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NONTRIVIAL LINEAR PROJECTIONS ON THE GRASSMANNIAN $\mathrm{Gr}_3(\mathbb{C}^6)$

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Abstract. A typical linear projection of the Grassmannian in its Plücker embedding is injective, unless its image is a projective space. A notable exception are self-adjoint linear projections, which have even degree. We consider linear projections of $\mathrm{Gr}_3\mathbb{C}^6$ with low-dimensional centers of projection. When the center has dimension less than five, we show that the projection has degree 1. When the center has dimension five and the projection has degree greater than 1, we show that it is self-adjoint.

1. Introduction

Consider a linear ordinary differential operator (ODO) of order n

$$L\,x(t) \;=\; x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t)\,, \tag{1.1}$$

where $a_0, \ldots, a_{n-1}$ are complex-valued continuous functions on an interval $I \subset \mathbb{R}$. Let V_L be the space of complex-valued solutions of the homogeneous equation $Lx = 0$.

The *Wronskian* of m smooth functions $f_1(t), \ldots, f_m(t)$ on I is the determinant

$$\mathrm{Wr}\big(f_1(t), f_2(t), \ldots, f_m(t)\big) \;:=\; \det \begin{pmatrix} f_1(t) & f_2(t) & \cdots & f_m(t) \\ f_1'(t) & f_2'(t) & \cdots & f_m'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(m-1)}(t) & f_2^{(m-1)}(t) & \cdots & f_m^{(m-1)}(t) \end{pmatrix}.$$

The Wronskian $\mathrm{Wr}\big(f_1(t), \ldots, f_m(t)\big)$ is not identically zero when $f_1(t), \ldots, f_m(t)$ form a basis of an m-dimensional subspace Λ in V_L. If $g_1(t), \ldots, g_m(t)$ is another basis, then

$$\mathrm{Wr}\big(g_1(t), g_2(t), \ldots, g_m(t)\big) \;=\; c\mathrm{Wr}\big(f_1(t), f_2(t), \ldots, f_m(t)\big)\,,$$

where c is the determinant of the transition matrix between the bases. Therefore, the one-dimensional linear subspace of $C^\infty(I)$ spanned by the Wronskian $\mathrm{Wr}\big(f_1(t), \ldots, f_m(t)\big)$ depends only upon Λ. This element of the projective space $\mathbb{P}C^\infty(I)$ is called the *Wronskian of the subspace* Λ. This defines the *Wronski map* $\mathrm{Wr}_{L,m}$ from the Grassmannian $\mathrm{Gr}_m V_L$ of m-dimensional subspaces of V_L to $\mathbb{P}C^\infty(I)$.

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For complex algebraic varieties X, Y of the same dimension and a dominant map $F\colon X \to Y$, the number of points in a preimage $F^{-1}(y)$ for $y \in Y$ is constant over an open dense subset of Y. This constant number is the *degree* of the map F [6].

Consider this for the Wronski map $\mathrm{Wr}_{L,m}$ when the image of $\mathrm{Gr}_m V_L$ has the same dimension as $\mathrm{Gr}_m V_L$. For generic linear ODO L of order n and any $m \in \{2, \ldots, n-1\}$ the Wronski map $\mathrm{Wr}_{L,m}$ is injective (see Remark 1.1) and so $\mathrm{Wr}_{L,m}$ has degree 1. For any L, is it injective when $m = 1$ or $m = n-1$. We are interested in the following question.

Question 1. *Under what conditions on a linear ODO L of order n and on $1 < m < n-1$ does the Wronski map $\mathrm{Wr}_{L,m}$ have degree greater than 1?*

The classical Wronski map is when V is the space of polynomials of degree $n-1$. This corresponds to the ODO $L_0\, x(t) = x^{(n)}(t)$. Work of Schubert in 1886 [9], combined with a result of Eisenbud and Harris in 1983 [4] shows that the Wronski map $\mathrm{Wr}_{L_0,m}$ has degree

$$\frac{1!2!\cdots(n-m-1)!\cdot(m(n-m))!}{m!(m+1)!\cdots(n-1)!}\,. \tag{1.2}$$

The degree exceeds 1 except in the trivial cases of $m = 1$ or $m = n-1$.

Three of us addressed Question 1 in a previous paper [7]. The operator

$$L^*\, x(t) \;:=\; (-1)^n x^{(n)}(t) \;+\; \sum_{i=1}^{n-1} (-1)^i (a_i x)^{(i)}(t) \tag{1.3}$$

is *(formally) adjoint* to the operator L (1.1). An operator L is a *(formally) self-adjoint differential operator* if $L^* = L$. When L is self-adjoint, its order n is even.

Two linear ODOs L and $\widetilde{L}$ on I are *equivalent* if there exists a smooth nonvanishing function μ on I such that

$$\widetilde{L}x \;=\; \frac{1}{\mu} L(\mu x)\,.$$

We paraphrase two results from [7]. For both, L is a linear ODO of order n.

Theorem 2.9 of [7] *If L is equivalent to a self-adjoint operator and $n = 2m$, then the Wronski map $\mathrm{Wr}_{L,m}$ has even degree.*

Corollary 1.8 of [7] *If the Wronski map $\mathrm{Wr}_{L,m}$ has degree 2, then $n = 2m$ and L is equivalent to a self-adjoint linear operator.*

The proof of [7, Thm. 2.9] is based on two observations. First, if L is equivalent to a self-adjoint operator then the space V_L is endowed with a canonical (up to a nonzero scaling) symplectic structure σ_L. Second, if $\Lambda^{\angle}$ is the skew-orthogonal complement of an m-dimensional subspace Λ of V_L with respect to the form σ_L, then

$$\mathrm{Wr}_{L,m}(\Lambda^{\angle}) \;=\; \mathrm{Wr}_{L,m}(\Lambda), \tag{1.4}$$

so that the Wronskian is preserved under taking skew-orthogonal complement.

From (1.2) it follows that for the ODO $L_0\, x(t) = x^{(n)}(t)$ with $n \geq 5$ and $m \notin \{1, n-1\}$ the Wronski map $\mathrm{Wr}_{L_0,m}$ has degree greater than 2. Thus $n = 2m$ is not necessary for the degree of the Wronski map to exceed 1.

Question 2. *When $n = 2m$, does the statement of* [7, Cor. 1.8] *generalize as follows: If the Wronski map $\mathrm{Wr}_{L,m}$ of a $2m$-th order linear ODO L has degree greater than 2, is L equivalent to a self-adjoint operator?*

We address a generalization of Question 2. The Grassmannian $\mathrm{Gr}_m V_L$ is a subvariety of Plücker space $\mathbb{P}\bigwedge^m V_L$. Given a linear subspace $\mathbb{P}Z \subset \mathbb{P}\bigwedge^m V_L$ (Z is a linear subspace of $\bigwedge^m V_L$), the linear projection with center $\mathbb{P}Z$ is the map $\mathbb{P}\bigwedge^m V_L \smallsetminus \mathbb{P}Z \to \mathbb{P}(\bigwedge^m V_L)/Z$ induced by the map $\bigwedge^m V_L \to (\bigwedge^m V_L)/Z$. When $\mathbb{P}Z$ is disjoint from the Grassmannian, it induces the *linear projection* $\pi_Z\colon \mathrm{Gr}_m V_L \to \mathbb{P}(\bigwedge^m V_L)/Z$.

Proposition 2.3 of [7] identifies the Wronski map with a linear projection. We explain that. Given a basis $f_1, \dots, f_n$ for V_L, let $f_1^*, \dots f_n^* \in V_L^*$ be its dual basis and set

$$c(t) \;:=\; \sum_{i=1}^{n} f_i(t) f_i^* \in V_L^*\,, \text{ for } t \in I\,. \tag{1.5}$$

Fix $m \in \{1, \dots, n-1\}$ and define the following subspace of $\bigwedge^m V_L^*$,

$$X_L \;:=\; \left\langle c(t) \wedge c'(t) \wedge \cdots \wedge c^{(m-1)}(t) \mid t \in I \right\rangle, \tag{1.6}$$

where

$$c^{(j)}(t) \;=\; \sum_{i=1}^{n} f_i^{(j)}(t) f_i^* \in V_L^*\,.$$

By [7, Prop. 2.3], the Wronski map takes values in the space X_L^* dual to X_L, which is $(\bigwedge^m V)/X_L^\perp$, where

$$X_L^\perp \;=\; \{w \in \textstyle\bigwedge^m V_L \mid \omega(v) = 0 \;\; \forall v \in X_L\}$$

is the annihilator of X_L (for details see [7, pp. 755-6]).

Remark 1.1. For generic linear ODO L, $X_L = \bigwedge^m V_L^*$, which implies that the Wronski map is injective (this is a consequence of Proposition 3.1 below, as $X_L^\perp = 0$). ◇

Remark 1.2. As a consequence of [7, Sect. 2.3], a linear ODO of order $2m$ is self-adjoint if and only if there exists a symplectic form σ on V_L^* such that

$$X_L^\perp \;\supseteq\; \mathbb{C}\sigma \wedge \textstyle\bigwedge^{m-2} V_L\,.$$

Moreover, the canonical symplectic form on V_L is induced by the form σ through the identification of V_L with V_L^* via σ. This inclusion implies that

$$\dim X_L^\perp \;\geq\; \dim \textstyle\bigwedge^{m-2} V_L^* \;=\; \binom{2m}{m-2},$$

with equality for a generic self-adjoint linear ODO of order $2m$. When $m = 3$ and L is self-adjoint, the minimal possible dimension of $X_L^\perp$ is 6. ◇

Let V be an even-dimensional complex vector space and $1 < m < \dim V$. A linear subspace $Z \subset \bigwedge^m V$ is *self-adjoint* if there exists a symplectic form σ on V^* such that

$$Z \;\supseteq\; \mathbb{C}\sigma \wedge \textstyle\bigwedge^{m-2} V\,.$$

We state our main results.

Theorem 3.6. *When $m = 2$ and $n = 4$, if $\mathbb{P}Z$ is a linear subspace disjoint from $\mathrm{Gr}_2\mathbb{C}^4$, then Z is self-adjoint.*

When $m = 3$ and $n = 6$, we consider centers Z of projective dimensions four or five.

Proposition 1.3. *Suppose that $m = 3$ and $n = 6$. Let $Z \subset \bigwedge^3\mathbb{C}^6$ be a linear subspace with $\mathbb{P}Z$ disjoint from $\mathrm{Gr}_3\mathbb{C}^6$.*

1. [Corollary 3.17] *If $\dim \mathbb{P}Z \leq 4$, then π_Z has degree 1.*
2. [Theorem 3.18] *If $\dim \mathbb{P}Z = 5$, then π_Z has degree greater than 1 if and only if Z is self-adjoint.*

We deduce our main results concerning Question 2.

Theorem 1.4. *Let L be a linear ODO of order 4. Then the degree of the Wronski map $\mathrm{Wr}_{L,2}$ exceeds 1 if and only if L is equivalent to a self-adjoint linear ODO.*

Theorem 1.5. *Let L be a linear ODO of order 6. Then the following statements hold.*

1. *If $\dim X_L^\perp \leq 5$, then the degree of the Wronski map $\mathrm{Wr}_{L,3}$ is equal to 1.*
2. *If $\dim X_L^\perp = 6$, then the degree of the Wronski map $\mathrm{Wr}_{L,3}$ exceeds 1 if and only if L is equivalent to a self-adjoint linear ODO.*

In the next section, we discuss an application of Theorems 1.4 and 1.5 to pole placement in linear systems theory. We prove our main results in Section 3.

2. APPLICATION TO POLE PLACEMENT FOR CONSTANT OUTPUT FEEDBACK

For a background on linear systems theory, see [2]. A *state-space realization* of a (strictly proper) m-input p-output linear system is a triple $\Sigma = (A, B, C)$ of matrices of sizes $N \times N$, $N \times m$, and $p \times N$. This defines a system of first order constant coefficient linear differential equations,

$$\dot{x} \;=\; Ax + Bu \qquad \text{and} \qquad y \;=\; Cx\,, \tag{2.1}$$

where $x \in \mathbb{C}^N$, $u \in \mathbb{C}^m$, and $y \in \mathbb{C}^p$ are functions of $t \in \mathbb{C}$ (and $\dot{x} = \frac{d}{dt}x$). Applying Laplace transform ($u(t) \mapsto \hat{u}(s)$) and assuming that $x(0) = 0$, we eliminate $\widehat{x}$ to obtain

$$\widehat{y}(s) \;=\; C(sI - A)^{-1}B\,\widehat{u}(s) \;=\; G(s)\,\widehat{u}(s)\,,$$

where $G(s) := C(sI - A)^{-1}B$ is the *transfer function* of (2.1). This $p \times m$ matrix of rational functions has poles at the eigenvalues of A.

A linear system may be controlled with output feedback, setting $u = Ky$, where K is a constant $m \times p$ matrix. Substitution in (2.1) and elimination gives the closed loop system,

$$\dot{x} \;=\; (A + BKC)x\,,$$

whose transfer function has poles at the zeroes of the characteristic polynomial

$$P_\Sigma(K) \;=\; P_\Sigma \;:=\; \det(sI \;-\; (A + BKC))\,. \tag{2.2}$$

The map $K \mapsto P_\Sigma(K)$ is called the *pole placement map*. Given a system (2.1) with state-space realization Σ and poles $z = \{z_1, \ldots, z_N\} \subset \mathbb{C}$, the pole placement problem asks for a matrix K such that $P_\Sigma(K)$ vanishes at the points of z. This is only possible for general z if $N \leq mp$ [1]. We are interested when $N \geq mp$ and the pole placement map is a nontrivial branched cover of its image.

Using the injection $\mathrm{Mat}_{m\times p}\mathbb{C} \to \mathrm{Gr}_p\mathbb{C}^{m+p}$ where K is sent to the column space of the matrix $\left(\begin{smallmatrix} K \\ I_p \end{smallmatrix}\right)$, standard manipulations show that the pole placement map is a linear projection of $\mathrm{Gr}_p\mathbb{C}^{m+p}$. The map that sends $s \in \mathbb{P}^1$ to the column space of $\left(\begin{smallmatrix} I_m \\ G(s) \end{smallmatrix}\right)$ defines the *Hermann-Martin curve* $\gamma_\Sigma \colon \mathbb{P}^1 \to \mathrm{Gr}_m\mathbb{C}^{m+p}$ [8]. Its degree is the McMillan degree, which is the minimal number N in a state-space realization giving the transfer function $G(s)$. Such a minimal representation is observable and controllable [2].

If $X_\Sigma \subset \bigwedge^m\mathbb{C}^{m+p}$ is the linear span of the image of the curve γ_Σ and $Z := X_\Sigma^\perp$ is its annihilator in $\bigwedge^p\mathbb{C}^{m+p}$, then the pole placement map P_Σ is the linear projection π_Z, and we may identify the quotient $X_\Sigma^* = (\bigwedge^p\mathbb{C}^{m+p})/Z$ with the space of polynomials of degree at most N. The pole placement map is *proper* if $\emptyset \neq \mathbb{P}Z$ is disjoint from the Grassmannian $\mathrm{Gr}_p\mathbb{C}^{m+p}$. This terminology is not standard in systems theory.

Consider the following change of coordinates in the state, input, and output spaces

$$x = R\widetilde{x}\,, \qquad u = Q\widetilde{y} + W\widetilde{u}\,, \quad \text{and} \quad y = T\widetilde{y}\,, \tag{2.3}$$

where R, W, and T are invertible matrices and Q is a $m \times p$ matrix. The transformation of the space $\mathbb{C}^N \times \mathbb{C}^m \times \mathbb{C}^p$ given by (2.3) is a *state-feedback transformation*. Substituting (2.3) into (2.1), we obtain a new state-space realization in $(\widetilde{x}, \widetilde{u}, \widetilde{y})$,

$$\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}\widetilde{u} \qquad \text{and} \qquad \widetilde{y} = \widetilde{C}\widetilde{x}\,,$$

given by the triple of matrices $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C})$, where

$$\widetilde{A} = R^{-1}(A + BQT^{-1}C)R\,, \quad \widetilde{B} = R^{-1}BW\,, \quad \text{and} \quad \widetilde{C} = T^{-1}CR\,. \tag{2.4}$$

Two realizations are *state-feedback equivalent* if one is a state-feedback transformation of the other. The following is standard.

Proposition 2.1. *Equivalent state-space realizations have equivalent Hermann-Martin curves, where the equivalence is induced by an element of* $\mathrm{GL}(\mathbb{C}^{m+p})$.

A state-space realization (2.1) is *symmetric* [5] if $A^T = A$ and $C = B^T$.

Proposition 2.2. [7, Sect. 3.2] *For a controllable and observable linear system with state-space realization Σ (2.1), the corresponding center Z is self-adjoint if and only if the realization Σ is state-feedback equivalent to a symmetric realization.*

The degree of the pole placement map of a symmetric state-space realization is at least 2, because $P_\Sigma(K^T) = P_\Sigma(K)$. The following corollaries are consequences of Theorem 3.6, of Corollary 3.17, and of Theorem 3.18.

Corollary 2.3. *If a controllable and observable linear system with $m = p = 2$ has a proper pole placement map, then any state-space realization* (2.1) *is state-feedback equivalent to a symmetric realization.*

Corollary 2.4. *Suppose that Σ is a state-space realization* (2.1) *of a controllable and observable linear system with $m = p = 3$ whose pole placement map is proper and has degree greater than* 1. *If the center Z of the pole placement map has dimension at most six, then* $\dim Z = 6$, *and Σ is state-feedback equivalent to a symmetric realization.*

3. Linear projections of the Grassmannian

For a finite-dimensional vector space W, let W^* be its linear dual. Write $\mathbb{P}W$ for its projective space of one-dimensional linear subspaces. Then $\mathbb{P}W^*$ is identified with the set of hyperplanes in W. For a vector subspace $Z \subset W$, $\mathbb{P}Z$ is a linear subspace of $\mathbb{P}W$. We will often write Z for $\mathbb{P}Z$, and α for a nonzero vector in W, for the linear subspace $\langle\alpha\rangle$, and for the corresponding point of $\mathbb{P}W$. Context will determine which we intend.

Let m, n be positive integers with $m < n$ and let V be an n-dimensional complex vector space. For a proper linear subspace $Z \subsetneq \mathbb{P}\bigwedge^m V$, the *projection* with *center* Z,

$$\mathbb{P}{\textstyle\bigwedge}^m V \smallsetminus Z \ \longrightarrow\ \mathbb{P}\left({\textstyle\bigwedge}^m V\right)/Z\,, \tag{3.1}$$

is induced by the quotient map $\bigwedge^m V \twoheadrightarrow (\bigwedge^m V)/Z$. This projection is a rational map on $\mathbb{P}\bigwedge^m V$ as it is not defined on Z.

The Grassmannian $\mathrm{Gr}_m V$ of m-dimensional subspaces of V is embedded into $\mathbb{P}\bigwedge^m V$ via the *Plücker embedding* which sends an m-dimensional space Λ with basis $v_1, \dots, v_m$ to the span of its Plücker vector $v_1 \wedge \cdots \wedge v_m$, written Λ. Elements of $\bigwedge^m V$ representing points of $\mathrm{Gr}_m V$ are *decomposable*. Whether we intend $\Lambda \in \mathrm{Gr}_m V$ to be a point of $\mathbb{P}\bigwedge^m V$ or a linear subspace of V will often be determined by context.

Let $Z \subset \mathbb{P}\bigwedge^m V$ be a linear subspace disjoint from $\mathrm{Gr}_m V$. Write π_Z for the restriction of the corresponding linear projection (3.1) to $\mathrm{Gr}_m V$. In [7] such a linear projection was called a *generalized Wronski map*, a terminology motivated by the following result.

Proposition 3.1 ([7, Prop. 2.3])**.** *The Wronski map $\mathrm{Wr}_{L,m}$ of an nth order linear ODO L is the projection π_Z with center $Z = X_L^{\perp}$, where X_L is defined by* (1.6).

Remark 3.2. Note that $X_L^{\perp}$ is disjoint from the Grassmannian $\mathrm{Gr}_m V_L$. This is because Wronskians are not identically zero and the formulation (1.5). ◇

Assume that $\dim V = 2m$. A 2-form $\sigma \in \bigwedge^2 V$ is an element of the tensor space $V \otimes V$. It is a linear map $V^* \to V$ which is given by contraction, $v \mapsto v \lrcorner \sigma$. The *rank* of σ is its rank as a linear map, and this is an even integer. When σ has rank $2m$, it is a symplectic form on V^*. Then corresponding map $V^* \to V$ is an isomorphism and σ induces a symplectic form $\sigma^* \in \bigwedge^2 V^*$ on V. The *skew-orthogonal complement* to $\Lambda \in \mathrm{Gr}_m V$ is the linear subspace

$$\Lambda^{\angle} \ :=\ \{w \in V \mid \sigma^*(w, v) = 0 \quad \forall v \in \Lambda\}\,.$$

This also has dimension m, so $\Lambda^{\angle} \in \mathrm{Gr}_m V$.

A linear subspace $Z \subset \mathbb{P}\bigwedge^m V$ is *self-adjoint* if there exists a symplectic form σ on V^* such that

$$Z \ \supseteq\ \mathbb{P}\bigl(\mathbb{C}\sigma \wedge {\textstyle\bigwedge}^{m-2} V\bigr)\,. \tag{3.2}$$

By [7, Cor. 1.5],

$$\pi_Z(\Lambda^{\angle}) \ =\ \pi_Z(\Lambda)\,, \quad \forall \Lambda \in \mathrm{Gr}_m V\,, \tag{3.3}$$

Thus when Z is self-adjoint, the degree of π_Z is even and hence exceeds 1. We address the converse: *Does degree of π_Z exceeding 1 imply that the center Z is self-adjoint?*

3.1. **Projection from a point.** Let π_ω be the linear projection with center $\omega \in \mathbb{P}\bigwedge^m V$.

Lemma 3.3. *Suppose that* $Z \subset \mathbb{P}\bigwedge^m V$ *is a linear subspace disjoint from the Grassmannian* $\mathrm{Gr}_m V$*. For* $\Lambda, \Lambda' \in \mathrm{Gr}_m V$*, we have* $\pi_Z(\Lambda) = \pi_Z(\Lambda')$ *if and only if there exists a point* $\omega \in Z$ *such that* $\pi_\omega(\Lambda) = \pi_\omega(\Lambda')$ *if and only if* Z *meets the line* $\langle \Lambda, \Lambda' \rangle$ *in* $\mathbb{P}\bigwedge^m V$ *containing the points* Λ, Λ'*.*

Proof. If $\pi_\omega(\Lambda) = \pi_\omega(\Lambda')$, then for any subspace Z containing ω, $\pi_Z(\Lambda) = \pi_Z(\Lambda')$. For the other direction, suppose that $\pi_Z(\Lambda) = \pi_Z(\Lambda')$ with $\Lambda \neq \Lambda'$ in $\mathrm{Gr}_m V$. Then the line $\langle \Lambda, \Lambda' \rangle$ they span meets Z. If $\omega \in \langle \Lambda, \Lambda' \rangle \cap Z$, then $\pi_\omega(\Lambda) = \pi_\omega(\Lambda')$. □

For a center $Z \subset \mathbb{P}\bigwedge^m V$ disjoint from the Grassmannian $\mathrm{Gr}_m V$, define

$$\mathcal{S}_Z \ := \ \{\Lambda \in \mathrm{Gr}_m(V) \mid \exists \Lambda' \neq \Lambda \text{ such that } \pi_Z(\Lambda) = \pi_Z(\Lambda')\}\,, \tag{3.4}$$

and for $\omega \in \mathbb{P}\bigwedge^m V$, similarly define $\mathcal{S}_\omega$. Lemma 3.3 is equivalent to

$$\mathcal{S}_Z \ = \ \bigcup_{\omega \in Z} \mathcal{S}_\omega\,. \tag{3.5}$$

Remark 3.4. Lemma 3.3 motivates our approach to study the degree of the map π_Z. First, for each $\omega \in Z$, describe all $\Lambda \in \mathrm{Gr}_m V$ such that there exist $\Lambda' \neq \Lambda$ in $\mathrm{Gr}_m V$ with $\pi_\omega(\Lambda) = \pi_\omega(\Lambda')$. Then take a union of all such Λ for $\omega \in Z$. If this union does not contain an open dense set of $\mathrm{Gr}_m V$ then π_Z has degree 1.

The group $\mathrm{GL}(V)$ of invertible linear transformations on V acts on $\mathrm{Gr}_m V$ and $\mathbb{P}\bigwedge^m V$, and for $\omega \in \mathbb{P}\bigwedge^m V$, $\Lambda, \Lambda' \in \mathrm{Gr}_m V$, and $g \in \mathrm{GL}(V)$, we have

$$\pi_\omega(\Lambda) = \pi_\omega(\Lambda') \qquad \text{if and only if} \qquad \pi_{g.\omega}(g.\Lambda) = \pi_{g.\omega}(g.\Lambda')\,.$$

Therefore, to find the set of pairs $\Lambda, \Lambda' \in \mathrm{Gr}_m(V)$ with the same image under π_ω it is enough to find this set for one representative of the $\mathrm{GL}(V)$-orbit of ω. ◇

Remark 3.5. Suppose that $\dim V = 4$. The Grassmannian $\mathrm{Gr}_2 V \subset \mathbb{P}\bigwedge^2 V \simeq \mathbb{P}^5$ is a quadratic hypersurface. Thus, if $\omega \in \mathbb{P}\bigwedge^2 V \smallsetminus \mathrm{Gr}_2 V$, then $\pi_\omega \colon \mathrm{Gr}_2 V \to \mathbb{P}(\bigwedge^2 V)/\omega \simeq \mathbb{P}^4$ has degree two. In particular, $\mathcal{S}_\omega \subset \mathrm{Gr}_2 V$ is dense and therefore has dimension four.

This will be relevant in Section 3.2, where we show that for $\omega \in \mathbb{P}\bigwedge^3 \mathbb{C}^6 \smallsetminus \mathrm{Gr}_3 \mathbb{C}^6$, either $\mathcal{S}_\omega$ is either zero-dimensional, empty, or four-dimensional, and the last case may be understood to be a consequence of the projection map on $\mathrm{Gr}_2 V$. ◇

This degree two projection $\mathrm{Gr}_2 V \to \mathbb{P}^4$ is intrinsically related to symplectic structures.

Theorem 3.6. *When* $\dim V = 4$*, any* $\sigma \in \mathbb{P}\bigwedge^2 V \smallsetminus \mathrm{Gr}_2 V$ *is a symplectic form on* V^**. For* $\Lambda, \Lambda' \in \mathrm{Gr}_2 V$ *with* $\Lambda \neq \Lambda'$*, we have that* $\pi_\sigma(\Lambda) = \pi_\sigma(\Lambda')$ *if and only if* $\Lambda' = \Lambda^\angle$*, the skew-orthogonal complement of* Λ *with respect to the symplectic form* σ^**.*

3.2. **Projection from a point when $m = 3$ and $n = 6$.** Assume that $\dim V = 6$. When convenient, we identify V with $\mathbb{C}^6$ with the standard basis $\{e_1, \dots, e_6\}$ and let $\{e_1^*, \dots, e_6^*\}$ be the dual basis for V^*. Following Remark 3.4, we first study the action of $\mathrm{GL}(V)$ on $\mathbb{P}\bigwedge^3 V$. The orbits under this action were described by Segre in 1918 [10]. For i, j, k, write e_{ijk} for $e_i \wedge e_j \wedge e_k$ and e_{ij} for $e_i \wedge e_j$. Then e_{123} is the Plücker vector of $\langle e_1, e_2, e_3 \rangle$.

Theorem 3.7 (Segre [10], see also [3])**.** *The action of* $\mathrm{GL}(V)$ *on* $\mathbb{P}\bigwedge^3 V$ *has four orbits* O_0, O_1, O_5*, and* O_{10}*, where* O_i *has codimension* i*. A normal form for an element* $\omega_i \in O_i$ *of each orbit is as follows.*

(1) $\omega_0 = e_{123} + e_{456}$*, a point on the line between* e_{123} *and* e_{456}*.*
(2) $\omega_1 = e_{126} - e_{153} + e_{234}$*, a general point in the tangent space to* $\mathrm{Gr}_3 V$ *at* e_{123}*.*
(3) $\omega_5 = e_1 \wedge (e_{23} + e_{45})$*, a point on the line between* e_{123} *and* e_{145}*.*
(4) $\omega_{10} = e_{123}$*, a point on the Grassmannian* $\mathrm{Gr}_3 V$*.*

Remark 3.8. For a 3-plane $\Lambda \in \mathrm{Gr}_3 V$, the tangent space $T_\Lambda \mathrm{Gr}_3 V$ to the Grassmannian is $\mathrm{Hom}(\Lambda, V/\Lambda)$. A general point of $T_\Lambda \mathrm{Gr}_3 V$ corresponds to an isomorphism $\Lambda \xrightarrow{\sim} V/\Lambda$. The normal form in Theorem 3.7(2) is the point of $T_{e_{123}} \mathrm{Gr}_3 V$ corresponding to the isomorphism that sends e_i to $e_{i+3} \mod \langle e_1, e_2, e_3 \rangle$. It is the tangent vector at $t = 0$ to the curve

$$\Lambda(t) \;=\; e_1(t) \wedge e_2(t) \wedge e_3(t)\,, \tag{3.6}$$

where $e_i(t) = e_i + t e_{i+3}$ for $i = 1, 2, 3$. ◇

Remark 3.9. The tangent variety $\mathcal{T}X$ of a projective variety $X \subset \mathbb{P}^N$ is the union of all lines tangent to X. The orbits from Theorem 3.7 are described geometrically as follows.

(1) The orbit O_0 is the complement of the tangent variety $\mathcal{T}\mathrm{Gr}_3 V$ of $\mathrm{Gr}_3 V \subset \mathbb{P}\bigwedge^3 V$.
(2) Let $\mathcal{T}_1$ be the union of all lines in $\mathbb{P}\bigwedge^3 V$ connecting two points in $\mathrm{Gr}_3 V$ whose corresponding subspaces in V have nonzero intersection. Then

$$\mathrm{Gr}_3 V \;\subset\; \mathcal{T}_1 \;\subset\; \mathcal{T}\mathrm{Gr}_3 V\,,$$

and O_1 is the complement of $\mathcal{T}_1$ in $\mathcal{T}\mathrm{Gr}_3 V$.
(3) The orbit O_5 is the complement of $\mathrm{Gr}_3 V$ in $\mathcal{T}_1$.
(4) The orbit O_{10} is $\mathrm{Gr}_3 V$. ◇

We describe $\mathcal{S}_\omega$ for $\omega \in \mathbb{P}\bigwedge^3 V \smallsetminus \mathrm{Gr}_3 V$.

Proposition 3.10. *Let* $\omega \in \mathbb{P}\bigwedge^3 V \smallsetminus \mathrm{Gr}_3 V$*. Then*

(1) *If* $\omega \in O_0$*, then* $\mathcal{S}_\omega$ *is finite and* $\dim \mathcal{S}_\omega = 0$
(2) *If* $\omega \in O_1$*, then* π_ω *is injective, so that* $\mathcal{S}_\omega = \emptyset$*.*
(3) *If* $\omega \in O_5$*, then* $\dim \mathcal{S}_\omega = 4$*.*

Proof. By Lemma 3.3, $\Lambda \in \mathcal{S}_\omega$ if and only if there is a $\Lambda' \in \mathrm{Gr}_3 V$ with $\Lambda \neq \Lambda'$ such that $\omega \in \langle \Lambda, \Lambda' \rangle$, the line in $\mathbb{P}\bigwedge^2 V$ spanned by the Plücker vectors of Λ and Λ'.

Let $\Lambda \neq \Lambda'$ be distinct 3-planes in $\mathrm{Gr}_3 V$ and $\omega \in \langle \Lambda, \Lambda' \rangle \smallsetminus \mathrm{Gr}_3 V$. By Remark 3.9(2), $\omega \notin O_1$, which proves (2). We argue by the dimension of $\Lambda \cap \Lambda'$. If $\dim \Lambda \cap \Lambda' = 0$, then $\omega \in O_0$, by Theorem 3.7(1). Since $\dim \mathrm{Gr}_3 V = 9$ and $\dim \mathbb{P}\bigwedge^3 V = 19$, dimension-counting shows that for a point $\omega \in O_0$, $\mathcal{S}_\omega$ is zero-dimensional and hence finite, proving (1). If $\dim \Lambda \cap \Lambda' = 1$, then $\omega \in O_5$, by Theorem 3.7(3). Statement (3) is Lemma 3.14 below. If $\dim \Lambda \cap \Lambda' = 2$, then $\langle \Lambda, \Lambda' \rangle \subset \mathrm{Gr}_3 V$. □

An element $\omega \in \bigwedge^3 V$ defines two linear maps

$$\begin{array}{rccccc} \wedge\omega & : & V & \longrightarrow & \bigwedge^4 V \\ & & v & \longmapsto & v \wedge \omega \end{array} \qquad\qquad \begin{array}{rccccc} \lrcorner\,\omega & : & V^* & \longrightarrow & \bigwedge^2 V \\ & & v & \longmapsto & v \lrcorner\, \omega \end{array} .$$

Lemma 3.11. *If $\omega \in O_5$, then both $\wedge\omega$ and $\lrcorner\,\omega$ have one-dimensional kernels.*

Proof. Computations using the normal form of $\omega \in O_5$ given by Theorem 3.7(3) show that the kernel of $\wedge\omega_5$ is $\langle e_1\rangle$ and the kernel of $\lrcorner\,\omega_5$ is $\langle e_6^*\rangle$. ☐

For $\omega \in O_5$, write $\alpha_\omega \in \mathbb{P}V$ for the kernel of $\wedge\omega$ and $A_\omega \in \mathbb{P}V^*$ for the kernel of $\lrcorner\,\omega$. We regard α_ω as a 1-dimensional linear subspace of V and A_ω as a hyperplane in V.

Corollary 3.12. *Let $\omega \in O_5$. Then $\alpha_\omega \subset A_\omega$, A_ω is the smallest subspace W of V such that $\omega \in \bigwedge^3 W$, and if $\pi_\omega(\Lambda) = \pi_\omega(\Lambda')$ for $\Lambda \neq \Lambda' \in \mathrm{Gr}_3V$, then $\alpha_\omega = \Lambda \cap \Lambda'$ and $A_\omega = \langle\Lambda, \Lambda'\rangle$ (their span in V). Finally, there is an indecomposable 2-form $\sigma \in \bigwedge^2 A_\omega$ such that $\omega = \alpha_\omega \wedge \sigma$, with α_ω and σ well-defined up to scalars.*

Proof. By the normal form of Theorem 3.7(3) and the proof of Proposition 3.10, $\alpha_\omega = \Lambda\cap\Lambda'$, so that $\langle\Lambda, \Lambda'\rangle$ is a hyperplane in V. Since $\omega, \Lambda, \Lambda'$ are collinear in $\mathbb{P}\bigwedge^3 V$, $\omega \in \bigwedge^3 A_\omega$. For any four dimensional subspace W of V, $\bigwedge^3 W \subset \mathrm{Gr}_3V$, which shows the minimality of A_ω. The last statement follows from these identifications and Theorem 3.7(3). ☐

By Corollary 3.12, if $\omega \in O_5$, then $\omega \in \mathbb{C}\alpha_\omega \wedge \bigwedge^2 A_\omega \simeq \bigwedge^2(A_\omega/\alpha_\omega)$. Notice that $\Lambda \mapsto \Lambda/\alpha_\omega$ identifies the Schubert variety

$$\Omega_\omega := \{\Lambda \in \mathrm{Gr}_3V \mid \alpha_\omega \in \Lambda \subset A_\omega\} \tag{3.7}$$

with $\mathrm{Gr}_2(A_\omega/\alpha_\omega) \simeq \mathrm{Gr}_2\mathbb{C}^4$.

Let $\mathrm{Fl}(1,5;V) \subset \mathbb{P}V \times \mathbb{P}V^*$ be the flag variety whose points are pairs (α, A) with $\alpha \subset A$; the one-dimensional linear subspace α lies in the hyperplane A. The projection of $\mathrm{Fl}(1,5;V)$ to each projective space factor is a $\mathbb{P}^4$ bundle. Let $L \to \mathrm{Fl}(1,5;V)$ be the subbundle of $\mathbb{P}\bigwedge^3 V \times \mathrm{Fl}(1,5;V)$ whose fiber over (α, A) is $\mathbb{P}(\alpha\wedge\bigwedge^2 A) \simeq \mathbb{P}^5$. The Schubert variety Ω_ω (3.7) depends only upon the flag $\alpha_\omega \subset A_\omega$ and it lies in $\mathbb{P}(\alpha_\omega \wedge \bigwedge^2 A_\omega)$. Write $\Omega(\alpha, A)$ for the Schubert variety corresponding to the flag $\alpha \subset A$. A consequence of this definition and Corollary 3.12 is the following.

Corollary 3.13. *For $\omega \in O_5$, the map $\omega \mapsto (\alpha_\omega, A_\omega) \in \mathrm{Fl}(1,5;V)$ realizes O_5 as a bundle over $\mathrm{Fl}(1,5;V)$, which is a dense open subset of L. The points in the fiber above (α, A) consist of points in $\mathbb{P}(\alpha \wedge \bigwedge^2 A)$ in the complement of $\Omega(\alpha, A)$.*

Lemma 3.14. *For $\omega \in O_5$, $\mathcal{S}_\omega$ is a dense subset of Ω_ω and therefore has dimension four.*

Proof. In the proof of Corollary 3.12, we observed that if $\Lambda \neq \Lambda'$ are 3-planes in Gr_3V with $\pi_\omega(\Lambda) = \pi_\omega(\Lambda')$, then $\alpha_\omega \subset \Lambda \subset A_\omega$. This implies that $\mathcal{S}_\omega \subset \Omega_\omega$.

Consider the restriction of π_ω to $\Omega_\omega \subset \mathrm{Gr}_3V$. Both ω and Ω_ω lie in $\mathbb{P}(\alpha_\omega \wedge \bigwedge^2 A_\omega)$, which is identified with $\mathbb{P}\bigwedge^2(A_\omega/\alpha_\omega)$. Write $\omega = \alpha_\omega \wedge \sigma$ with $\sigma \in \bigwedge^2(A_\omega/\alpha_\omega)$. Identifying Ω_ω with $\mathrm{Gr}_2(A_\omega/\alpha_\omega)$, the map π_ω on Ω_ω becomes π_σ, which has degree 2, by Remark 3.5. This completes the proof. ☐

Theorem 3.6 and the proof of Lemma 3.14 imply the following corollary.

Corollary 3.15. *Let $\omega = \alpha \wedge \sigma \in O_5$. If $\Lambda \neq \Lambda'$ are 3-planes then $\pi_\omega(\Lambda) = \pi_\omega(\Lambda')$ if and only if $\Lambda, \Lambda' \in \Omega_\omega$ and $\Lambda'/\alpha = \big(\Lambda/\alpha\big)^{\angle\sigma}$.*

3.3. **The center has dimension less than five.** By Proposition 3.10 and (3.5), if $Z \subset \mathbb{P}\bigwedge^3 V$ is a linear subspace that does not meet the Grassmannian Gr_3V, then

$$\mathcal{S}_Z \;=\; \bigcup_{\omega\in Z\cap O_0} \mathcal{S}_\omega \;\cup\; \bigcup_{\omega\in Z\cap O_5} \mathcal{S}_\omega\,, \tag{3.8}$$

and it follows that

$$\dim \mathcal{S}_Z \;\leq\; \max\{\dim(Z\cap O_0), \dim(Z\cap O_5)+4\}\,. \tag{3.9}$$

Since $\dim \mathrm{Gr}_3V = 9$, the last relation implies the following result.

Theorem 3.16. *If $Z \subset \mathbb{P}\bigwedge^3 V$ is a linear subspace that does not meet the Grassmannian Gr_3V, $\dim Z < 9$, and $\dim Z\cap O_5 \leq 4$, then π_Z has degree 1 on Gr_3V.*

Proof. From the assumptions and (3.9), we have that $\dim \mathcal{S}_Z \leq 8$. Thus $\mathrm{Gr}_3V \smallsetminus \mathcal{S}_Z$ contains a nonempty Zariski open set and therefore π_Z has degree 1. □

Corollary 3.17. *If Z does not meet the Grassmannian Gr_3V and $\dim Z \leq 4$, then π_Z has degree 1.*

3.4. **Five-dimensional center.** Let $Z \subset \mathbb{P}\bigwedge^3 V$ be a linear subspace such that the following three conditions hold,

(*i*) $\dim Z = 5$,
(*ii*) $\dim Z\cap O_5 \geq 5$, which together with (*i*) is equivalent to $\dim Z\cap O_5 = 5$, and
(*iii*) Z does not meet the Grassmannian Gr_3V, so that $Z \subset O_5$.

We establish the following result.

Theorem 3.18. *If $Z \subset \mathbb{P}\bigwedge^3 V$ is a linear subspace that does not meet the Grassmannian Gr_3V, $\dim Z = 5$, and the degree of π_Z exceeds 1, then Z is self-adjoint.*

The hypotheses imply that $Z \subset O_5$. We begin with a lemma about lines in O_5. For this, ω_i, σ_i, ρ_i, α_i, v_i, w_i for $i = 1,2$, and v are vectors and not points in projective space.

Lemma 3.19. *Assume that $\omega_1, \omega_2 \in O_5$ and the line they span lies in O_5. If $\omega_i = \alpha_i \wedge \sigma_i$ for $i = 1,2$ as in Corollary 3.12, then one of the following cases holds.*

1. *$\langle\alpha_1\rangle = \langle\alpha_2\rangle$.*
2. *α_1 and α_2 are linearly independent and $\langle\sigma_1\rangle \equiv \langle\sigma_2\rangle \mod \langle\alpha_1,\alpha_2\rangle$. There is a 2-form $\sigma \in \bigwedge^2 V$ such that, up to a scalar factor, $\omega_i = \alpha_i\wedge\sigma$ for $i = 1,2$.*
3. *There exist $v, w_1, w_2, v_1, v_2 \in V$ where $\alpha_1, \alpha_2, v, v_1, v_2$ are linearly independent with $\langle v, v_1, v_2\rangle = \langle v, v_1, w_1\rangle = \langle v, v_2, w_2\rangle$ such that*

$$\omega_1 \;=\; \alpha_1\wedge(\alpha_2\wedge w_1 + v\wedge v_1) \qquad \textit{and} \qquad \omega_2 \;=\; \alpha_2\wedge(\alpha_1\wedge w_2 + v\wedge v_2)\,.$$

Proof. Suppose that (1) does not hold, so that α_1 and α_2 are linearly independent. Let us suppose that $\alpha_1 = e_1$ and $\alpha_2 = e_2$. Let $U := \langle e_1, e_2\rangle$ and $W = \langle e_3,\dots,e_6\rangle \simeq \mathbb{C}^4$, which are transversal. We express σ_1, σ_2 in terms of e_2 and e_1 respectively. We have

$$\begin{aligned} \omega_1 &= e_1\wedge\sigma_1 = e_1\wedge(e_2\wedge w_1 + \rho_1)\,,\\ \omega_2 &= e_2\wedge\sigma_2 = e_2\wedge(e_1\wedge w_2 + \rho_2)\,, \end{aligned} \tag{3.10}$$

where $w_1, w_2 \in W$, and $\rho_1, \rho_2 \in \bigwedge^2 W$ are the terms in σ_1, σ_2 that do not contain e_2 and e_1 respectively. For $i = 1, 2$, since σ_i is indecomposable, neither ρ_i nor $w_i \wedge \rho_i$ is zero.

Let $\lambda, \mu \in \mathbb{C}$ be nonzero. Since $\lambda\omega_1 + \mu\omega_2 \in O_5$, it has the form $\alpha \wedge \sigma$, where $0 \neq \alpha \in V$ is defined up to a scalar by $\alpha \wedge (\lambda\omega_1 + \mu\omega_2) = 0$. Let us write $\alpha = ae_1 + be_2 + v$, where $v \in W$. The vector v and the coefficients a and b are functions of λ and μ, up to a common scalar, and at least one of a, b, and v is nonzero. We use (3.10) to rewrite $\alpha \wedge (\lambda\omega_1 + \mu\omega_2) = 0$ as

$$(ae_1 + be_2 + v) \wedge (\lambda e_{12} \wedge w_1 + \lambda e_1 \wedge \rho_1 - \mu e_{12} \wedge w_2 + \mu e_2 \wedge \rho_2) \;=\; 0\,.$$

Recall that $e_{12} = e_1 \wedge e_2$. Expanding gives

$$e_{12} \wedge (\mu a \rho_2 - \lambda b \rho_1 + v \wedge (\lambda w_1 - \mu w_2)) \;-\; \lambda e_1 \wedge v \wedge \rho_1 \;-\; \mu e_2 \wedge v \wedge \rho_2 \;=\; 0\,. \tag{3.11}$$

These summands lie in $e_{12} \wedge \bigwedge^2 W$, $e_1 \wedge \bigwedge^3 W$, and $e_2 \wedge \bigwedge^3 W$, respectively, and are therefore linearly independent. This gives the following three equations,

$$\mu a \rho_2 - \lambda b \rho_1 \;=\; v \wedge (\mu w_2 - \lambda w_1)\,, \tag{3.12}$$

$$v \wedge \rho_1 \;=\; 0\,, \qquad \text{and} \tag{3.13}$$

$$v \wedge \rho_2 \;=\; 0\,. \tag{3.14}$$

The last two are linear equations for $v \in W$. Note that each ρ_i is either decomposable (lies in $\mathrm{Gr}_2 W$) or indecomposable, corresponding to having rank 2 or rank 4. If either ρ_1 or ρ_2 is indecomposable and hence of rank 4, then $v = 0$ is the only solution.

Suppose first that $v = 0$ is a solution to (3.13) and (3.14). Then (3.12) implies that $\langle \rho_1 \rangle = \langle \rho_2 \rangle$. (We cannot have $ab = 0$, for then (3.12) and $(a, b) \neq (0, 0)$ implies that one of ρ_1 or ρ_2 is zero.) Scaling ω_1 and ω_2 if necessary, $\rho_1 = \rho_2 = \rho$, and using (3.10) we may set $\sigma = e_2 \wedge w_1 + e_1 \wedge w_2 + \rho$. Then Case (2) holds.

Suppose that (3.13) and (3.14) admit a nonzero solution, v. Thus ρ_1 and ρ_2 are each decomposable, and they have the form $\rho_i = v_i \wedge v$, for nonzero $v_1, v_2 \in W$. Then

$$\lambda\omega_1 + \mu\omega_2 \;=\; e_{12} \wedge (\lambda w_1 - \mu w_2) + (\lambda e_1 \wedge v_1 + \mu e_2 \wedge v_2) \wedge v. \tag{3.15}$$

This is indecomposable for $(\lambda, \mu) \neq (0, 0)$.

Suppose that $\langle \rho_1 \rangle = \langle \rho_2 \rangle$, which corresponds to a 2-plane $H \subset W$. Then (3.12) for all λ, μ implies that $w_1, w_2 \in H$. In particular, $\rho_1 = v' \wedge w_1$, for some $v' \in H$. But then $\sigma_1 = (e_1 + v') \wedge w_1$, which contradicts its being indecomposable.

Now suppose that ρ_1 and ρ_2 are linearly independent. If $H_i \in \mathrm{Gr}_2 W$ is the 2-plane corresponding to ρ_i, then $\langle v \rangle = H_1 \cap H_2$, and thus v is independent of λ, μ (up to a scalar), and we also see that v, v_1, v_2 are linearly independent. We establish Case (3) by showing that $\langle v, v_1, v_2 \rangle = \langle v, v_1, w_1 \rangle = \langle v, v_2, w_2 \rangle$.

Consider the 2-forms $\mu a \rho_2 - \lambda b \rho_1$ for all λ, μ. If these are all 0, then $a = b = 0$ as ρ_1 and ρ_2 are linearly independent. Then (3.12) implies that v, w_1, w_2 are proportional, which implies that σ_1 and σ_2 are decomposable, a contradiction.

Thus, for general λ, μ, the 2-form $\mu a \rho_2 - \lambda b \rho_1 \in \bigwedge^2 \langle v, v_1, v_2 \rangle$ is nonzero. By (3.12), for all λ, μ we have that $\mu w_2 - \lambda w_1 \in \langle v, v_1, v_2 \rangle$. Since σ_1 is indecomposable, w_1 is independent of v, v_1, and the same holds for w_2, v, v_2, which completes the proof. ☐

A line in O_5 has type (i) if it satisfies condition (i) of Lemma 3.19.

Corollary 3.20. *Let $\ell \subset O_5$ be a line. If ℓ has type* (1)*, then α_ω is the same point in $\mathbb{P}V$ for every $\omega \in \ell$. If ℓ has type* (3)*, then $A_\omega \in \mathbb{P}V^*$ is the same hyperplane for every $\omega \in \ell$.*

Proof. The claim about lines of type (1) follows from their definition and Lemma 3.19(1). Suppose ℓ has type (3). Recall that for $\omega \in O_5$, A_ω is the unique hyperplane of V with $\omega \in \bigwedge^3 A_\omega$. By the normal form for points on a line of type (3) from Lemma 3.19(3), we see that $A_\omega = \langle \alpha_1, \alpha_2, v, v_1, v_2\rangle$ for all $\omega \in \ell$. □

Now let us define

$$\begin{aligned} E_Z &:= \{\alpha_\omega \in \mathbb{P}V \mid \text{ for } \omega \in Z\}, \quad \text{and} \\ F_Z &:= \{A_\omega \in \mathbb{P}V^* \mid \text{ for } \omega \in Z\}. \end{aligned} \tag{3.16}$$

Lemma 3.21. *If Z is a linear subspace of $\mathbb{P}\bigwedge^3 V$ of dimension five with $Z \subset O_5$ such that the degree of π_Z exceeds* 1*, then $E_Z = \mathbb{P}V$ and $F_Z = \mathbb{P}V^*$.*

The proof we give uses the following fact about maps between projective spaces.

Proposition 3.22. *If $\phi\colon \mathbb{P}^r \to \mathbb{P}^r$ is a nonconstant map, then it is onto.*

Proof. Suppose that $\phi(\mathbb{P}^r) \neq \mathbb{P}^r$. Since the image is closed, we may compose ϕ with the linear projection from a point $x \notin \phi(\mathbb{P}^r)$, obtaining a map $\psi\colon \mathbb{P}^r \to \mathbb{P}^{r-1}$. This is given by r homogeneous forms $f_1, \dots, f_r$ of the same degree d with no common zeroes; for $z \in \mathbb{P}^r$, $\psi(z) = [f_1(z), \dots, f_r(z)]$. We must have $d > 0$, as ϕ and hence ψ is nonconstant. This contradicts $f_1, \dots, f_r$ having no common zeroes, as r forms of degree d define a subvariety in $\mathbb{P}^r$ of codimension at most r. □

Proof of Lemma 3.21. Recall the map $O_5 \to \mathrm{Fl}(1,5;V)$ that sends ω to the flag $\alpha_\omega \subset A_\omega$. Then E_Z is the image of Z under the further map to $\mathbb{P}V$ and F_Z is its image under the map to $\mathbb{P}V^*$. As Z, $\mathbb{P}V$, and $\mathbb{P}V^*$ are all projective spaces of dimension five, for each of $\mathbb{P}V$ and $\mathbb{P}V^*$, the image of Z is either a point, or the map is surjective.

By Corollary 3.12, if $\Lambda \in \mathcal{S}_\omega$ for $\omega \in O_5$, then $\alpha_\omega \subset \Lambda \subset A_\omega$. If E_Z is a point α, then $\mathcal{S}_Z \subset \{\Lambda \in \mathrm{Gr}_3 V \mid \alpha \subset \Lambda\}$, which is a proper subvariety of $\mathrm{Gr}_3 V$, and thus π_Z has degree 1. Similarly, if F_Z is a point, then π_Z has degree 1. □

We have another technical lemma.

Lemma 3.23. *Given $k{+}1$ linearly independent elements $\{\alpha_i\}_{i=1}^{k+1}$ in V, if $\rho \in \bigwedge^2 V$ satisfies*

$$\rho \equiv 0 \mod \langle \alpha_i, \alpha_{k+1}\rangle, \quad \forall i \in \{1, \dots, k\}, \tag{3.17}$$

then up to a nonzero constant,

$$\rho \equiv \begin{cases} \alpha_1 \wedge \alpha_2 \mod \alpha_{k+1} & k = 2, \\ 0 \mod \alpha_{k+1} & k > 2. \end{cases} \tag{3.18}$$

Proof. From (3.17) it follows that for any i there exist β_i, γ_i in V such that

$$\rho = \alpha_i \wedge \beta_i + \alpha_{k+1} \wedge \gamma_i .$$

Therefore for any $1 \le i \neq j \le k$

$$\alpha_i \wedge \beta_i - \alpha_j \wedge \beta_j + \alpha_{k+1} \wedge (\gamma_i - \gamma_j) = 0 . \tag{3.19}$$

Since $\alpha_i, \alpha_j, \alpha_{k+1}$ are linearly independent, by the classical Cartan lemma we have

$$\beta_i \;\in\; \langle \alpha_i, \alpha_j, \alpha_{k+1} \rangle \,. \tag{3.20}$$

If $k > 2$, then for any $i \in \{1, \ldots, k\}$, as there is more than one choice of $j \in \{1, \ldots, k\} \smallsetminus \{i\}$ in (3.20), we obtain that

$$\beta_i \;\in\; \langle \alpha_i, \alpha_{k+1} \rangle \,, \tag{3.21}$$

which implies that $\rho \equiv 0 \mod \alpha_{k+1}$.

If $k = 2$ then again by (3.20) and (3.19), we have that

$$\beta_1 \;=\; c\alpha_2 \mod \langle \alpha_1, \alpha_3 \rangle \,, \qquad \beta_2 \;=\; -c\alpha_1 \mod \langle \alpha_2, \alpha_3 \rangle \,,$$

for some constant c, which completes the proof. □

With these lemmas in place, we give the proof of Theorem 3.18.

Proof of Theorem 3.18. For this proof, $Z \subset \bigwedge^3 V$ is a linear subspace of dimension six and $\mathbb{P}Z$ is its image in $\mathbb{P}\bigwedge^3 V$. By (3.2), to show that Z is self-adjoint, we must produce a form $\sigma \in \bigwedge^2 V$ such that $Z = V \wedge \mathbb{C}\sigma$.

By Lemma 3.21, the maps from $\mathbb{P}Z$ to each of $\mathbb{P}V$ and $\mathbb{P}V^*$ are surjective. Thus we may choose a basis $\{\omega_i\}_{i=1}^6$ for Z whose images in each of $\mathbb{P}V$ and $\mathbb{P}V^*$ are linearly independent. For each $i = 1, \ldots, 6$, write $\omega_i = \alpha_i \wedge \sigma_i$, so that α_i is the image of ω_i in $\mathbb{P}V$ and let A_i be its image in $\mathbb{P}V^*$. Then $\{\alpha_i \mid i = 1, \ldots, 6\}$ form a basis for V and $\{A_i \mid i = 1, \ldots, 6\}$ form a basis for V^*. These vectors ω_i, σ_i, and α_i are only defined up to scalar multiples, so we may freely replace any by a scalar multiple.

By Corollary 3.20, no line $\langle \omega_i, \omega_j \rangle$ for $i \neq j$ has type (1) or (3), as α_i and α_j are independent and $A_i \neq A_j$. Therefore, they all have type (2). By Lemma 3.19(2), there exists $\sigma \in \bigwedge^2 V$ such that $\alpha_i \wedge \sigma_i = \alpha_i \wedge \sigma$ for $i = 1, 2$. Applying Lemma 3.19(2) to $\langle \omega_1, \omega_3 \rangle$ and to $\langle \omega_2, \omega_3 \rangle$, after replacing σ and σ_3 (and possibly $\alpha_1, \alpha_2, \alpha_3$) by scalar multiples,

$$\sigma - \sigma_3 \;\equiv\; 0 \mod \langle \alpha_i, \alpha_3 \rangle \,, \quad \text{for} \quad i = 1, 2 \,.$$

By Lemma 3.23 for $\rho = \sigma - \sigma_3$ and $k = 2$, we have

$$\sigma - \sigma_3 \;\equiv\; c\alpha_1 \wedge \alpha_2 \mod \langle \alpha_3 \rangle$$

for some constant c. Consequently, there exists $\beta \in V$ such that

$$\sigma - c\alpha_1 \wedge \alpha_2 \;=\; \sigma_3 + \alpha_3 \wedge \beta \,.$$

Setting $\widetilde{\sigma} := \sigma - c\alpha_1 \wedge \alpha_2$ we get

$$\alpha_i \wedge \sigma_i \;=\; \alpha_i \wedge \widetilde{\sigma} \,, \text{ for } \quad i = 1, 2, 3 \,. \tag{3.22}$$

Since the lines between $\omega_4 = \alpha_4 \wedge \sigma_4$ and ω_i for $i = 1, 2, 3$ have type (2), Lemma 3.19(2) implies that after multiplying by scalars, we have

$$\widetilde{\sigma} \;\equiv\; \sigma_4 \mod \langle \alpha_i, \alpha_4 \rangle \,, \text{ for } \quad i = 1, 2, 3 \,. \tag{3.23}$$

Then, by Lemma 3.23 with $\rho = \widetilde{\sigma} - \sigma_4$ and $k = 3$ we have

$$\widetilde{\sigma} \;\equiv\; \sigma_4 \mod \alpha_4 \,,$$

which implies that in addition to (3.22) we have $\alpha_4 \wedge \sigma_4 = \alpha_4 \wedge \widetilde{\sigma}$. The same arguments applied to α_5 and α_6 imply that for all $1 \leq i \leq 6$, we have $\omega_i = \alpha_i \wedge \widetilde{\sigma}$. As $\alpha_1, \ldots, \alpha_6$ form a basis for V, we have that $Z = V \wedge \mathbb{C}\widetilde{\sigma}$, which implies that it is self-adjoint, and completes the proof of Theorem 3.18. □

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