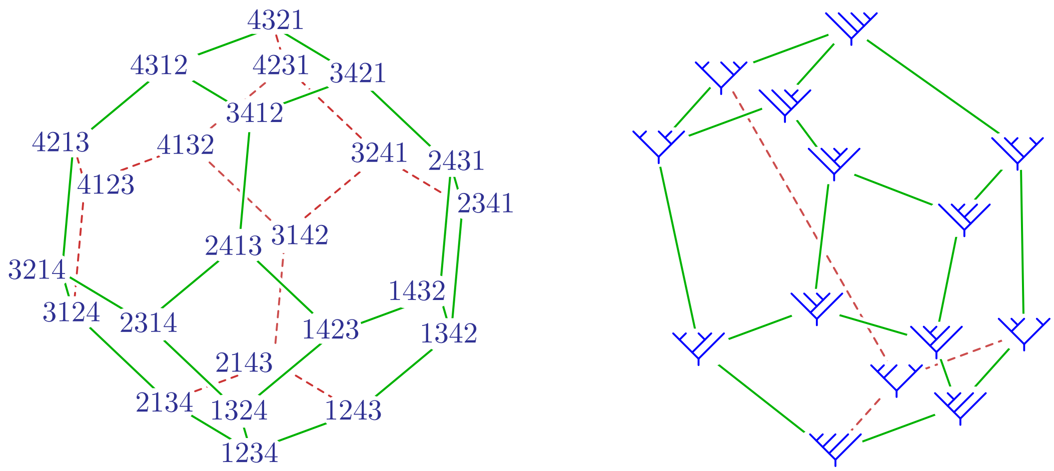
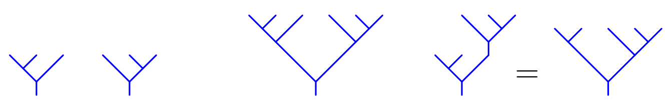


FIGURE 1: Weak order on $\mathfrak{S}_4$ and Tamari order on $\mathcal{Y}_4$

The unique tree in $\mathcal{Y}_1$ is Y. Given trees t_ℓ and t_r, form the tree $t_\ell \vee t_r$ by grafting the root of t_ℓ (respectively of t_r) to the left (respectively right) leaf of Y. Form the tree $t_\ell \backslash t_r$ by grafting the root of t_r to the rightmost leaf of t_ℓ. For example,



t_ℓ $\quad t_r$ $\quad t_\ell \vee t_r$ $\quad t_\ell \backslash t_r$.

Decompositions $t = t_1 \backslash t_2$ correspond to pruning t along the right branches from the root. A tree t is *indecomposable* if it has no nontrivial decomposition $t = t_1 \backslash t_2$ with $t_1, t_2 \neq |$. Equivalently, t is indecomposable if the root node is the rightmost node of t. Any tree t is uniquely decomposed $t = t_1 \backslash \cdots \backslash t_m$ into indecomposable trees $t_1, \ldots, t_m$.

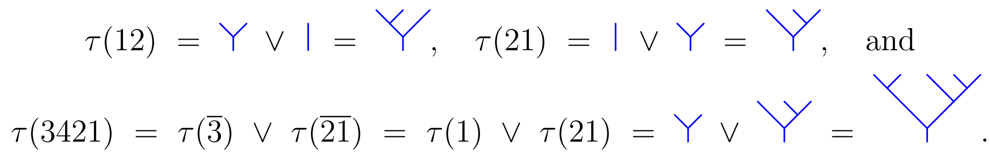
We define a poset map $\tau\colon \mathfrak{S}_n \to \mathcal{Y}_n$. First, given distinct integers $a_1, \ldots, a_k$, let $\overline{a} \in \mathfrak{S}_k$ be the unique permutation such that $\overline{a}(i) < \overline{a}(j)$ if and only if $a_i < a_j$. Thus $\overline{4726} = 2413$. Since $\mathfrak{S}_0$, $\mathcal{Y}_0$, $\mathfrak{S}_1$, and $\mathcal{Y}_1$ are singletons, we must have

$$\tau : \mathfrak{S}_0 \longrightarrow \mathcal{Y}_0 \quad \text{with} \quad \tau : \emptyset \longmapsto |, \quad \text{and}$$
$$\tau : \mathfrak{S}_1 \longrightarrow \mathcal{Y}_1 \quad \text{with} \quad \tau : 1 \longmapsto \text{Y}.$$

Let $n > 0$ and assume that τ has been defined on $\mathfrak{S}_k$ for $k < n$. For $w \in \mathfrak{S}_n$ suppose that $w(j) = n$, and define

$$\tau(w) \;:=\; \tau\big(\overline{w(1), \ldots, w(j{-}1)}\big) \;\vee\; \tau\big(\overline{w(j{+}1), \ldots, w(n)}\big)\,.$$

For example,



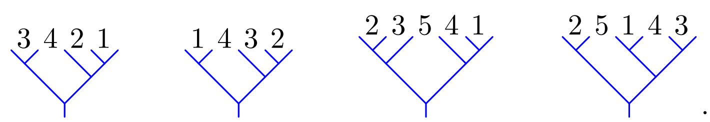
Loday and Ronco [15] show that the fibers $\tau^{-1}(t)$ of τ are intervals in the weak order. This gives two canonical sections of τ. For $t \in \mathcal{Y}_n$,

$$\mathsf{min}(t) \;:=\; \min\{w \mid \tau(w) = t\} \qquad \text{and} \qquad \mathsf{max}(t) \;:=\; \max\{w \mid \tau(w) = t\}\,,$$

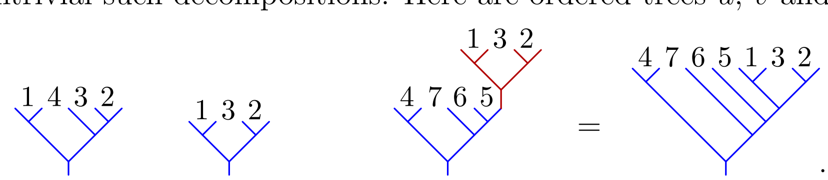
the minimum and maximum in the weak order. Equivalently, $\mathsf{min}(t)$ is the unique 231-avoiding permutation in $\tau^{-1}(t)$ and $\mathsf{max}(t)$ is the unique 132-avoiding permutation. These maps are order-preserving.

The 1-skeleta of $\mathfrak{S}_n$ and $\mathcal{Y}_n$ form the Hasse diagrams of the weak and Tamari orders, respectively. Since τ is an order-preserving surjection, it induces a cellular map between the 1-skeleta of these polytopes. Tonks [27] extended τ to the faces of $\mathfrak{S}_n$, giving a cellular surjection.

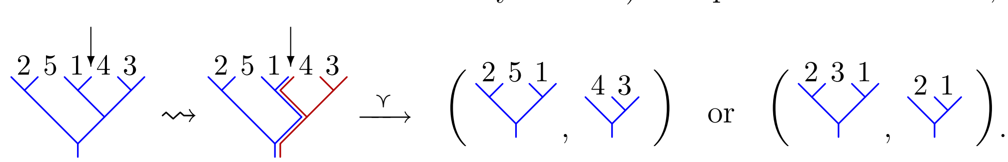
The nodes and internal edges of a tree are the Hasse diagram of a poset with the root node maximal. Labeling the nodes (equivalently, the gaps between the leaves) of $\tau(w)$ with the values of the permutation w gives a linear extension of the node poset of $\tau(w)$, and all linear extensions of a tree t arise in this way for a unique permutation in $\tau^{-1}(t)$. Such a linear extension w of a tree is an *ordered tree* and $\tau(w)$ is the corresponding unordered tree. In this way, $\mathfrak{S}_n$ is identified with the set of ordered trees with n nodes. Here are some ordered trees,



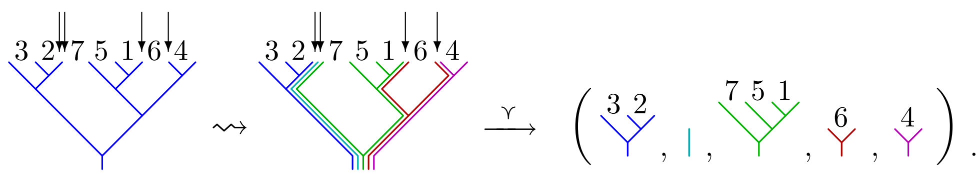
Given ordered trees u, v, form the ordered tree $u\backslash v$ by grafting the root of v to the rightmost leaf of u, where the nodes of u are greater than the nodes of v, but the relative orders within u and v are maintained. Thus we may decompose an ordered tree $w = u\backslash v$ whenever $\tau(w) = r\backslash s$ with $\tau(u) = r$, $\tau(v) = s$, and the nodes of r in w precede the nodes of s in w. An ordered tree w is *indecomposable* if it has no nontrivial such decompositions. Here are ordered trees u, v and $u\backslash v$,



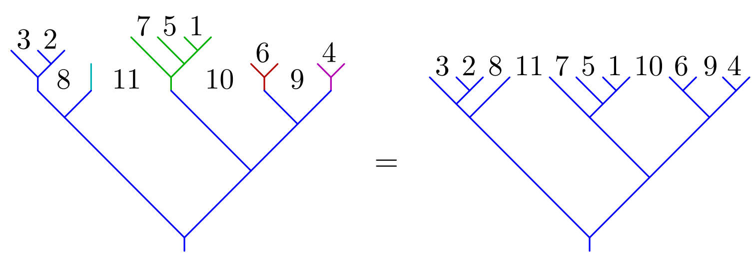
We may *split* an ordered tree w along a leaf to obtain either an ordered forest (where the nodes in the forest are totally ordered) or a pair of ordered trees,



Write $w \xrightarrow{\curlyvee} (w_0, w_1)$ to indicate that the ordered forest (w_0, w_1) (or pair of ordered trees) is obtained by splitting w along some leaf. (Context will determine how to interpret the result.) More generally, we may split an ordered tree w along a multiset of $m \geq 0$ of its leaves to obtain an ordered forest, or tuple of ordered trees, written $w \xrightarrow{\curlyvee} (w_0, \ldots, w_m)$. For example,

(1.1) 

Given $v \in \mathfrak{S}_m$ and an ordered forest $(w_0, \ldots, w_m)$, let $(w_0, \ldots, w_m)/v$ be the ordered tree obtained by grafting the root of w_i to the ith leaf of v, where the nodes of v are greater than all nodes of w, but the relative orders within the w_i and v are maintained. When v is the ordered tree corresponding to 1432 and $w \xrightarrow{\curlyvee} (w_0, \ldots, w_m)$ is the splitting (1.1), this grafting is



The notions of splitting and grafting also make sense for the unordered trees $\mathcal{Y}_n$ and we use the same notation, $\bullet \xrightarrow{\curlyvee} \bullet$ and $\bullet/\bullet$. (Simply delete the labels in the constructions above.) These operations of splitting and grafting are compatible with the map $\tau\colon \mathfrak{S}_\bullet \to \mathcal{Y}_\bullet$: if $w \xrightarrow{\curlyvee} (w_0, \ldots, w_m)$ then $\tau(w) \xrightarrow{\curlyvee} (\tau(w_0), \ldots, \tau(w_m))$ and all splittings in $\mathcal{Y}_\bullet$ are induced in this way from splittings in $\mathfrak{S}_\bullet$. The same is true for grafting, $\tau((w_0, \ldots, w_m)/v) = (\tau(w_0), \ldots, \tau(w_m))/\tau(v)$.

1.2. $\mathfrak{S}Sym$ **and** $\mathcal{Y}Sym$**.** For basics on Hopf algebras, see [19]. Let $\mathfrak{S}Sym := \bigoplus_{n \geq 0} \mathfrak{S}Sym_n$ be the graded $\mathbb{Q}$–vector space whose n^{th} graded piece has basis $\{F_w \mid w \in \mathfrak{S}_n\}$. Malvenuto and Reutenauer [16] defined a Hopf algebra structure on $\mathfrak{S}Sym$. For $w \in \mathfrak{S}_\bullet$, define the coproduct

$$\Delta F_w \;:=\; \sum_{w \xrightarrow{\curlyvee} (w_0, w_1)} F_{w_0} \otimes F_{w_1}\,,$$

where (w_0, w_1) is a pair of ordered trees. If $v \in \mathfrak{S}_m$, define the product

$$F_w \cdot F_v \;:=\; \sum_{w \xrightarrow{\curlyvee} (w_0, \ldots, w_m)} F_{(w_0, \ldots, w_m)/v}\,.$$

The counit is the projection $\varepsilon\colon \mathfrak{S}Sym \to \mathfrak{S}Sym_0$ onto the 0th graded piece, which is spanned by the unit, $1 = F_\emptyset$, for this multiplication.

Proposition 1.1 ([16]). *With these definitions of coproduct, product, counit, and unit, $\mathfrak{S}Sym$ is a graded, connected cofree Hopf algebra that is neither commutative nor cocommutative.*

Let $\mathcal{Y}Sym := \bigoplus_{n\geq 0}\mathcal{Y}Sym_n$ be the graded $\mathbb{Q}$–vector space whose n^{th} graded piece has basis $\{F_t \mid t \in \mathcal{Y}_n\}$. Loday and Ronco [14] defined a Hopf algebra structure on $\mathcal{Y}Sym$. For $t \in \mathcal{Y}_\bullet$, define the coproduct

$$\Delta F_t \;:=\; \sum_{t \overset{\curlyvee}{\to} (t_0,t_1)} F_{t_0} \otimes F_{t_1}\,,$$

and if $s \in \mathcal{Y}_m$, define the product

$$F_t \cdot F_s \;:=\; \sum_{t \overset{\curlyvee}{\to} (t_0,\dots,t_m)} F_{(t_0,\dots,t_m)/s}\,.$$

The counit is the projection $\varepsilon\colon \mathcal{Y}Sym \to \mathcal{Y}Sym_0$ onto the 0th graded piece, which is spanned by the unit, $1 = F_{|}$, for this multiplication. The map τ extends to a linear map $\boldsymbol{\tau}\colon \mathfrak{S}Sym \to \mathcal{Y}Sym$, defined by $\boldsymbol{\tau}(F_w) = F_{\tau(w)}$.

Proposition 1.2 ([14]). *With these definitions of coproduct, product, counit, and unit, $\mathcal{Y}Sym$ is a graded, connected cofree Hopf algebra that is neither commutative nor cocommutative and the map $\boldsymbol{\tau}$ a morphism of Hopf algebras.*

Some structures of the Hopf algebras $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$, particularly their primitive elements and coradical filtrations, are better understood with respect to a second basis. The Möbius function μ (or μ_P) of a poset P is defined for pairs (x,y) of elements of P with $\mu(x,y) = 0$ if $x \not\leq y$, $\mu(x,x) = 1$, and, if $x < y$, then

$$\mu(x,y) \;=\; -\sum_{x\leq z<y} \mu(x,z) \qquad \text{so that} \qquad 0 \;=\; \sum_{x\leq z\leq y} \mu(x,z)\,. \tag{1.2}$$

For $w \in \mathfrak{S}_\bullet$ and $t \in \mathcal{Y}_\bullet$, set

$$M_w \;:=\; \sum_{w\leq v} \mu(w,v) F_v \qquad \text{and} \qquad M_t \;:=\; \sum_{t\leq s} \mu(t,s) F_s\,, \tag{1.3}$$

where the first sum is over $v \in \mathfrak{S}_\bullet$, the second sum over $s \in \mathcal{Y}_\bullet$, and $\mu(\cdot,\cdot)$ is the Möbius function in the weak and Tamari orders.

Proposition 1.3 ([1, 2]). *If $w \in \mathfrak{S}_\bullet$, then*

$$\boldsymbol{\tau}(M_w) \;=\; \begin{cases} M_{\tau(w)}, & \textit{if } w = \mathsf{max}(\tau(w))\,, \\ 0, & \textit{otherwise} \end{cases} \tag{1.4}$$

and

$$\Delta M_w \;=\; \sum_{w=u\backslash v} M_u \otimes M_v\,. \tag{1.5}$$

If $t \in \mathcal{Y}_\bullet$, then

$$\Delta M_t \;=\; \sum_{t=r\backslash s} M_r \otimes M_s\,. \tag{1.6}$$

This implies that the set $\{M_w \mid w \in \mathfrak{S}_\bullet \text{ is indecomposable}\}$ is a basis for the primitive elements of $\mathfrak{S}Sym$ (and the same for $\mathcal{Y}Sym$), thereby explicitly realizing the cofree-ness of $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$.

1.3. **Möbius functions and interval retracts.** A pair $f\colon P \to Q$ and $g\colon Q \to P$ of poset maps is a *Galois connection* if f is left adjoint to g in that

$$\forall\, p \in P \text{ and } q \in Q\,, \quad f(p) \;\leq_Q\; q \iff p \;\leq_P\; g(q)\,.$$

When this occurs, Rota [21, Theorem 1] related the Möbius functions of P and Q:

$$\forall\, p \in P \text{ and } q \in Q\,, \quad \sum_{f(y)=q} \mu_P(p,y) \;=\; \sum_{g(x)=q} \mu_Q(x,q)\,.$$

Rota's formula was used in [2] to establish the coproduct formulas (1.4) and (1.6), as the maps $\tau\colon \mathfrak{S}_\bullet \to \mathcal{Y}_\bullet$ and $\mathsf{max}\colon \mathcal{Y}_\bullet \to \mathfrak{S}_\bullet$ form a Galois connection [4, Section 9].

We do not have a Galois connection between $\mathfrak{S}_\bullet$ and $\mathcal{M}_\bullet$, and so cannot use Rota's formula. Nevertheless, there is a useful relation between the Möbius functions of $\mathfrak{S}_\bullet$ and $\mathcal{M}_\bullet$ that we establish here in a general form. A surjective poset map $f\colon P \to Q$ from a finite lattice P is an *interval retract* if the fibers of f are intervals and if f admits an order-preserving section $g\colon Q \to P$ with $f \circ g = \mathrm{id}$.

Theorem 1.4. *If the poset map $f\colon P \to Q$ is an interval retract, then the Möbius functions μ_P and μ_Q of P and Q are related by the formula*

$$\mu_Q(x,y) \;=\; \sum_{\substack{f(a)=x \\ f(b)=y}} \mu_P(a,b) \qquad (\forall x,y \in Q). \tag{1.7}$$

In Section 2, we define an interval retract $\beta\colon \mathfrak{S}_n \to \mathcal{M}_n$.

We evaluate each side of (1.7) using Hall's formula, which expresses the Möbius function in terms of *chains*. A linearly ordered subset $C\colon x_0 < \cdots < x_r$ of a poset is a *chain* of *length $\ell(C) = r$* from x_0 to x_r. Given a poset P, let $\text{Ч}(P)$ be the set of all chains in P. A poset P is an *interval* if it has a unique maximum element and a unique minimum element. If $P = [x,y]$ is an interval, let $\text{Ч}'(P)$ denote the chains in P beginning in x and ending in y. Hall's formula states that

$$\mu(x,y) \;=\; \sum_{C \in \text{Ч}'[x,y]} (-1)^{\ell(C)}\,.$$

Our proof rests on the following two lemmas.

Lemma 1.5. *If P is an interval, then* $\displaystyle\sum_{C \in \text{Ч}(P)} (-1)^{\ell(C)} = 1.$

Proof. Suppose that $P = [x,y]$ and append new minimum and maximum elements to P to get $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$. Then the definition of Möbius function (1.2) gives

$$\mu(\hat{0},\hat{1}) \;=\; -\sum_{\hat{0} \leq z \leq y} \mu(\hat{0},z)\,,$$

which is zero by (1.2). By Hall's formula,

$$0 \;=\; \mu(\hat{0},\hat{1}) \;=\; \sum_{C\in \text{Ч}'[\hat{0},\hat{1}]} (-1)^{\ell(C)} \;=\; -1 + \sum_{C\in \text{Ч}(P)} (-1)^{\ell(C)+2}\,,$$

where the term -1 comes from the chain $\hat{0} < \hat{1}$. This proves the lemma. □

Call a partition $P = K_0 \sqcup \cdots \sqcup K_r$ of P into subposets K_i monotone if $x < y$ with $x \in K_i$ and $y \in K_j$ implies that $i \leq j$. Given $\emptyset \subsetneq I \subseteq [0,r]$, write $\text{Ч}_I(P)$ for the subset of chains C in $\text{Ч}(P)$ such that $C \cap K_i \neq \emptyset$ if and only if $i \in I$.

Lemma 1.6. *Let $P = K_0 \sqcup \cdots \sqcup K_r$ be a monotonic partition of a poset P. If $\bigcup_{i\in I} K_i$ is an interval for all $I \subseteq [0,r]$, then*

$$\sum_{C\in \text{Ч}_{[0,r]}(P)} (-1)^{\ell(C)} \;=\; (-1)^r\,. \tag{1.8}$$

Proof. We argue by induction on r. Lemma 1.5 is the case $r = 0$ (wherein $K_0 = P$), so we consider the case $r \geq 1$.

Form the poset $\hat{P} = P \cup \{\hat{0},\hat{1}\}$ as in the proof of Lemma 1.5. Since P is an interval, we have $\sum_{C\in \text{Ч}'[\hat{0},\hat{1}]} (-1)^{\ell(C)} = 0$. As $\text{Ч}'[\hat{0},\hat{1}] = \bigsqcup_I \text{Ч}_I(P)$ we have,

$$0 \;=\; -1 \;+\; \sum_{\emptyset\subsetneq I\subsetneq[0,r]} \Bigg(\sum_{C\in \text{Ч}_I(P)} (-1)^{\ell(C)} \Bigg) \;+\; \sum_{C\in \text{Ч}_{[0,r]}(P)} (-1)^{\ell(C)}\,,$$

where the term -1 counts the chain $\hat{0} < \hat{1}$. Applying induction, we have

$$0 = \sum_{k=0}^{r} \binom{r+1}{k} (-1)^{k-1} + \sum_{C\in \text{Ч}_{[0,r]}(P)} (-1)^{\ell(C)}\,.$$

Comparing this to the binomial expansion of $(1-1)^{r+1}$ completes the proof. □

Proof of Theorem 1.4. Fix $x < y$ in Q. We use Hall's formula to rewrite the right-hand side of (1.7) as

$$\sum_{\substack{f(a)=x\\ f(b)=y}} \; \sum_{C\in \text{Ч}'[a,b]} (-1)^{\ell(C)}\,. \tag{1.9}$$

Fix a chain $D\colon x = q_0 < \cdots < q_r = y$ in $\text{Ч}'[x,y]$ and let $P|_D$ be the subposet of P consisting of elements that occur in some chain of P that maps to D under f. This is nonempty as f has section. Furthermore, the sets $K_i := f^{-1}(q_i) \cap P|_D$, for $i = 0,\ldots,r$, form a monotonic partition of $P|_D$. We claim that $\bigcup_{i\in I} K_i$ is an interval for all $I \subseteq [0,r]$. If so, let us first rewrite (1.9) as a sum over chains D in Q,

$$\sum_{D\in \text{Ч}'[x,y]} \; \sum_{C\in \text{Ч}_{[0,\ell(D)]}(P|_D)} (-1)^{\ell(C)}\,.$$

By Lemma 1.6, the inner sum becomes $(-1)^{\ell(D)}$, which completes the proof.

To prove the claim, suppose that $I = \{i_0 < \cdots < i_s\}$. Each set K_i $(i \in I)$ is an interval, as it is the intersection of two intervals in the lattice P. Thus K_{i_0} and

K_{i_s} are intervals with minimum and maximum elements m and M, respectively. Any chain in $\bigcup_{i\in I} K_i$ can be extended to a chain beginning with m and ending at M, so $\bigcup_{i\in I} K_i$ is an interval. □

Remark 1.7. In light of the proof of Theorem 1.4, the result holds with a weaker hypothesis. The condition in the definition of interval retract that P be a lattice may be replaced by the order map $f\colon P \to Q$ having the property that the set $f^{-1}(q) \cap P|_D$ is an interval whenever D is a chain in Q and $q \in D$.

2. The Multiplihedra $\mathcal{M}_\bullet$

The map $\tau\colon \mathfrak{S}_\bullet \to \mathcal{Y}_\bullet$ forgets the linear ordering of the node poset of an ordered tree, and it induces a morphism of Hopf algebras $\boldsymbol{\tau}\colon \mathfrak{S}Sym \to \mathcal{Y}Sym$. In fact, one may take the (ahistorical) view that the Hopf structure on $\mathcal{Y}Sym$ is induced from that on $\mathfrak{S}Sym$ via the map τ. Forgetting some, but not all, of the structure on a tree in $\mathfrak{S}_\bullet$ factorizes the map τ. Here, we study combinatorial consequences of one such factorization, and later treat its algebraic consequences.

2.1. **Bi-leveled trees.** A *bi-leveled tree* $(t;\mathsf{T})$ is a planar binary tree $t \in \mathcal{Y}_n$ together with an (upper) order ideal T of its node poset, where T contains the leftmost node of t as a minimal element. Thus T contains all nodes along the path from the leftmost leaf to the root, and none above the leftmost node. Numbering the gaps between the leaves of t by $1,\dots,n$ from left to right, T becomes a subset of $\{1,\dots,n\}$.

Saneblidze and Umble [22] introduced bi-leveled trees to describe a cellular projection from the permutahedra to Stasheff's multiplihedra $\mathcal{M}_\bullet$, with the bi-leveled trees on n nodes indexing the vertices $\mathcal{M}_n$. Stasheff used a different type of tree for the vertices of $\mathcal{M}_\bullet$. These alternative trees lead to a different Hopf structure which we explore in a forthcoming paper [9]. We remark that $\mathcal{M}_0 = \{\mathsf{I}\}$.

The partial order on $\mathcal{M}_n$ is defined by $(s;\mathsf{S}) \le (t;\mathsf{T})$ if $s \le t$ in $\mathcal{Y}_n$ and $\mathsf{S} \supseteq \mathsf{T}$. The Hasse diagrams of the posets $\mathcal{M}_n$ are 1-skeleta for the multiplihedra. We represent a bi-leveled tree by drawing the underlying tree t and circling the nodes in T. The Hasse diagram of $\mathcal{M}_4$ appears in Figure 2.

2.2. **Poset maps.** Forgetting the order ideal in a bi-leveled tree, $(t;\mathsf{T}) \mapsto t$, is a poset map $\phi\colon \mathcal{M}_\bullet \to \mathcal{Y}_\bullet$. We define a map $\beta\colon \mathfrak{S}_\bullet \to \mathcal{M}_\bullet$ so that

$$\mathfrak{S}_\bullet \xrightarrow{\ \beta\ } \mathcal{M}_\bullet \xrightarrow{\ \phi\ } \mathcal{Y}_\bullet$$

factors the map $\tau\colon \mathfrak{S}_\bullet \to \mathcal{Y}_\bullet$, and we define a right inverse (section) ι of β.

Let $w \in \mathfrak{S}_\bullet$ be an ordered tree. Define the set

$$\mathsf{T}(w) \ := \ \{i \mid w(i) \ge w(1)\}\,. \tag{2.1}$$

Observe that $\beta(w) := (\tau(w);\mathsf{T}(w))$ is a bi-leveled tree. Indeed, as w is a linear extension of $\tau(w)$, $\mathsf{T}(w)$ is an upper order ideal which by definition (2.1) contains the leftmost node as a minimal element. Since covers in the weak order can only decrease the subset $\mathsf{T}(w)$ and τ is also a poset map, we see that β is a poset map.

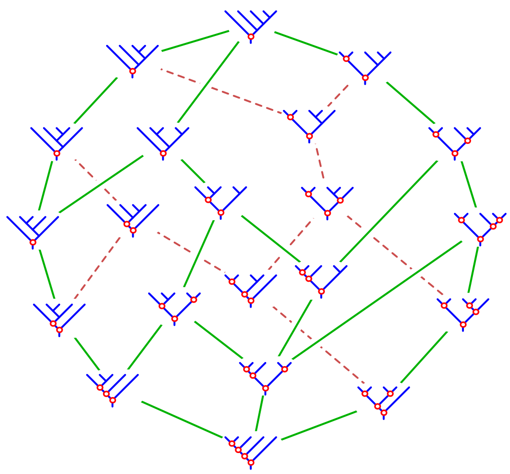


FIGURE 2: The 1-skeleton of the multiplihedron $\mathcal{M}_4$.

Theorem 2.1. *The maps* $\beta\colon \mathfrak{S}_\bullet \to \mathcal{M}_\bullet$ *and* $\phi\colon \mathcal{M}_\bullet \to \mathcal{Y}_\bullet$ *are surjective poset maps with* $\tau = \phi \circ \beta$.

The fibers of the map β are intervals (indeed, products of intervals); see Figure 3. We prove this using an equivalent representation of a bi-leveled tree and

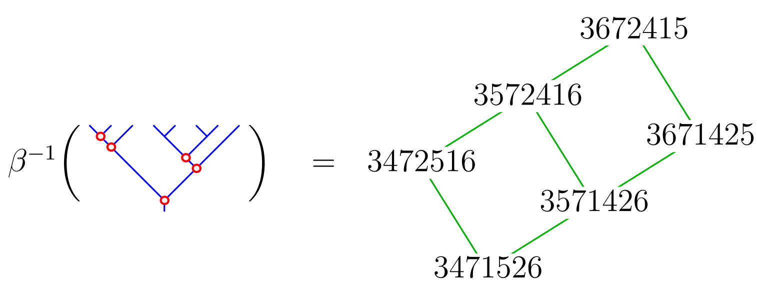


FIGURE 3: The preimages of β are intervals.

a description of the map β in that representation. If we prune a bi-leveled tree $b = (t; \mathsf{T})$ above the nodes in T (but not on the leftmost branch) we obtain a tree t_0' (the order ideal) on r nodes and a planar forest $\boldsymbol{t} = (t_1, \ldots, t_r)$ of r trees. If we prune t_0' just below its leftmost node, we obtain the tree Y (from the pruning) and a tree t_0, and t_0' is obtained by grafting Y onto the leftmost leaf of t_0. We may recover b from this tree t_0 on $r-1$ nodes and the planar forest $\boldsymbol{t} = (t_1, \ldots, t_r)$, and so we also write $b = (t_0, \boldsymbol{t})$. We illustrate this correspondence in Figure 4.

We describe the map β in terms of this second representation of bi-leveled trees. Given a permutation w with $\beta(w) = (t; \mathsf{T})$ and $|\mathsf{T}| = r$, let $u_1 u_2 \ldots u_r$ be the restriction of w to the set T. We may write the values of w as $w = u_1 v^1 u_2 \cdots u_r v^r$, where v^i is the (possibly empty) subword of w between the numbers u_i and u_{i+1} and

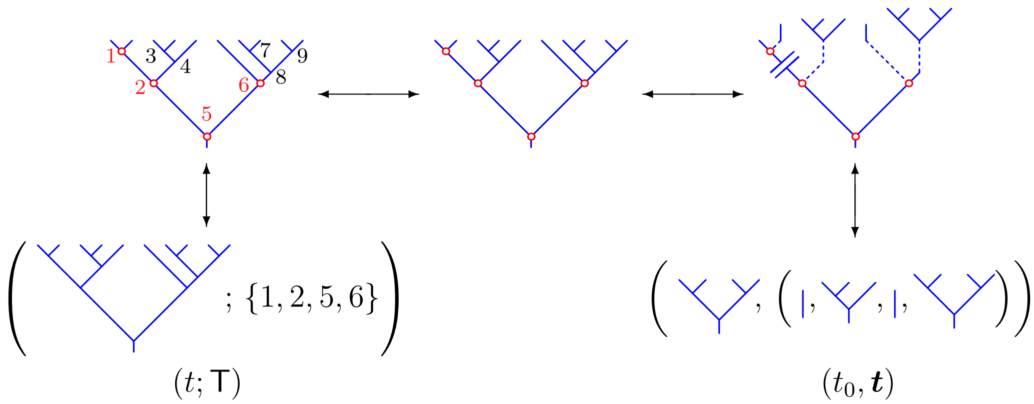


FIGURE 4: Two representations of bi-leveled trees.

v^r is the word after u_r. Call this the *bi-leveled factorization* of w. For example,

$$654789132 \longmapsto (\underset{u_1}{6}\,,\ \underset{v^1}{54}\,,\ \underset{u_2}{7}\,,\ \underset{v^2}{\emptyset}\,,\ \underset{u_3}{8}\,,\ \underset{v^3}{\emptyset}\,,\ \underset{u_4}{9}\,,\ \underset{v^4}{132})\,.$$

Note that $\beta(w) = (\tau(\overline{u_2 \ldots u_r}), (\tau(\overline{v^1}), \ldots, \tau(\overline{v^r})))$.

Theorem 2.2. *For any $b \in \mathcal{M}_n$ the fiber $\beta^{-1}(b) \subseteq \mathfrak{S}_n$ is a product of intervals.*

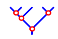
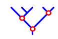
Proof. Let $b = (t_0, (t_1, \ldots, t_r)) = (t; \mathsf{T}) \in \mathcal{M}_n$ be a bi-leveled tree. A permutation $w \in \beta^{-1}(b) \in \mathfrak{S}_n$ has a bi-leveled factorization $w = u_1 v^1 u_2 \ldots u_r v^r$ with

$$\begin{aligned} &(i) \quad w|_{\mathsf{T}} = u_1 u_2 \ldots u_r,\ u_1 = n{+}1{-}r,\ \tau(\overline{u_2 \ldots u_r}) = t_0, \quad \text{and} \\ &(ii) \quad \tau(\overline{v^i}) = t_i,\ \text{for } i = 1, \ldots, r\,. \end{aligned} \tag{2.2}$$

Since $u_1 < u_2, \ldots, u_r$ are the values of w in the positions of T, and $u_1 = n{+}1{-}r$ exceeds all the letters in $v^1, \ldots, v^r$, which are the values of w in the positions in the complement of T, these two parts of the bi-leveled factorization may be chosen independently to satisfy (2.2), which shows that $\beta^{-1}(b)$ is a product.

To see that the factors are intervals, and thus $\beta^{-1}(b)$ is an interval, we examine the conditions (i) and (ii) separately. Those $u_1 \ldots u_r = w|_{\mathsf{T}}$ for w in the fiber $\beta^{-1}(b)$ are exactly the set of $n{+}1{-}r, u_2, \ldots, u_r$ with $\{u_2, \ldots, u_r\} = \{n{+}2{-}r, \ldots, n\}$ and $\tau(\overline{u_2 \ldots u_r}) = t_0$. This is a poset under the restriction of the weak order, and it is in natural bijection with the interval $\tau^{-1}(t_0) \subset \mathfrak{S}_{r-1}$. Its minimal element is $\mathsf{min}_0(b) = u_1 u_2 \ldots u_r$, where $u_2 \ldots u_r$ is the unique 231-avoiding word on $\{n{+}1{-}r, \ldots, n\}$ satisfying (i), and its maximal element is $\mathsf{max}_0(b) = u_1 u_2 \ldots u_r$, where now $u_2 \ldots u_r$ is the unique 132-avoiding word on $\{n{+}1{-}r, \ldots, n\}$ satisfying (i).

Now consider sequences of words $v^1, \ldots, v^r$ on distinct letters $\{1, \ldots, n{-}r\}$ satisfying (ii). This is also a poset under the restriction of the weak order. It has a minimal element, which is the unique such sequence $\underline{\mathsf{min}}(b)$ satisfying (ii) where the letters of v^i preceed those of v^j whenever $i < j$, and where each v^i is 231-avoiding. Its maximal element is the unique sequence $\underline{\mathsf{max}}(b)$ satisfying (ii) where the letters of v^i are greater than those of v^j when $i < j$ and v^i is 132-avoiding. $\square$

The fibers of β are intervals so that consistently choosing the minimum or maximum in a fiber gives two set-theoretic sections. Neither section is order-preserving as may be seen from Figure 5. We have  $<$  but the maxima in their fibers

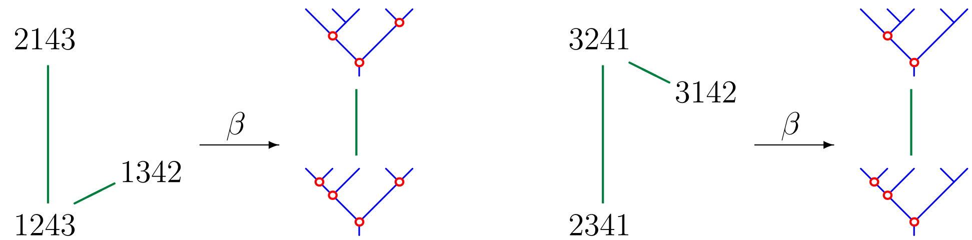
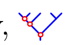
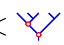
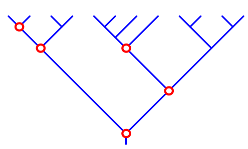
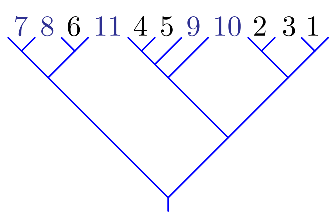


FIGURE 5: Fibers of β.

under β, 1342 and 2143, are incomparable. Similarly,  $<$  but the minima in their fibers under β, 2341 and 3142, are incomparable. This shows that the map $\beta\colon \mathfrak{S}_\bullet \to \mathcal{M}_\bullet$ is not a lattice congruence (unlike the map $\tau\colon \mathfrak{S}_\bullet \to \mathcal{Y}_\bullet$ [20]).

In the notation of the proof, given a bi-leveled tree $b = (t_0, (t_1, \ldots, t_r))$, let $\iota(b)$ be the permutation $w \in \beta^{-1}(b)$ with bi-leveled factorization $w = u_1 v^1 u_2 \ldots u_r v^r$ where $u_1 u_2 \ldots u_r = \mathsf{min}_0(b)$ and $(v^1, \ldots, v^r) = \underline{\mathsf{max}}(b)$. This defines a map $\iota\colon \mathcal{M}_n \to \mathfrak{S}_n$ that is a section of the map β. For example,

$\iota\Bigg($  $\Bigg) =$ 

$= 786\,11\,459\,10\,231\,.$

Remark 2.3. This map ι may be characterized in terms of pattern avoidance: the permutation $\iota(b)$ is the unique $w \in \beta^{-1}(b)$ avoiding the pinned patterns

$$\left\{\underline{2}031,\ \underline{0}231,\ \underline{3}021\right\},$$

where the underlined letter must be the first letter of a permutation. To see this, note that the first pattern forces the letters in v^i to be larger than those in v^{i+1} for $1 \le i < r$, the second pattern forces $u_2 \ldots u_r$ to be 231-avoiding, and the last pattern forces each v^i to be 132-avoiding.

Theorem 2.4. *The map ι is injective, right-inverse to β, and order-preserving. That is, $\beta\colon \mathfrak{S}_n \to \mathcal{M}_n$ is an interval retract.*

Since $\mathfrak{S}_n$ is a lattice [10], the fibers of β are intervals, and ι is a section of β. That is, we need only verify that ι is order-preserving. We begin by describing the covers in $\mathcal{M}_\bullet$. Since β is a surjective poset map, every cover in $\mathcal{M}_n$ is the image of some cover $w \lessdot w'$ in $\mathfrak{S}_n$.

Lemma 2.5. *If a cover $w \lessdot w' \in \mathfrak{S}_n$ does not collapse under β, i.e., $\beta(w) \neq \beta(w')$, then it yields one of three types of covers $\beta(w) \lessdot \beta(w')$ in $\mathcal{M}_n$.*

(*i*) *In exactly one tree* t_i *in* $\beta(w) = (t_0, (t_1, \ldots, t_r))$, *a node is moved from left to right across its parent to obtain* $\beta(w')$. *That is,* $t_i \lessdot t'_i$.
(*ii*) *In* $\beta(w) = (t; \mathsf{T})$, *the leftmost node of* t *is moved across its parent, which has no other child in the order ideal* T, *and deleted from* T *to obtain* $\beta(w')$.
(*iii*) *If* $\mathsf{T}(w) = \{1 = T_1 < \cdots < T_r\}$, *then* $\tau(w') = \tau(w)$ *and* $\mathsf{T}(w') = \mathsf{T}(w) \setminus \{T_j\}$ *for some* $j > 2$.

Proof. Put $w' = (k, k+1)w$, with $k, k+1$ appearing in order in w. Let $(t; \mathsf{T})$ and $(t_0, (t_1, \ldots, t_r))$ be the two representations of $\beta(w)$. Write $\mathsf{T} = \{T_1 < \cdots < T_r\}$ (with $T_1 = 1$) and $w|_{\mathsf{T}} = u_1 u_2 \ldots u_r$. If $w \lessdot w'$ and $\beta(w) \lessdot \beta(w')$, then k appears within w in one of three ways: (*i*) $u_1 \neq k$, (*ii*) $u_1 = k$ and $u_2 = k+1$, or (*iii*) $u_1 = k$ and $u_j = k+1$ for some $j > 2$. These yield the corresponding descriptions in the statement of the lemma. (Note that in type (*i*), $\mathsf{T}(w') = \mathsf{T}$, so if we set $\beta(w') = (t'_0, (t'_1, \ldots, t'_r))$, then $t_i = t'_i$, except for one index i, where $t_i \lessdot t'_i$.) □

Figure 6 illustrates these three types of covers, labeled by their type.

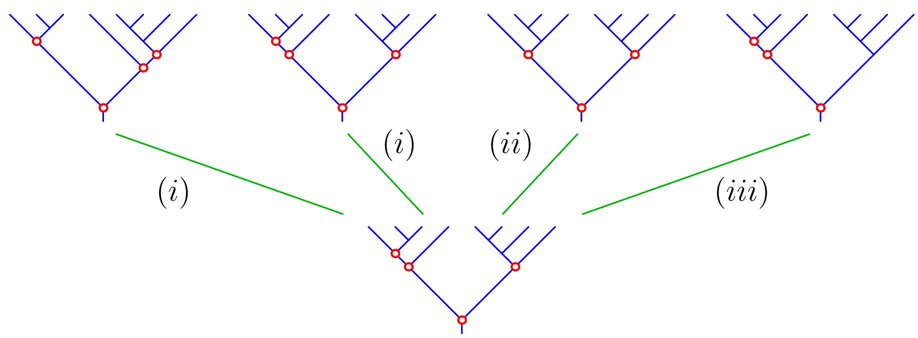


FIGURE 6: Some covers in $\mathcal{M}_7$.

For $\mathsf{T} \subset \{1, \ldots, n\}$ with $1 \in \mathsf{T}$, let $\mathfrak{S}_n(\mathsf{T}) := \{w \in \mathfrak{S}_n \mid \mathsf{T}(w) = \mathsf{T}\}$. Let $\mathcal{M}_n(\mathsf{T})$ be those bi-leveled trees whose order ideal consists of the nodes in T. Note that $\beta(\mathfrak{S}_n(\mathsf{T})) = \mathcal{M}_n(\mathsf{T})$ and $\beta^{-1}(\mathcal{M}_n(\mathsf{T})) = \mathfrak{S}_n(\mathsf{T})$.

Lemma 2.6. *The map* $\iota\colon\ \mathcal{M}_n(\mathsf{T}) \to \mathfrak{S}_n(\mathsf{T})$ *is a map of posets.*

Proof. Let $\mathsf{T} = \{1 = T_1 < \cdots < T_r\}$. Setting $T_{r+1} = n+1$, define $a_i := T_{i+1} - T_i - 1$ for $i = 1, \ldots, r$. Then $b \mapsto (t_0, (t_1, \ldots, t_r))$ gives an isomorphism of posets,

$$\mathcal{M}_n(\mathsf{T}) \ \xrightarrow{\sim}\ \mathcal{Y}_{r-1} \times \mathcal{Y}_{a_1} \times \cdots \times \mathcal{Y}_{a_r}\,.$$

As the maps $\mathsf{min}, \mathsf{max}\colon \mathcal{Y}_a \to \mathfrak{S}_a$ are order-preserving, the proof of Theorem 2.2 gives the desired result. □

Proof of Theorem 2.4. Let $b \lessdot c$ be a cover in $\mathcal{M}_n$. We will show that $\iota(b) \leq \iota(c)$ in $\mathfrak{S}_n$. Suppose that $b = (t; \mathsf{T})$, with $\mathsf{T} = \{1 = T_1 < \cdots < T_r\}$. Let $\iota(b)$ have bi-leveled factorization $\iota(b) = u_1 v^1 u_2 v^2 \ldots u_r v^r$, and set $k := n + 1 - |\mathsf{T}|$.

The result is immediate if the cover $b \lessdot c$ is of type (*i*), for then $b, c \in \mathcal{M}_n(\mathsf{T})$ and $\iota\colon \mathcal{M}_n(\mathsf{T}) \to \mathfrak{S}_n$ is order-preserving, as observed in Lemma 2.6.

Now suppose that $b \lessdot c$ is a cover of type (ii). Set $w := \iota(b)$. We claim that $w \lessdot (k,k{+}1)w$ and $\iota(c) = (k,k{+}1)w$. Now, $u_1 = k$ labels the leftmost node of b, so the first claim is immediate. Note that u_2 labels the parent of the node labeled u_1. This parent has no other child in T, so we must have $u_2 < u_3$. As $u_2u_3\dots u_r$ is 231-avoiding and contains $k{+}1$, we must have $u_2 = k{+}1$. This shows that

$$\iota(c) \;=\; (k,k{+}1)w \;=\; u_2\,(v^1u_1v^2)\,u_3\dots u_rv^r\,.$$

Indeed, u_2 is minimal among $u_2,\dots,u_r$ and $u_3\dots u_r$ is 231-avoiding, thus $\min_0(c) = u_2\dots u_r$. The bi-leveled factorization of $(k,k{+}1)w$ gives $(v^1u_1v^2, v^3,\dots,v^r)$, which we claim is $\underline{\mathsf{max}}(c)$. As u_1 is the largest letter in the sequence, we need only check that $v^1u_1v^2$ is 132-avoiding. But this is true for v^1 and v^2 and there can be no 132-pattern involving u_1 as the letters in v^1 are all greater than those in v^2.

Finally, suppose that $b \lessdot c$ is of type (iii). Then $c = (t;\mathsf{T}\setminus\{T_j\})$ for some $j > 2$. We will find a permutation $w' \in \beta^{-1}(b)$ satisfying $(k,k{+}1)w' \in \beta^{-1}(c)$ and

$$\iota(b) \;\le\; w' \;\lessdot (k,k{+}1)w' \;\le\; \iota(c)\,. \tag{2.3}$$

Let $w' \in \beta^{-1}(b)$ be the minimal permutation having bi-leveled factorization

$$w' \;=\; u'_1v^1u'_2\dots u'_rv^r\,, \quad \text{with} \quad u'_j = k{+}1\,.$$

Here $(v^1,\dots,v^r) = \underline{\mathsf{max}}(b)$ is the same sequence as in $\iota(b)$. The structure of $\beta^{-1}(b)$ implies that $\iota(b) \le w'$. We also have

$$w' \;\lessdot\; (k,k{+}1)w' \qquad \text{and} \qquad \beta((k,k{+}1)w') \;=\; c\,.$$

While $\iota(c)$ and $(k,k{+}1)w'$ are not necessarily equal, we do have that

$$(k,k{+}1)w'|_{\mathsf{T}\setminus\{T_j\}} \;=\; u'_ju'_2\dots u'_{j-1}u'_{j+1}\dots u'_r$$

and $u'_2\dots u'_{j-1}u'_{j+1}\dots u'_r$ is 231-avoiding. That is, $(k,k{+}1)w'|_{\mathsf{T}\setminus\{T_j\}} = \iota(c)|_{\mathsf{T}\setminus\{T_j\}}$. Otherwise, w' would not be minimal. The bi-leveled factorization of $(k,k{+}1)w'$ is

$$u'_j\,v^1\,u'_2\,\dots\,u'_{j-1}\,(v^{j-1}u'_1v^j)\,u'_{j+1}\,\dots\,u'_r\,v^r\,,$$

and we necessarily have $(v^1,\dots,v^{j-1}u'_1v^j,\dots,v^r) \le \underline{\mathsf{max}}(c)$, which imples that $(k,k{+}1)w' \le \iota(c)$. We thus have the chain (2.3) in $\mathfrak{S}_n$, completing the proof. $\square$

If $b \lessdot c$ is the cover of type (iii) in Figure 6, the chain (2.3) from $\iota(b)$ to $\iota(c)$ is

$$4357126 \;\le\; 4367125 \;\lessdot\; 5367124 \;\le\; 5467123\,.$$

2.3. **Tree enumeration.**

Let

$$\mathbf{S}(q) \;:=\; \sum_{n\ge 0} n!q^n \;=\; 1 + q + 2q^2 + 6q^3 + 24q^4 + 120q^5 + \cdots$$

be the enumerating series of permutations, and define $\mathbf{M}(q)$ and $\mathbf{Y}(q)$ similarly

$$\mathbf{M}(q) \;:=\; \sum_{n\ge 0} A_nq^n \;=\; 1 + q + 2q^2 + 6q^3 + 21q^4 + 80q^5 + \cdots\;, \tag{2.4}$$

$$\mathbf{Y}(q) \;:=\; \sum_{n\ge 0} C_nq^n \;=\; 1 + q + 2q^2 + 5q^3 + 14q^4 + 42q^5 + \cdots\;,$$

where $A_n := |\mathcal{M}_n|$ and $C_n := |\mathcal{Y}_n|$ are the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$, whose enumerating series satisfies

$$\mathbf{Y}(q) \;=\; \frac{1-\sqrt{1-4q}}{2q} \;=\; \frac{2}{1+\sqrt{1-4q}}\,.$$

Bi-leveled trees are Catalan-like [8, Theorem 3.1]: for $n \geq 1$, $A_n = C_{n-1} + \sum_{i=1}^{n-1} A_i\, A_{n-i}$. See also [24, A121988]. Their enumerating series satisfies

$$\mathbf{M}(q) \;=\; 1 + q\mathbf{Y}(q)\cdot\mathbf{Y}(q\mathbf{Y}(q)).$$

We will also be interested in $\mathbf{M}_+(q) := \sum_{n>0} A_n q^n = q\mathbf{Y}(q)\cdot\mathbf{Y}(q\mathbf{Y}(q))$.

Theorem 2.7. *The only nontrivial quotients among the enumerating series* $\mathbf{S}(q)$*,* $\mathbf{M}(q)$*,* $\mathbf{M}_+(q)$*, and* $\mathbf{Y}(q)$ *whose expansions have nonnegative coefficients are*

$$\mathbf{S}(q)/\mathbf{M}(q), \quad \mathbf{S}(q)/\mathbf{Y}(q), \quad \mathbf{M}_+(q)/\mathbf{Y}(q), \quad \textit{and} \quad \mathbf{M}(q)/\mathbf{Y}(q).$$

Proof. We prove the positivity of the quotient $\mathbf{S}(q)/\mathbf{M}(q)$ in Section 4.2. The positivity of $\mathbf{S}(q)/\mathbf{Y}(q)$ was established after [2, Theorem 7.2], which shows that $\mathfrak{S}Sym$ is a smash product over $\mathcal{Y}Sym$.

For the positivity of $\mathbf{M}_+(q)/\mathbf{Y}(q)$, we use [3, Proposition 3], which computes $\mathbf{Y}(q\mathbf{Y}(q)) = \sum_{n>0} B_n q^{n-1}$, where

$$B_1 \;:=\; C_0 \quad\text{and}\quad B_n \;:=\; \sum_{k=0}^{n-1} \frac{k}{n-1}\binom{2n-k-3}{n-k-1} C_k \quad \text{for } n>1\,. \tag{2.5}$$

In particular, $B_n \geq 0$ for all $n \geq 0$. Returning to the quotient, we have

$$\frac{\mathbf{M}_+(q)}{\mathbf{Y}(q)} \;=\; \frac{q\mathbf{Y}(q)\cdot\mathbf{Y}(q\mathbf{Y}(q))}{\mathbf{Y}(q)} \;=\; q\mathbf{Y}(q\mathbf{Y}(q))\,,$$

so $\mathbf{M}_+(q)/\mathbf{Y}(q) = \sum_{n>0} B_n q^n$ has nonnegative coefficients.

For $\mathbf{M}(q)/\mathbf{Y}(q)$, use the identity $1/\mathbf{Y}(q) = 1 - q\mathbf{Y}(q)$ to obtain

$$\frac{\mathbf{M}(q)}{\mathbf{Y}(q)} \;=\; \mathbf{M}_+(q) + 1 - q\mathbf{Y}(q) \;=\; 1 + \sum_{n>0}(B_n - C_{n-1})q^n\,.$$

Positivity is immediate as $B_n - C_{n-1} \geq 0$ for $n > 0$.

We leave the proof that the remaining quotients have negative coefficients to the reader's computer. ☐

Remark 2.8. Up to an index shift, the quotient $\mathbf{M}_+(q)/\mathbf{Y}(q)$ corresponds to the sequence [24, A127632] begining with $(1, 1, 3, 11, 44, 185, 804)$. We give a new combinatorial interpretation of this sequence in Corollary 4.3.

3. The Algebra $\mathcal{M}Sym$

Let $\mathcal{M}Sym := \bigoplus_{n\geq 0} \mathcal{M}Sym_n$ denote the graded $\mathbb{Q}$–vector space whose n^{th} graded piece has the basis $\{F_b \mid b \in \mathcal{M}_n\}$. The maps $\beta\colon \mathfrak{S}_\bullet \to \mathcal{M}_\bullet$ and $\phi\colon \mathcal{M}_\bullet \to \mathcal{Y}_\bullet$ of graded sets induce surjective maps of graded vector spaces

$$\mathfrak{S}Sym \xrightarrow{\;\boldsymbol{\beta}\;} \mathcal{M}Sym \xrightarrow{\;\boldsymbol{\phi}\;} \mathcal{Y}Sym \qquad F_w \;\mapsto\; F_{\beta(w)} \;\mapsto\; F_{\phi(\beta(w))}\,, \tag{3.1}$$

which factor the Hopf algebra map $\boldsymbol{\tau}\colon \mathfrak{S}Sym \to \mathcal{Y}Sym$, as $\phi(\beta(w)) = \tau(w)$. We will show how the maps $\boldsymbol{\beta}$ and $\boldsymbol{\tau}$ induce on $\mathcal{M}Sym$ the structures of an algebra, of a $\mathfrak{S}Sym$-module, and of a $\mathcal{Y}Sym$-comodule so that the composition (3.1) factors the map $\boldsymbol{\tau}$ as maps of algebras, of $\mathfrak{S}Sym$-modules, and of $\mathcal{Y}Sym$-comodules.

3.1. **Algebra structure on $\mathcal{M}Sym$.** For $b, c \in \mathcal{M}_{\bullet}$ define

$$F_b \cdot F_c \;=\; \boldsymbol{\beta}(F_w \cdot F_v)\,, \tag{3.2}$$

where w, v are permutations in $\mathfrak{S}_{\bullet}$ with $b = \beta(w)$ and $c = \beta(v)$.

Theorem 3.1. *The operation $F_b \cdot F_c$ defined by* (3.2) *is independent of choices of w, v with $\beta(w) = b$ and $\beta(v) = c$ and it endows $\mathcal{M}Sym$ with the structure of a graded connected algebra such that the map $\boldsymbol{\beta}\colon \mathfrak{S}Sym \to \mathcal{M}Sym$ is a surjective map of graded connected algebras.*

If the expression $\boldsymbol{\beta}(F_w \cdot F_v)$ is independent of choice of $w \in \beta^{-1}(b)$ and $v \in \beta^{-1}(c)$, then the map $\boldsymbol{\beta}$ is automatically multiplicative. The associative and unital properties for $\mathcal{M}Sym$ are then inherited from those for $\mathfrak{S}Sym$, and the theorem follows. To prove independence (in Lemma 3.2), we formulate a description of (3.2) in terms of splittings and graftings of bi-leveled trees.

Let $s \xrightarrow{\curlyvee} (s_0, \ldots, s_m)$ be a splitting on the underlying tree of a bi-leveled tree $b = (s; \mathsf{S}) \in \mathcal{M}_n$. Then the nodes of s are distributed among the nodes of the partially ordered forest $(s_0, \ldots, s_m)$ so that the order ideal S gives a sequence of order ideals in the trees s_i. Write $b \xrightarrow{\curlyvee} (b_0, \ldots, b_m)$ for the corresponding splitting of the bi-leveled tree b, viewing b_i as $(s_i; \mathsf{S}|_{s_i})$. (Note that only b_0 is guaranteed to be a bi-leveled tree.) Given $c = (t; \mathsf{T}) \in \mathcal{M}_m$ and a splitting $b \xrightarrow{\curlyvee} (b_0, \ldots, b_m)$ of $b \in \mathcal{M}_n$, form a bi-leveled tree $(b_0, \ldots, b_m)/c$ whose underlying tree is $(s_0, \ldots, s_m)/t$ and whose order ideal is either

$$\begin{array}{rl} (i) & \mathsf{T},\ \text{ if } b_0 \in \mathcal{M}_0, \text{ or} \\ (ii) & \mathsf{S} \cup \{\text{the nodes of } t\},\ \text{ if } b_0 \notin \mathcal{M}_0. \end{array} \tag{3.3}$$

Lemma 3.2. *The product* (3.2) *is independent of choices of w, v with $\beta(w) = b$ and $\beta(v) = c$. For $b \in \mathcal{M}_n$ and $c \in \mathcal{M}_m$, we have*

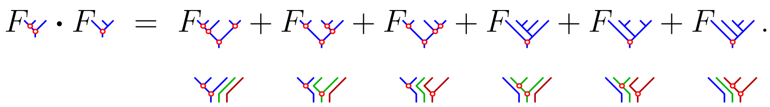
$$F_b \cdot F_c \;=\; \sum_{b \xrightarrow{\curlyvee} (b_0, \ldots, b_m)} F_{(b_0, \ldots, b_m)/c}\,.$$

Proof. Fix any $w \in \beta^{-1}(b)$ and $v \in \beta^{-1}(c)$. The bi-leveled tree $\beta((w_0, \ldots, w_m)/v)$ associated to a splitting $w \xrightarrow{\curlyvee} (w_0, \ldots, w_m)$ has underlying tree $(s_0, \ldots, s_m)/t$, where $s \xrightarrow{\curlyvee} (s_0, \ldots, s_m)$ is the induced splitting on the underlying tree $s = \tau(w) = \phi(b)$. Each node of $(w_0, \ldots, w_m)/v$ comes from a node of either w or v, with the labels of nodes from w all smaller than the labels of nodes from v. Consequently, the leftmost node of $(w_0, \ldots, w_m)/v$ comes from either

(i) v, and then $\mathsf{T}((w_0, \ldots, w_m)/v) = \mathsf{T}(v) = \mathsf{T}(c)$, or
(ii) w, and then $\mathsf{T}((w_0, \ldots, w_m)/v) = \mathsf{T}(w) = \mathsf{T}(b) \cup \{\text{the nodes of } v\}$.

The first case is when $w_0 \in \mathfrak{S}_0$ and the second case is when $w_0 \notin \mathfrak{S}_0$. □

Here is the product F $\cdot$ F, together with the corresponding splittings of ,



3.2. $\mathfrak{S}Sym$ **module structure on** $\mathcal{M}Sym$**.** Since $\boldsymbol{\beta}$ is a surjective algebra map, $\mathcal{M}Sym$ becomes a $\mathfrak{S}Sym$-bimodule with the action

$$F_w \cdot F_b \cdot F_v \;=\; F_{\beta(w)} \cdot F_b \cdot F_{\beta(v)}\ .$$

The map $\boldsymbol{\tau}$ likewise induces on $\mathcal{Y}Sym$ the structure of a $\mathfrak{S}Sym$-bimodule, and the maps $\boldsymbol{\beta}$, $\boldsymbol{\phi}$, and $\boldsymbol{\tau}$ are maps of $\mathfrak{S}Sym$-bimodules.

Curiously, we may use the map $\iota\colon \mathcal{M}_\bullet \to \mathfrak{S}_\bullet$ to define the structure of a right $\mathfrak{S}Sym$-comodule on $\mathcal{M}Sym$,

$$F_b \;\longmapsto\; \sum_{\iota(b)\xrightarrow{\curlyvee}(w_0,w_1)} F_{\beta(w_0)} \otimes F_{w_1}\,.$$

This induces a right comodule structure, because if $\iota(b) \xrightarrow{\curlyvee} (w_0, w_1)$, then $w_0 = \iota(\beta(w_0))$, which may be checked using the characterization of ι in terms of pattern avoidance, as explained in Remark 2.3.

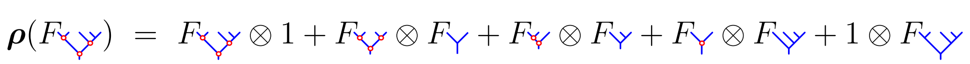
While $\mathcal{M}Sym$ is both a right $\mathfrak{S}Sym$-module and right $\mathfrak{S}Sym$-comodule, it is not an $\mathfrak{S}Sym$–Hopf module. For if it were a Hopf module, then the fundamental theorem of Hopf modules (see Remark 4.4) would imply that the series $\mathbf{M}(q)/\mathbf{S}(q)$ has positive coefficients, which contradicts Theorem 2.7.

3.3. $\mathcal{Y}Sym$**-comodule structure on** $\mathcal{M}Sym$**.** For $b \in \mathcal{M}_\bullet$, define the linear map $\boldsymbol{\rho}\colon \mathcal{M}Sym \to \mathcal{M}Sym \otimes \mathcal{Y}Sym$ by

$$\boldsymbol{\rho}(F_b) \;=\; \sum_{b\xrightarrow{\curlyvee}(b_0,b_1)} F_{b_0} \otimes F_{\phi(b_1)}\,. \tag{3.4}$$

By $\phi(b_1)$, we mean the tree underlying b_1.

Example 3.3. In the fundamental bases of $\mathcal{M}Sym$ and $\mathcal{Y}Sym$, we have



Theorem 3.4. *Under $\boldsymbol{\rho}$, $\mathcal{M}Sym$ is a right $\mathcal{Y}Sym$-comodule.*

Proof. This is counital as $(b, |)$ is a splitting of b. Coassociativity is also clear as both $(\boldsymbol{\rho} \otimes 1)\boldsymbol{\rho}$ and $(1 \otimes \Delta)\boldsymbol{\rho}$ applied to F_b for $b \in \mathcal{M}_\bullet$ are sums of terms $F_{b_0} \otimes F_{\phi(b_1)} \otimes F_{\phi(b_2)}$ over all splittings $b \xrightarrow{\curlyvee} (b_0, b_1, b_2)$. □

Careful bookkeeping of the terms in $\boldsymbol{\rho}(F_b \cdot F_c)$ show that it equals $\boldsymbol{\rho}(F_b) \cdot \boldsymbol{\rho}(F_c)$ for all $b, c \in \mathcal{M}_\bullet$ and thus $\mathcal{M}Sym$ is a $\mathcal{Y}Sym$–comodule algebra. Hence, $\boldsymbol{\phi}$ is a map of $\mathcal{Y}Sym$–comodule algebras, and in fact $\boldsymbol{\beta}$ is also a map of $\mathcal{Y}Sym$–comodule algebras. We leave this to the reader, and will not pursue it further.

Since $\boldsymbol{\tau}\colon \mathfrak{S}Sym \to \mathcal{Y}Sym$ is a map of Hopf algebras, $\mathfrak{S}Sym$ is naturally a right $\mathcal{Y}Sym$-comodule where the comodule map is the composition

$$\mathfrak{S}Sym \;\xrightarrow{\ \Delta\ }\; \mathfrak{S}Sym \otimes \mathfrak{S}Sym \;\xrightarrow{\ 1\otimes\boldsymbol{\tau}\ }\; \mathfrak{S}Sym \otimes \mathcal{Y}Sym\,.$$

With these definitions, the following lemma is immediate.

Lemma 3.5. *The maps $\boldsymbol{\tau}$ and $\boldsymbol{\phi}$ are maps of right $\mathcal{Y}Sym$-comodules.*

In particular, we have the equality of maps $\mathfrak{S}Sym \to \mathcal{M}Sym \otimes \mathcal{Y}Sym$,

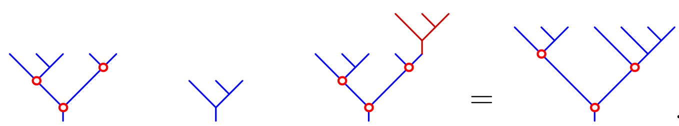
$$\boldsymbol{\rho} \circ \boldsymbol{\beta} \;=\; (\boldsymbol{\beta} \otimes \boldsymbol{\tau}) \circ \Delta\,. \tag{3.5}$$

3.4. **Coaction in the monomial basis.** The coalgebra structures of $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ were elucidated by considering a second basis related to the fundamental basis via Möbius inversion. For $b \in \mathcal{M}_n$, define

$$M_b \;:=\; \sum_{b \leq c} \mu(b,c) F_c\,, \tag{3.6}$$

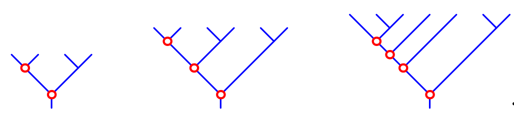
where $\mu(\cdot,\cdot)$ is the Möbius function on the poset $\mathcal{M}_n$.

Given $b \in \mathcal{M}_m$ and $s \in \mathcal{Y}_q$, write $b\backslash s$ for the bi-leveled tree with $p+q$ nodes whose underlying tree is formed by grafting the root of s onto the rightmost leaf of b, but whose order ideal is that of b. Here is an example of b, s, and $b\backslash s$,



Observe that we cannot have $b = |$ in this construction.

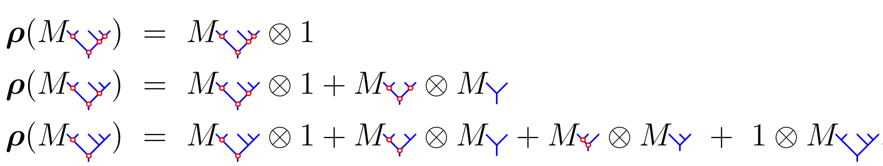
The maximum bi-leveled tree with a given underlying tree t is $\beta(\mathsf{max}(t))$, which has order ideal T consisting only of the nodes of t along its leftmost branch. Here are three such trees of the form $\beta(\mathsf{max}(t))$,



Theorem 3.6. *Given $b = (t;\mathsf{T}) \in \mathcal{M}_\bullet$, we have*

$$\boldsymbol{\rho}(M_b) \;=\; \begin{cases} \displaystyle\sum_{b=c\backslash s} M_c \otimes M_s & \text{if } b \neq \beta(\mathsf{max}(t)) \\ \displaystyle\sum_{b=c\backslash s} M_c \otimes M_s \;+\; 1 \otimes M_t & \text{if } b = \beta(\mathsf{max}(t)) \end{cases}.$$

For example,



Our proof of Theorem 3.6 uses Proposition 1.3 and the following results.

Lemma 3.7. *For any bi-leveled tree $b \in \mathcal{M}_\bullet$, we have*

$$\boldsymbol{\beta}\Big(\sum_{\beta(w)=b} M_w\Big) = M_b\,.$$

Proof. Expand the left hand side in terms of the fundamental bases to get

$$\boldsymbol{\beta}\Big(\sum_{\beta(w)=b}\ \sum_{w\le v}\mu_{\mathfrak{S}}(w,v)F_v\Big) \;=\; \sum_{\beta(w)=b}\ \sum_{w\le v}\mu_{\mathfrak{S}}(w,v)F_{\beta(v)}\,.$$

As β is surjective, we may change the index of summation to $b\le c$ in $\mathcal{M}_\bullet$ to obtain

$$\sum_{b\le c}\Big(\sum_{\substack{\beta(w)=b\\ \beta(v)=c}}\mu_{\mathfrak{S}}(w,v)\Big)F_c\,.$$

By Theorems 1.4 and 2.4, the inner sum is $\mu_{\mathcal{M}}(b,c)$, so this sum is M_b. □

Recall that $w=u\backslash v$ only if $\tau(w)=\tau(u)\backslash\tau(v)$ and the values of w in the nodes of u exceed the values in the nodes of v. We always have the trivial decomposition $w=(\emptyset,w)$. Suppose that $w=u\backslash v$ with $u\neq\emptyset$ a nontrivial decomposition. If $\beta(w)=b=(t;\mathsf{T})$, then T is a subset of the nodes of u so that $\beta(u)=(\tau(u);\mathsf{S})$ and $b=\beta(u)\backslash\tau(v)$. Moreover, for every decomposition $b=c\backslash s$ and every u,v with $\beta(u)=c$ and $\tau(v)=s$, we have $b=\beta(u\backslash v)$. Thus, for $b\in\mathcal{M}_\bullet$, we have

$$\bigsqcup_{\beta(w)=b}\ \bigsqcup_{\substack{w=u\backslash v\\ u\neq\emptyset}}(u,v) \;=\; \bigsqcup_{b=c\backslash t}\ \bigsqcup_{\beta(u)=c}\ \bigsqcup_{\tau(v)=t}(u,v)\,. \tag{3.7}$$

Proof of Theorem 3.6. Let $b=(t;\mathsf{T})$ with $t\neq\mathsf{I}$. Using Lemma 3.7, we have

$$\boldsymbol{\rho}(M_b) \;=\; \boldsymbol{\rho}\boldsymbol{\beta}\Big(\sum_{\beta(w)=b}M_w\Big) \;=\; \sum_{\beta(w)=b}\boldsymbol{\rho}\boldsymbol{\beta}M_w\,.$$

By (3.5), (3.7), and (1.5), this equals

$$\sum_{\beta(w)=b}\ \sum_{\substack{w=u\backslash v\\ u\neq\emptyset}}\boldsymbol{\beta}(M_u)\otimes\boldsymbol{\tau}(M_v) \quad+\quad \sum_{\beta(w)=b}\boldsymbol{\beta}(M_\emptyset)\otimes\boldsymbol{\tau}(M_w)$$

$$=\;\sum_{b=c\backslash s}\Big(\sum_{\beta(u)=c}\boldsymbol{\beta}(M_u)\Big)\otimes\Big(\sum_{\tau(v)=s}\boldsymbol{\tau}(M_t)\Big)\quad+\quad\sum_{\beta(w)=b}1\otimes\boldsymbol{\tau}(M_w)\,.$$

By Lemma 3.7 and (1.4), the first sum becomes $\sum_{b=c\backslash s}M_c\otimes M_s$ and the second sum vanishes unless $b=\beta(\mathsf{max}(t))$. This completes the proof. □

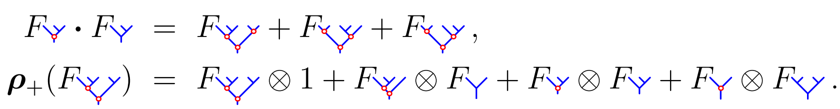
4. Hopf Variations

4.1. **The $\mathcal{Y}Sym$–Hopf module $\mathcal{M}Sym_+$.** Let $\mathcal{M}_+ := (\mathcal{M}_n)_{n\ge1}$ be the bi-leveled trees with at least one internal node and define $\mathcal{M}Sym_+$ to be the positively graded part of $\mathcal{M}Sym$, which has bases indexed by $\mathcal{M}_+$. A *restricted splitting* of $b\in\mathcal{M}_+$ is a splitting $b\xrightarrow{\curlyvee_+}(b_0,\dots,b_m)$ with $b_0\in\mathcal{M}_+$, i.e., $b_0\neq\mathsf{I}$. Given $b\xrightarrow{\curlyvee_+}(b_0,\dots,b_m)$ and $t\in\mathcal{Y}_m$, form the bi-leveled tree $(b_0,\dots,b_m)/t$ by grafting the ordered forest $(b_0,\dots,b_m)$ onto the leaves of t, with order ideal consisting of the nodes of t together with the nodes of the forest coming from the order ideal of b, as in (3.3)(ii).

We define an action and coaction of $\mathcal{Y}Sym$ on $\mathcal{M}Sym_+$ that are similar to the product and coaction on $\mathcal{M}Sym$. They come from a second collection of polytope maps $\mathcal{M}_n \twoheadrightarrow \mathcal{Y}_{n-1}$ arising from viewing the vertices of $\mathcal{M}_n$ as *painted trees* on $n-1$ nodes (see [5, 8]). For $b \in \mathcal{M}_+$ and $t \in \mathcal{Y}_m$, set

$$
\begin{aligned}
F_b \cdot F_t &= \sum_{b \xrightarrow{\curlyvee_+} (b_0,\dots,b_m)} F_{(b_0,\dots,b_m)/t}\,, \\
\boldsymbol{\rho}_+(F_b) &= \sum_{b \xrightarrow{\curlyvee_+} (b_0,b_1)} F_{b_0} \otimes F_{\phi(b_1)}\,.
\end{aligned}
\tag{4.1}
$$

For example, in the fundamental bases of $\mathcal{M}Sym_+$ and $\mathcal{Y}Sym$, we have



Theorem 4.1. *The operations in* (4.1) *define a $\mathcal{Y}Sym$–Hopf module structure on $\mathcal{M}Sym_+$.*

Proof. The unital and counital properties are immediate. We check only that the action is associative, the coaction is coassociative, and the two structures commute with each other.

Associativity. Fix $b = (t;\mathsf{T}) \in \mathcal{M}_+$, $r \in \mathcal{Y}_m$, and $s \in \mathcal{Y}_n$. A term in the expression $(F_b \cdot F_r) \cdot F_s$ corresponds to a restricted splitting and grafting $b \xrightarrow{\curlyvee_+} (b_0,\dots,b_m) \rightsquigarrow (b_0,\dots,b_m)/r = c$, followed by another $c \xrightarrow{\curlyvee_+} (c_0,\dots,c_n) \rightsquigarrow (c_0,\dots,c_n)/t$. The order ideal for this term equals $\mathsf{T} \cup \{\text{the nodes of } r \text{ and } s\}$. Note that restricted splittings of c are in bijection with pairs of splittings

$$\Big(b \xrightarrow{\curlyvee_+} (b_0,\dots,b_{m+n})\,,\; r \xrightarrow{\curlyvee} (r_0,\dots,r_n)\Big).$$

Terms of $F_b \cdot (F_r \cdot F_s)$ also correspond to these pairs of splittings. The order ideal for this term is again $\mathsf{T} \cup \{\text{the nodes of } r \text{ and } s\}$. That is, $(F_b \cdot F_r) \cdot F_s$ and $F_b \cdot (F_r \cdot F_s)$ agree term by term.

Coassociativity. Fix $b = (t;\mathsf{T}) \in \mathcal{M}_+$. Terms $F_c \otimes F_r \otimes F_s$ in $(\boldsymbol{\rho}_+ \otimes \mathbb{1})\boldsymbol{\rho}_+(F_b)$ and $(\mathbb{1} \otimes \Delta)\boldsymbol{\rho}_+(F_b)$ both correspond to restricted splittings $b \xrightarrow{\curlyvee_+} (c, c_1, c_2)$, where $\phi(c_1) = r$ and $\phi(c_2) = s$. In either case, the order ideal on c is $\mathsf{T}|_c$.

Commuting structures. Fix $b = (s;\mathsf{S}) \in \mathcal{M}_+$ and $t \in \mathcal{Y}_m$. A term $F_{c_0} \otimes F_{\phi(c_1)}$ in $\boldsymbol{\rho}_+(F_b \cdot F_t)$ corresponds to a choice of a restricted splitting and grafting $b \xrightarrow{\curlyvee_+} (b_0,\dots,b_m) \rightsquigarrow (b_0,\dots,b_r)/t = c$, followed by a restricted splitting $c \xrightarrow{\curlyvee_+} (c_0,c_1)$. The order ideal on c_0 equals the nodes of c_0 inherited from S, together with the nodes of c_0 inherited from t. The restricted splittings of c are in bijection with pairs of splittings $\big(b \xrightarrow{\curlyvee_+} (b_0,\dots,b_{m+1}), t \xrightarrow{\curlyvee} (t_0,t_1)\big)$. If $t_0 \in \mathcal{Y}_n$, then the pair of graftings $c_0 = (b_0,\dots,b_n)/t_0$ and $c_1 = (b_{n+1},\dots,b_m)/t_1$ are precisely the terms appearing in $\boldsymbol{\rho}_+(F_b) \cdot \Delta(F_t)$. ∎

The similarity of (4.1) to the coaction (3.4) of $\mathcal{Y}Sym$ on $\mathcal{M}Sym$ gives the following result, whose proof we leave to the reader.

Corollary 4.2. *For $b \in \mathcal{M}_+$, we have*

$$\boldsymbol{\rho}_+(M_b) \;=\; \sum_{b=c\backslash s} M_c \otimes M_s\,.$$

This elucidates the structure of $\mathcal{M}Sym_+$. Let $\mathcal{B} \subset \mathcal{M}_+$ be the indecomposable bi-leveled trees—those with only trivial decompositions, $b = b\backslash |$. Then $(t;\mathsf{T}) \in \mathcal{B}$ if and only if T contains the rightmost node of t. Every tree c in $\mathcal{M}_+$ has a unique decomposition $c = b\backslash s$ where $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\bullet$. Indeed, pruning c immediately above the rightmost node in its order ideal gives a decomposition $c = b\backslash s$ where $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\bullet$. This induces a bijection of graded sets,

$$\mathcal{M}_+ \;\longleftrightarrow\; \mathcal{B} \times \mathcal{Y}_\bullet\,.$$

Moreover, if $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\bullet$, then Corollary 4.2 and (1.6) together imply that

$$\boldsymbol{\rho}_+(M_{b\backslash s}) \;=\; \sum_{s=r\backslash t} M_{b\backslash r} \otimes M_t\,. \tag{4.2}$$

Note that $\mathbb{Q}\mathcal{B} \otimes \mathcal{Y}Sym$ is a graded right $\mathcal{Y}Sym$-comodule with structure map,

$$b \otimes M_s \;\longmapsto\; b \otimes (\Delta M_s)\,,$$

for $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\bullet$. Comparing this with (4.2), we deduce the following algebraic and combinatorial facts.

Corollary 4.3. *The map $\mathbb{Q}\mathcal{B} \otimes \mathcal{Y}Sym \to \mathcal{M}Sym_+$ defined by $b \otimes M_s \mapsto M_{b\backslash s}$ is an isomorphism of graded right $\mathcal{Y}Sym$ comodules.*

The quotient of enumerating series $\mathbf{M}(q)_+/\mathbf{Y}(q)$ is equal to the enumerating series of the graded set $\mathcal{B}$.

In particular, if $\mathcal{B}_n := \mathcal{B} \cap \mathcal{M}_n$, then $|\mathcal{B}_n| = B_n$ by (2.5).

Remark 4.4. The *coinvariants* in a right comodule M over a coalgebra C are $M^{\mathrm{co}} := \{m \in M \mid \boldsymbol{\rho}(m) = m \otimes 1\}$. We identify the vector space $\mathbb{Q}\mathcal{B}$ with $\mathcal{M}Sym_+^{\mathrm{co}}$ via $b \mapsto M_b$. The isomorphism $\mathbb{Q}\mathcal{B} \otimes \mathcal{Y}Sym \to \mathcal{M}Sym_+$ is a special case of the Fundamental Theorem of Hopf Modules [19, Theorem 1.9.4]: If M is a Hopf module over a Hopf algebra H, then $M \simeq M^{\mathrm{co}} \otimes H$ as Hopf modules.

4.2. **Hopf module structure on $\mathcal{M}Sym$.** We use Theorem 3.6 to identify the $\mathcal{Y}Sym$-coinvariants in $\mathcal{M}Sym$. Let $\mathcal{B}'$ be those indecomposable bi-leveled trees which are not of the form $\beta(\mathsf{max}(t))$, for some $t \in \mathcal{Y}_+$, together with $\{|\}$.

Corollary 4.5. *The $\mathcal{Y}Sym$-coinvariants of $\mathcal{M}Sym$ have a basis $\{M_b \mid b \in \mathcal{B}'\}$.*

For $n > 0$, the difference $\mathcal{B}_n \setminus \mathcal{B}'_n$ consists of indecomposable bi-leveled trees with n nodes of the form $\beta(\mathsf{max}(t))$. If $\beta(\mathsf{max}(t)) \in \mathcal{B}_n$, then $t = s \vee |$, for some $s \in \mathcal{Y}_{n-1}$, and so $|\mathcal{B}'_n| = B_n - C_{n-1}$, which we saw in the proof of Theorem 2.7.

For $t \in \mathcal{Y}_\bullet$, set $|\mathbin{\backslash\!\!\backslash} t := \beta(\mathsf{max}(t))$, and if $| \neq b \in \mathcal{B}'$, set $b \mathbin{\backslash\!\!\backslash} t := b\backslash t$. Every bi-leveled tree uniquely decomposes as $b \mathbin{\backslash\!\!\backslash} t$ with $b \in \mathcal{B}'$ and $t \in \mathcal{Y}_\bullet$. By Theorem 3.6, $M_b \otimes M_t \mapsto M_{b \mathbin{\backslash\!\!\backslash} t}$ induces an isomorphism of right $\mathcal{Y}Sym$-comodules,

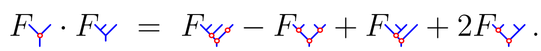
$$\mathcal{M}Sym^{\mathrm{co}} \otimes \mathcal{Y}Sym \;\longrightarrow\; \mathcal{M}Sym\,, \tag{4.3}$$

where the structure map on $\mathcal{M}Sym^{\mathrm{co}} \otimes \mathcal{Y}Sym$ is $M_b \otimes M_t \mapsto M_b \otimes \Delta(M_t)$. Treating $\mathcal{M}Sym^{\mathrm{co}}$ as a trivial $\mathcal{Y}Sym$-module, $M_b \cdot M_t = \varepsilon(M_t)M_b$, $\mathcal{M}Sym^{\mathrm{co}} \otimes \mathcal{Y}Sym$ becomes a right $\mathcal{Y}Sym$-module. As explained in [19, Example 1.9.3], this makes $\mathcal{M}Sym^{\mathrm{co}} \otimes \mathcal{Y}Sym$ into a $\mathcal{Y}Sym$–Hopf module.

We express this structure on $\mathcal{M}Sym$. Let $b \backslash\!\backslash t \in \mathcal{M}_\bullet$ and $s \in \mathcal{Y}_\bullet$, then

$$M_{b\backslash\!\backslash t} \cdot M_s \;=\; \sum_{r \in t \cdot s} M_{b \backslash\!\backslash r} \quad \text{and} \quad \boldsymbol{\rho}\big(M_{b\backslash\!\backslash t}\big) \;=\; \sum_{t = r \backslash s} M_{b\backslash\!\backslash r} \otimes M_s\,, \tag{4.4}$$

where $t \cdot s$ is the set of trees r indexing the product $M_t \cdot M_s$ in $\mathcal{Y}Sym$. The coaction is as before, but the product is new. It is not positive in the fundamental basis,



We complete the proof of Theorem 2.7.

Corollary 4.6. *The power series* $\mathbf{S}(q)/\mathbf{M}(q)$ *has nonnegative coefficients.*

Proof. Observe that

$$\mathbf{S}(q)/\mathbf{M}(q) \;=\; \Big(\mathbf{S}(q)/\mathbf{Y}(q)\Big) \Big/ \Big(\mathbf{M}(q)/\mathbf{Y}(q)\Big)\,.$$

Since both $\mathfrak{S}Sym$ and $\mathcal{M}Sym$ are right $\mathcal{Y}Sym$–Hopf modules, the two quotients of enumerating series on the right are generating series for their coinvariants, by the Fundamental Theorem of Hopf modules. Thus

$$\mathbf{S}(q)/\mathbf{M}(q) \;=\; \mathbf{S}^{\mathrm{co}}(q)/\mathbf{M}^{\mathrm{co}}(q)\,,$$

where $\mathbf{S}^{\mathrm{co}}(q)$ and $\mathbf{M}^{\mathrm{co}}(q)$ are the enumerating series for $\mathfrak{S}Sym^{\mathrm{co}}$ and $\mathcal{M}Sym^{\mathrm{co}}$. To show that $\mathbf{S}^{\mathrm{co}}(q)/\mathbf{M}^{\mathrm{co}}(q)$ is nonnegative, we index bases for these spaces by graded sets $\mathcal{S}$ and $\mathcal{B}'$, then establish a bijection $\mathcal{B}' \times \mathcal{S}' \to \mathcal{S}$ for some graded subset $\mathcal{S}' \subset \mathcal{S}$.

The set $\mathcal{B}'$ was identified in Corollary 4.5. The coinvariants $\mathfrak{S}Sym^{\mathrm{co}}$ were given in [2, Theorem 7.2] as a *left Hopf kernel*. The basis was identified as follows. Recall that pemutations $u \in \mathfrak{S}_\bullet$ may be written uniquely in terms of indecomposables,

$$u \;=\; u_1 \backslash \cdots \backslash u_r \tag{4.5}$$

(taking $r = 0$ for $u = \emptyset$). Let $\mathcal{S} \subset \mathfrak{S}_\bullet$ be those permutations u whose rightmost indecomposable component has a 132-pattern, and thus $u \neq \mathsf{max}(t)$ for any $t \in \mathcal{Y}_+$. (Note that $u = \emptyset \in \mathcal{S}$.) Then $\{M_u \mid u \in \mathcal{S}\}$ is a basis for $\mathfrak{S}Sym^{\mathrm{co}}$.

Fix a section $g\colon \mathcal{M}_\bullet \to \mathfrak{S}_\bullet$ of the map $\beta\colon \mathfrak{S}_\bullet \to \mathcal{M}_\bullet$ and define a subset $\mathcal{S}' \subset \mathcal{S}$ as follows. Given the decomposition $u = u_1 \backslash \cdots \backslash u_r$ in (4.5) with $r \geq 0$, consider the length $\ell \geq 0$ of the maximum initial sequence $u_1 \backslash \cdots \backslash u_\ell$ of indecomposables belonging to $g(\mathcal{B}')$. Put $u \in \mathcal{S}'$ if ℓ is even. Define the map of graded sets

$$\kappa \;:\; \mathcal{B}' \times \mathcal{S}' \longrightarrow \mathcal{S} \quad \text{by} \quad (b, v) \;\longmapsto\; g(b) \backslash v\,.$$

The image of κ lies in $\mathcal{S}$ as the last component of a nontrivial $g(b)\backslash v$ is either $g(b)$ or the last component of v, neither of which can be $\mathsf{max}(t)$ for $t \in \mathcal{Y}_+$.

We claim that κ is bijective. If $u \in \mathcal{S}'$, then $u = \kappa(|, u)$. If $u \in \mathcal{S} \setminus \mathcal{S}'$, then u has an odd number of initial components from $g(\mathcal{B}')$. Letting its first factor be $g(b)$, we see that $u = g(b)\backslash u' = \kappa(b, u')$ with $u' \in \mathcal{S}'$. This surjective map is injective as the expressions $\kappa(\mathsf{l}, u')$ and $\kappa(b, u')$ with $b \in \mathcal{B}'_+$ and $u' \in \mathcal{S}'$ are unique.

This isomorphism of graded sets identifies the enumerating series of the graded set $\mathcal{S}'$ as the quotient $\mathbf{S}^{\mathrm{co}}(q)/\mathbf{M}^{\mathrm{co}}(q)$, which completes the proof. □

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