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THE CRITICAL POINT DEGREE OF A PERIODIC GRAPH

M. FAUST, J. ROBINSON, AND F. SOTTILE

Abstract. The critical point degree of a periodic graph operator is the number of critical points of its complex Bloch variety. Determining it is a step towards the spectral edges conjecture and more generally understanding Bloch varieties. Previous work showed that it is bounded above by the volume of the Newton polytope of the graph, and that the inequality is strict when there are asymptotic critical points. We identify contributions from asymptotic critical points that arise from the structure of the graph, and show that the critical point degree is bounded above by the difference of the volume of the Newton polytope and these contributions. These results have implications for nonlinear optimization.

Introduction

Operators on $\mathbb{Z}^d$-periodic graphs are an abstraction of physical models of crystals. The spectrum of such an operator is a union of intervals in $\mathbb{R}$, which Floquet theory reveals to be the image of the Bloch variety of the operator under a coordinate projection. The Bloch variety may be understood as an algebraic hypersurface in $(S^1)^d \times \mathbb{R}$ and this coordinate projection to $\mathbb{R}$ is a function λ whose critical values include the edges of the spectral intervals.

The complexification of the Bloch variety is an algebraic hypersurface in $(\mathbb{C}^\times)^d \times \mathbb{C}$ defined by the dispersion polynomial. The *critical point degree* of the operator is the number of complex critical points of the function λ on the Bloch variety, counted with multiplicity. Results of [18] show that the critical point degree is bounded above by the normalized volume $\mathrm{nvol}(A)$ of the Newton polytope A of the dispersion polynomial, and this bound is not attained if the Newton polytope has vertical faces or if the Bloch variety has asymptotic singularities. This is reviewed in Section 1.3.

We improve that bound. Corollary 4.4 identifies a contribution N_{vert} from vertical faces of A. This is asymptotic behavior as λ remains bounded, but the Floquet multipliers become unbounded. In Section 2 we identify a structure of the graph (disconnected initial graph) which implies that the Bloch variety has asymptotic singularities. Corollary 5.5 identifies their contribution, N_{disc}. For these, both λ and the Floquet multipliers become unbounded. Both contributions N_{vert} and N_{disc} arise from structural aspects of the underlying graph, which we study in Section 2. We state a simplified version of our main theorem.

Theorem 3.6. *For $d = 1$, 2, or 3, the critical point degree of a Bloch variety of an operator on a $\mathbb{Z}^d$-periodic graph is at most* $\mathrm{nvol}(A) - N_{\mathrm{vert}} - N_{\mathrm{disc}}$.

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This restriction to $d \leq 3$ is because the singular locus of a complex Bloch variety often has dimension at least $d-3$. (See Remark 1.4.)

Studying critical points of the function λ on the Bloch variety is a topic of current interest [3, 11, 18, 19, 37]. One motivation is the *spectral edges conjecture* from mathematical physics, which concerns the structure of the Bloch variety above the edges of the spectral bands. It posits that for a sufficiently general operator H, the local extrema of the function λ on the real Bloch variety are nondegenerate in that their Hessians are nondegenerate quadratic forms, and they occur in distinct bands. This is stated more precisely in [33, Conj. 5.25], and it also appears in [8, 22, 32, 38, 39]. Important notions, such as effective mass in solid state physics, the Liouville property, Green's function asymptotics, Anderson localization, homogenization, and many other assumed properties in physics, depend upon this conjecture. This conjecture and some recent progress is discussed in [11, 18, 19]. Determining the critical point degree for a general operator was important to those results.

This work is related to nonlinear optimization. The results in [11, 18] were among the first to show that well-known combinatorial bounds (BKK bound [5] for general sparse systems) for the number of solutions to certain polynomial optimization problems are attained, despite the corresponding systems of polynomials being far from general. This directly inspired [7] and subsequent works [34, 35, 36, 45], which have shown that this phenomenon is quite common. This is related to recent work [13, 28, 40] refining the BKK bound for structured polynomial systems. The structural understanding of the critical point degree that we derive here may indicate the next steps in this program.

Section 1 provides some background on periodic graph operators and their Bloch varieties. In Section 2 we study asymptotic properties of the dispersion polynomial and define initial graphs. Section 3 develops technical aspects of projective toric varieties, including local properties and singularities, and we prove Theorem 3.6, using results from the subsequent two sections. Section 4 studies the contribution N_{vert} from vertical faces. Finally, in Section 5 we study the contribution N_{disc} of asymptotic singularities that arise when an initial graph of Γ is disconnected. We end with Example 5.6 suggesting further refinements to our analysis of asymptotic critical points using the structure of the graph. This approach of understanding the asymptotic behavior of the Bloch variety/operator has long been used to study operators on periodic graphs [2, 14, 15, 16, 20, 21, 22, 27, 37].

We derive another result of interest. Theorem 2.15 describes restrictions of the dispersion polynomial to faces of its Newton polytope in terms of initial subgraphs of Γ. This refines [14, Lem. 4.2] and will be used in [17].

1. Algebraic Aspects of Discrete Periodic Operators

Let n and d be positive integers. Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{T}$ be, respectively, the integers, the real numbers, the complex numbers, and the unit complex numbers. Unless otherwise noted, a graph will be simple (no loops or multiple edges), undirected, and have bounded vertex degree. A more comprehensive development is found in [4, Ch. 4] and [41].

1.1. Discrete Periodic Operators.

A $\mathbb{Z}^d$-periodic graph Γ is a graph equipped with a free cocompact action of $\mathbb{Z}^d$. Cocompactness means there are finitely many orbits of vertices $\mathcal{V}(\Gamma)$

and of edges $\mathcal{E}(\Gamma)$. An edge between vertices u and v is written (u,v); we have $(u,v)=(v,u)$, as Γ is undirected, and when $(u,v)\in\mathcal{E}(\Gamma)$ we may write $u{\sim}v$. Figure 1 shows three $\mathbb{Z}^2$-periodic graphs. These have 2, 3, and 4 orbits of $\mathbb{Z}^2$ on $\mathcal{V}(\Gamma)$, respectively.

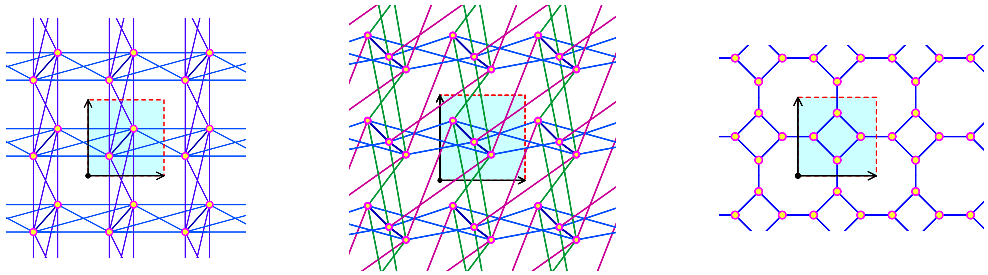


FIGURE 1. Three periodic graphs.

A periodic graph Γ is a discrete model of a crystal whose vertices represent atoms and edges represent interactions among atoms. Interaction strengths are modeled by a $\mathbb{Z}^d$-periodic function $e\colon\mathcal{E}(\Gamma)\to\mathbb{R}$ called edge weights, and we fix a $\mathbb{Z}^d$-periodic function $V\colon\mathcal{V}(\Gamma)\to\mathbb{R}$, a *potential*. The tight-binding model [1, 29] involves the discrete Schrödinger operator H acting on functions $f\colon\mathcal{V}(\Gamma)\to\mathbb{C}$. This is the sum of a multiplication operator and a weighted graph Laplacian. The function Hf has value at $v\in\mathcal{V}(\Gamma)$,

$$(Hf)(v)\ =\ V(v)\cdot f(v)\ +\ \sum_{v\sim u} e_{(v,u)}(f(v)-f(u))\,. \tag{1}$$

Let $\ell_2(\Gamma)$ be the Hilbert space of square summable functions $f\colon\mathcal{V}(\Gamma)\to\mathbb{C}$. Then H is a bounded self-adjoint operator on $\ell_2(\Gamma)$ whose spectrum $\sigma(H)$ is a finite union of closed and bounded intervals in $\mathbb{R}$, giving the familiar structure of energy bands and band gaps.

Remark 1.1. For $v\in\mathcal{V}(\Gamma)$ we absorb the sum $\sum_{v\sim u} e_{(v,u)}$ into $V(v)$. Then (1) becomes

$$(Hf)(v)\ =\ V(v)\cdot f(v)\ -\ \sum_{v\sim u} e_{(v,u)}f(u)\,. \tag{2}$$ ⋄

More structure of the spectrum is revealed by Floquet theory [4, Ch. 4], [30, Sect. 1.3], or [32]. As V and e are $\mathbb{Z}^d$-periodic, the operator H commutes with the $\mathbb{Z}^d$-action. The points $z=(z_1,\dots,z_d)\in\mathbb{T}^d$ are unitary characters of $\mathbb{Z}^d$ under the map $(z,\alpha)\mapsto z^\alpha:=z_1^{\alpha_1}\cdots z_d^{\alpha_d}\in\mathbb{T}$. Let us consider the Fourier transform of functions in $\ell_2(\Gamma)$. Let $L^2(\mathbb{T}^d)$ be the Hilbert space of square-integrable functions on $\mathbb{T}^d$. Writing $\alpha+v$ for the action of $\alpha\in\mathbb{Z}^d$ on a vertex $v\in\mathcal{V}(\Gamma)$, the Fourier transform of $f\colon\mathcal{V}(\Gamma)\to\mathbb{C}$ is a function $\hat{f}\colon\mathcal{V}(\Gamma)\to L^2(\mathbb{T}^d)$ that is quasi-periodic as follows. For $v\in\mathcal{V}(\Gamma)$ and $\alpha\in\mathbb{Z}^d$,

$$\hat{f}(\alpha+v)\ =\ z^\alpha\cdot\hat{f}(v)\ , \tag{3}$$

as functions of the *Floquet multipliers* $z=(z_1,\dots,z_d)\in\mathbb{T}^d$. Note that $z=e^{\sqrt{-1}\,k}$, where $k\in\mathbb{R}^d$ are quasimomenta.

By (3), a quasiperiodic function $\hat{f}$ is determined by its values at one vertex in each $\mathbb{Z}^d$-orbit on $\mathcal{V}(\Gamma)$. Let $W\subset\mathcal{V}(\Gamma)$ be such a choice of orbit representatives, called a *fundamental* *domain*. These are highlighted for the graphs in Figure 1. Then the Fourier transform is an isometry $\ell_2(\Gamma) \xrightarrow{\sim} L^2(\mathbb{T}^d)^W$, where $L^2(\mathbb{T}^d)^W$ is the space of functions from W to $L^2(\mathbb{T}^d)$.

As H commutes with the $\mathbb{Z}^d$-action on Γ, for $\hat{f}\colon W \to L^2(\mathbb{T}^d)$, the value of $H\hat{f}$ at a vertex $v \in W$ is (using (1) and (3)),

$$\big(H\hat{f}\big)(v) \;=\; V(v)\cdot \hat{f}(v) \;-\; \sum_{v\sim\alpha+u} e_{(v,\alpha+u)}\, z^\alpha \hat{f}(u)\ . \tag{4}$$

Example 1.2. Let Γ be the hexagonal lattice shown in Figure 2 with a labeling in a neighborhood of its fundamental domain. Thus, $W = \{u, v\}$ consists of two vertices and there are

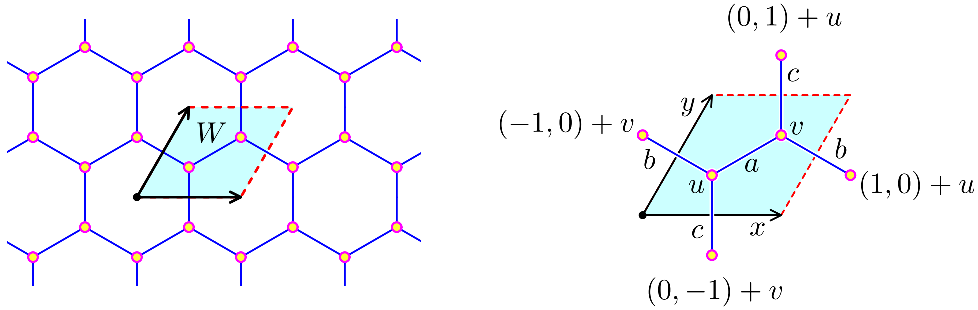


FIGURE 2. The hexagonal lattice and a labeling in a neighborhood of W.

three (orbits of) edges, with labels a, b, c. Let $(x, y) \in \mathbb{T}^2$. The operator H is

$$\begin{aligned}(H\hat{f})(u) &= V(u)\hat{f}(u) - a\hat{f}(v) - bx^{-1}\hat{f}(v) - cy^{-1}\hat{f}(v)\ ,\\ (H\hat{f})(v) &= V(v)\hat{f}(v) - a\hat{f}(u) - bx\hat{f}(u) - cy\hat{f}(u)\ .\end{aligned}$$

Collecting coefficients of $\hat{f}(u)$ and $\hat{f}(v)$, we represent H by the 2×2-matrix,

$$H \;=\; H(x,y) \;=\; \begin{pmatrix} V(u) & -a - bx^{-1} - cy^{-1} \\ -a - bx - cy & V(v)\end{pmatrix}\ , \tag{5}$$

whose entries are Laurent polynomials in x, y. For $(x, y) \in \mathbb{T}^2$, $(\overline{x}, \overline{y}) = (x^{-1}, y^{-1})$. Then $H^T = \overline{H}$, so that H is Hermitian. $\diamond$

The *Floquet matrix $H(z)$* is the $W \times W$ matrix of Laurent polynomials in z whose rows and columns are indexed by elements of W, and whose entry in row v and column u is

$$\delta_{v,u} V(v) \;-\; \sum_{v\sim\alpha+u} e_{(v,\alpha+u)}\, z^\alpha\ . \tag{6}$$

(*cf.* the right hand side of (4)). The operator H acts on $L^2(\mathbb{T}^d)^W$ via multiplication by the Floquet matrix. Write $c = (e, V)$ for the edge weights and potential, and $H_c(z)$ for the Floquet matrix when needed to indicate the parameters. As e is $\mathbb{Z}^d$-periodic and the edges are undirected, $e_{(v,\alpha+u)} = e_{(-\alpha+v,u)} = e_{(u,-\alpha+v)}$, which implies the symmetry, $H(z)^T = H(z^{-1})$.

Remark 1.3. The potential $V(v)$ only occurs in the diagonal entry in position (v, v) . The edge parameter $e_{(v,\alpha+u)} = e_{(u,-\alpha+v)}$ only occurs in positions (v, u) and (u, v) with the term $e_{(v,\alpha+u)}\, z^\alpha$ in position (v, u) and $e_{(v,\alpha+u)}\, z^{-\alpha}$ in position (u, v). $\diamond$

1.2. **Bloch and Fermi Varieties.** The *dispersion polynomial* $\Phi(z,\lambda) = \Phi_c(z,\lambda)$ is the characteristic polynomial of the Floquet matrix $H(z)$,

$$\Phi_c(z,\lambda) \; := \; \det\left(\lambda Id_W \; - \; H_c(z)\right) . \tag{7}$$

This is a monic polynomial in λ of degree $|W|$ whose coefficients are Laurent polynomials in z. The dispersion polynomial of the Floquet matrix $H(x,y)$ of (5) is

$$\begin{aligned}\lambda^2 \; - \lambda\big(V(u)+V(v)\big) + V(u)V(v) - (a^2+b^2+c^2)\\ - \; ab(x+x^{-1}) \; - \; ac(y+y^{-1}) \; - \; bc(xy^{-1}+yx^{-1}) \; .\end{aligned} \tag{8}$$

The *Bloch variety* $BV = BV_c \subset \mathbb{T}^d \times \mathbb{R}$ is defined by the dispersion polynomial

$$BV_c \; = \; Var(\Phi_c) \; := \; \{(z,\lambda) \in \mathbb{T}^d \times \mathbb{R} \mid \Phi_{(e,V)}(z,\lambda) \; = \; 0\} \, .$$

(We write *$Var(f)$* for the vanishing set of a function f.) Figure 3 shows three Bloch varieties for the hexagonal lattice from Figure 2. The parameters (a,b,c,u,v) of each are $(1,1,1,1,0)$,

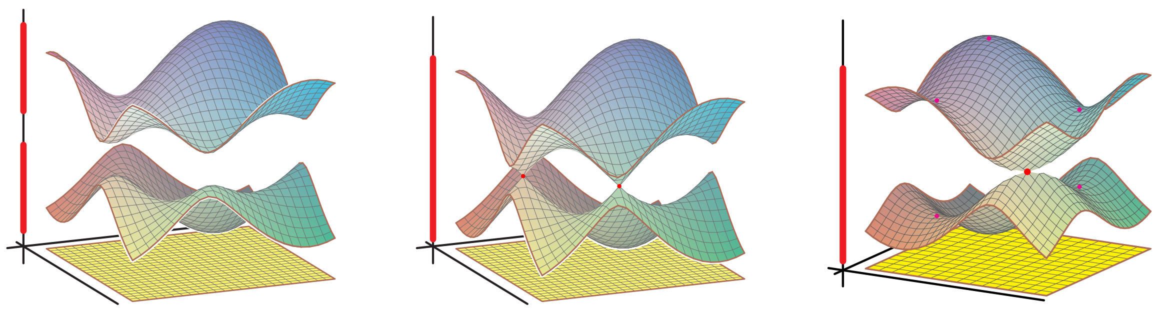


FIGURE 3. Three Bloch varieties for the hexagonal lattice

$(1,1,1,0,0)$, and $(5,3,2,0,0)$, respectively. These are displayed over a square representing the fundamental domain $[-\frac{\pi}{2},\frac{3\pi}{2}]^2$ of $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, and the spectra $\sigma(H)$ are the thickened intervals along the vertical axes. Each Bloch variety is a 2-sheeted cover of $\mathbb{T}^2$.

For a given $z \in \mathbb{T}^d$, the points λ such that $(z,\lambda) \in BV$ are the eigenvalues of the matrix $H(z)$. Since $z^{-1} = \overline{z}$ (the complex conjugate of z) and $H(z)^T = H(z^{-1})$, the Floquet matrix $H(z)$ is Hermitian. Consequently, it has $|W|$ real eigenvalues, which implies that the Bloch variety $BV \subset \mathbb{T}^d \times \mathbb{R}$ is a $|W|$-sheeted cover of $\mathbb{T}^d$ (counting multiplicities of eigenvalues). Figure 4 shows Bloch varieties corresponding to the graphs of Figure 1. The jth sheet is the graph of the jth *band function* $\lambda_j \colon \mathbb{T}^d \to \mathbb{R}$, whose image is a spectral band of H. Rather than treating these individually, we instead (see [11, 18, 41]) consider the restriction of the coordinate function λ to the Bloch variety, which is a function $\lambda\colon BV \to \mathbb{R}$. Then the spectrum of H is the image of the Bloch variety under this function λ. The spectra in Figure 4 are the thickened intervals along the vertical axes.

The endpoints of the intervals are the *spectral edges* of H. These are images of local extrema of the function λ on the Bloch variety. The *spectral edges conjecture* [33, Conj. 5.25] posits, among other things, that the Hessian of λ is a nondegenerate quadratic form at each extremum. This is implied by the critical points conjecture [18]: all critical points of

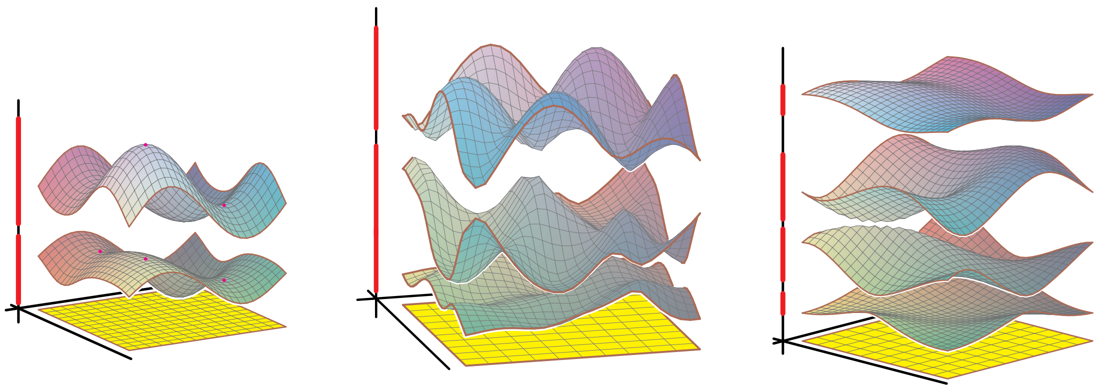


FIGURE 4. Three more Bloch varieties

λ are isolated and multiplicity 1 (and consequently nonsingular). By Lemma 5.1 of [18], a nonsingular critical point has a nondegenerate Hessian.

Remark 1.4. The singular locus of the determinant hypersurface has codimension 4 in $\mathrm{Mat}_{n\times n}$. Thus, we expect that the singular locus of the Bloch variety has dimension at least $d+1-4 = d-3$. When $d \geq 4$, this is positive, giving infinitely many critical points. As singular points are critical points, counting critical points often becomes moot when $d \geq 4$. The paper [19] concerns cases when $d > 3$ and yet the critical points are discrete. ◇

If we write the dispersion polynomial $\Phi(z, \lambda)$ (7) as a linear combination of monomials,

$$\Phi(z,\lambda) \;=\; \sum_{(\alpha,j)\in\mathbb{Z}^d\times\mathbb{N}} c_{(\alpha,j)} z^{\alpha}\lambda^{j}\ , \tag{9}$$

the set $\mathcal{A}(\Phi)$ of exponent vectors (α, j) with non-zero coefficients $c_{(\alpha,j)}$ is its *support*. For general parameters V, e, the origin $\mathbf{0}$ lies in $\mathcal{A}(\Phi)$ as the product of the potentials occurs in Φ. The dispersion polynomial for the hexagonal lattice has support the columns of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

The convex hull $\mathcal{N}(\Phi) = \mathrm{conv}(\mathcal{A}(\Phi))$ of the support is the *Newton polytope* of Φ. Figure 5 shows Newton polytopes for dispersion polynomials from the hexagonal lattice and the graphs of Figure 1. All are symmetric with respect to the vertical axis as $\Phi(z, \lambda) = \Phi(z^{-1}, \lambda)$, which follows from $H(z)^T = H(z^{-1})$.

1.3. **Critical Points of Discrete Periodic Operators.** As the Bloch variety is defined by a polynomial Φ with real coordinates, it is natural to consider its complexification. This is the set of points (z, λ) in $(\mathbb{C}^\times)^d \times \mathbb{C}$ such that $\Phi(z, \lambda) = 0$. This complexified Bloch variety is the Zariski closure of the real Bloch variety defined in Section 1.2.

A goal of [11, 18] was to give bounds for the number of critical points and use that to prove the critical points conjecture for some periodic graphs. Our goal is to refine those bounds. We begin with equations that define the set of critical points of λ on the Bloch variety.

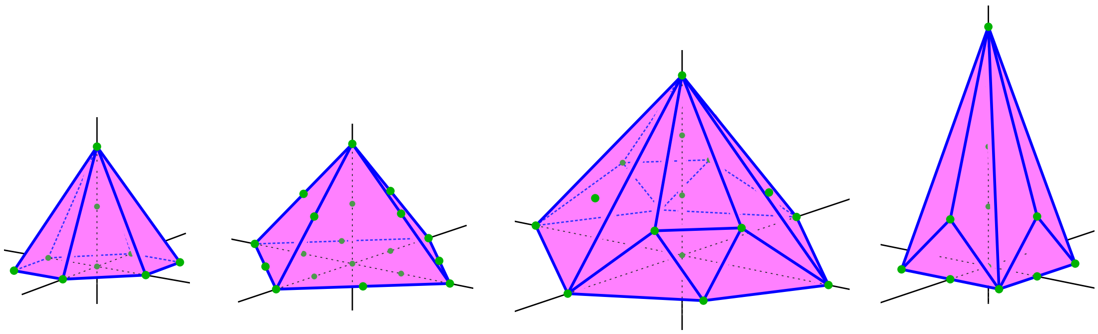


FIGURE 5. Some Newton polytopes

Proposition 1.5 (Prop. 2.1 of [18]). *A point $(z,\lambda)\in(\mathbb{C}^\times)^d\times\mathbb{C}$ is a critical point of the function λ on the Bloch variety if and only if it is a solution to the system of equations*

$$\Phi(z,\lambda) \;=\; z_1\frac{\partial\Phi(z,\lambda)}{\partial z_1} \;=\; z_2\frac{\partial\Phi(z,\lambda)}{\partial z_2} \;=\; \cdots \;=\; z_d\frac{\partial\Phi(z,\lambda)}{\partial z_d} \;=\; 0\,. \tag{10}$$

Write Ψ for this system of polynomial equations (10), called the *critical point equations*. The number of critical points (counted with multiplicity, see Proposition 3.3) on a Bloch variety is its *critical point degree*. This is 0 if there are infinitely many critical points. Standard arguments in algebraic geometry imply that there is a (Zariski) dense open subset U of the space of parameters for operators on a graph Γ with the following property: All operators with parameters from U have the same critical point degree, and this common degree is maximal among all operators on Γ. This maximum is called the *critical point degree of* Γ, written $\mathrm{cpdeg}(\Gamma)$. The following is a consequence of [11, Thm. 16] and [18, Lem. 5.1] (by the same arguments as [18, Thm. 5.2]).

Theorem 1.6. *If one operator on a periodic graph Γ has* $\mathrm{cpdeg}(\Gamma)$ *distinct critical points, then the spectral edges conjecture holds for Γ.*

There is a well-known lower bound for the critical point degree (when it is nonzero). As $\Phi(z,\lambda)=\Phi(z^{-1},\lambda)$, we have

$$\frac{\partial\Phi}{\partial z_i}(z,\lambda) \;=\; -\frac{1}{z_i^2}\frac{\partial\Phi}{\partial z_i}(z^{-1},\lambda) \qquad \text{for } i=1,\dots,d\,. \tag{11}$$

A fixed point of the involution $z\mapsto z^{-1}$ is a 2-torsion point ($z^2=1$) and is called a *corner point*. There are 2^d corner points, $\{\pm1\}^d$. By (11), if z is a corner point, then

$$\frac{\partial\Phi}{\partial z_i}(z,\lambda) \;=\; 0\,, \qquad \text{for } i=1,\dots,d\,.$$

This implies the following well-known fact.

Proposition 1.7. *Any point $(z,\lambda)\in BV$ with z a corner point is a critical point.*

Corollary 1.8. *Let Γ be a periodic graph with fundamental domain W. If an operator on Γ has finitely many critical points, then its critical point degree is at least $2^d|W|$.*

Proof. For a corner point z_0, there are m points (z_0,λ) on the Bloch variety, counted with multiplicity. This gives $2^d|W|$ such critical points on the Bloch variety. □

The main results in [18] concern bounds for the critical point degree. Given a polytope A of dimension n in $\mathbb{R}^n$ with vertices in the integer lattice $\mathbb{Z}^n$ (a lattice polytope), its *normalized volume* is $\mathrm{nvol}(A) := n!\mathrm{vol}(A)$, where $\mathrm{vol}(A)$ is the Euclidean volume of A, normalized so that a fundamental domain of $\mathbb{Z}^n$ has volume one. The normalized volume is always an integer.

Proposition 1.9 (Cor. 2.5 [18])**.** *The critical point degree of a discrete periodic operator with dispersion polynomial* Φ *is at most* $\mathrm{nvol}(\mathcal{N}(\Phi))$.

A second result of [18] gives conditions implying that the bound is attained. This is in terms of the asymptotic behavior of the Bloch variety. As we explain in Section 3, this asymptotic behavior is controlled by the Newton polytope $\mathcal{N}(\Phi)$. Dot product with an integer vector $\eta\in\mathbb{Z}^{d+1}$ is a linear form on $\mathbb{R}^{d+1}$. The subset of $\mathcal{N}(\Phi)$ on which this linear form is minimized is the face of $\mathcal{N}(\Phi)$ *exposed* by η. Every face of $\mathcal{N}(\Phi)$ is exposed by some vector $\eta\in\mathbb{Z}^{d+1}$. The dimension of the linear span of the vectors exposing a face F is its *codimension*, $\mathrm{codim}(F) := d{+}1-\dim(F)$. A face F of $\mathcal{N}(\Phi)$ has a normalized volume, $\mathrm{nvol}(F) = (\dim(F))!\,\mathrm{vol}(F)$, where $\mathrm{vol}(F)$ is normalized with respect to the intersection of $\mathbb{Z}^{d+1}$ with the affine span of F (this intersection is isomorphic to $\mathbb{Z}^{\dim(F)}$).

Restricting the sum (9) to the face F exposed by η gives the *initial form* of Φ,

$$\mathrm{in}_\eta\Phi \;:=\; \sum_{(\alpha,j)\in\mathcal{A}(\Phi)\cap F} c_{(\alpha,j)} z^\alpha\lambda^j\,.$$

This describes the asymptotic behavior of the Bloch variety in the logarithmic direction η, and is studied in Section 2.2. A vector $(\eta,a)\in\mathbb{Z}^{d+1}$ gives an action of $\mathbb{C}^\times$ on $(\mathbb{C}^\times)^d\times\mathbb{C}$,

$$t\in\mathbb{C}^\times\,,\ (z,\lambda)\in(\mathbb{C}^\times)^d\times\mathbb{C}\ \longmapsto\ t.(z,\lambda)\ :=\ (t^{\eta_1}z_1,t^{\eta_2}z_2,\dots,t^{\eta_d}z_d\,,\,t^a\lambda)\,.$$

We have the following result about initial forms.

Lemma 1.10. *Let* $(\eta,a)\in\mathbb{Z}^{d+1}$ *and* r *be the minimal value it takes on* $\mathcal{N}(\Phi)$*. We have the following.*

(i) $\mathrm{in}_{(\eta,a)}\Phi(t.(z,\lambda)) = t^r\,\mathrm{in}_{(\eta,a)}\Phi(z,\lambda)$.
(ii) $t^{-r}\Phi(t.(z,\lambda)) = \mathrm{in}_{(\eta,a)}\Phi(z,\lambda) +$ *higher order terms in* t.
(iii) $r\,\mathrm{in}_{(\eta,a)}\Phi \;=\; \sum_{i=1}^d \eta_i z_i\frac{\partial \mathrm{in}_{(\eta,a)}\Phi}{\partial z_i} \;+\; a\lambda\frac{\partial \mathrm{in}_{(\eta,a)}\Phi}{\partial\lambda}$.
(iv) *For each* $i=1,\dots,d$, $\mathrm{in}_{(\eta,a)}z_i\frac{\partial}{\partial z_i}\Phi = z_i\frac{\partial}{\partial z_i}\mathrm{in}_{(\eta,a)}\Phi$.

Proof. Statements (i) and (ii) follow from the calculation

$$(t.(z,\lambda))^{(\alpha,j)} \;=\; t^{\eta\cdot\alpha+aj}z^\alpha\lambda^j\,,$$

and (iii) holds as $z_i\frac{\partial}{\partial z_i}z^\alpha = \alpha_i z^\alpha$ and $\lambda\frac{\partial}{\partial\lambda}\lambda^j = j\lambda^j$. This implies that monomials are eigenvectors for the operators $z_i\frac{\partial}{\partial z_i}$ and $\lambda\frac{\partial}{\partial\lambda}$, which implies (iv). □

As monomials are eigenvectors for the operators $z_i \frac{\partial}{\partial z_i}$, each polynomial in (10) has support a subset of $\mathcal{N}(\Phi)$. Thus, we may replace each polynomial f in the critical point equations (10) by its initial form $\mathrm{in}_{(\eta,a)} f$, giving its *initial subsystem*, $\mathrm{in}_{(\eta,a)}\Psi$. By Lemma 1.10 (iv), the initial subsystem $\mathrm{in}_{(\eta,a)}\Psi$ is the system of critical point equations for the initial form $\mathrm{in}_{(\eta,a)}\Phi$ of Φ.

We distinguish three types of faces of $\mathcal{N}(\Phi)$. The *base* is the face exposed by the vector $(\mathbf{0}, 1)$ parallel to the last coordinate axis. The corresponding initial form of Φ is $\det H(z)$ and is obtained by setting $\lambda = 0$. A face F is *vertical* if it contains a vector parallel to $(\mathbf{0}, 1)$. Any face containing a vertical face is vertical. Vectors exposing a vertical face are horizontal, having the form $(\eta, 0)$. The fourth Newton polytope of Figure 5 has four vertical facets, each of which is a triangle of normalized volume (area) two. Lemma 1.10 implies the following.

Corollary 1.11. *Suppose that $(\eta, 0)$ exposes a vertical face F of $\mathcal{N}(\Phi)$. Then the initial subsystem $\mathrm{in}_{(\eta,0)}\Psi$ of the critical point equations* (10) *satisfies* $\mathrm{codim}(F)$*-many independent linear equations given by the vectors that expose F.*

Proof. Vectors exposing F have the form $(\zeta, 0)$, so that Lemma 1.10 (iii) becomes

$$r \cdot \mathrm{in}_{(\zeta,0)}\Phi \;=\; \sum_{i=1}^{d} \zeta_i z_i \frac{\partial \mathrm{in}_{(\zeta,0)}\Phi}{\partial z_i}\,, \tag{12}$$

a linear equation on $\mathrm{in}_{(\zeta,0)}\Psi$. □

An *oblique face* is one that is neither vertical nor the base. We collect some results about the critical point degree and initial subsystems of Ψ.

Proposition 1.12.

(i) *The bound on the critical point degree of Proposition 1.9 is attained if and only if for every vector η that is not parallel to $(\mathbf{0}, 1)$, the initial subsystem $\mathrm{in}_\eta\Psi$ of the critical point equations has no solutions.*

(ii) *If η exposes an oblique face, then any solution to the facial subsystem $\mathrm{in}_\eta\Psi$ is a singular point of the initial Bloch variety Var$(\mathrm{in}_\eta\Phi)$.*

(iii) *If η exposes a vertical face, then $\mathrm{in}_\eta\Psi$ has a solution.*

Statement (i) is [18, Thm. 3.8], while (ii) is [18, Lem. 3.7]. The third statement follows from Corollary 1.11 and Lemma 4.2. While statement (iii) was not stated in [18], its main results excluded Newton polytopes with vertical faces. Our main result, Theorem 3.6, is a refinement of Proposition 1.12. These results all involve studying the critical point equations on a compactification of the Bloch variety in a projective toric variety. The next two sections develop the algebraic combinatorics and algebraic geometry needed for this.

2. The Initial Graph of a periodic operator

We establish a purely combinatorial result about initial forms of a dispersion polynomial. This enables the definition of an initial graph; a fundamental concept for Sections 4 and 5. Part of this appeared in [14], and it is key to [17].

2.1. **Cancellation-free determinants.** Let $M = (f_{i,j})_{i,j=1}^n$ be a matrix of polynomials $f_{i,j}(y)$ in some (set of) indeterminates y. Recall the formula for the determinant of a matrix,

$$\det M \;=\; \sum_{w\in S_n} \mathrm{sgn}(w) f_{1,w(1)}(y)\, f_{2,w(2)}(y) \,\cdots\, f_{n,w(n)}(y) \;=\; \sum_{w\in S_n} \mathrm{sgn}(w) M_w \ .$$

This is the sum over all permutations $w \in S_n$, $\mathrm{sgn}(w) \in \{\pm 1\}$ is the sign of w, and M_w is the corresponding product of entries. We say that $\det M$ is *cancellation-free* if, for all $w \in S_n$ such that M_w is a non-zero polynomial, its support is a subset of the support of $\det M$.

The *generic Floquet matrix* $H(z, e, V)$ for a graph Γ is the Floquet matrix (6) where the edge parameters $e(u, v)$ and the potentials $V(v)$ are indeterminates. The *generic dispersion polynomial* for Γ is

$$\Phi(z, \lambda, e, V) \;:=\; \det(\lambda Id_n \;-\; H(z, e, V))\,,$$

where $H(z, e, V)$ is the generic Floquet matrix for Γ, and we identify W with $[n]$.

Proposition 2.1. *The generic dispersion polynomial is cancellation-free.*

To minimize discussion of signs, replace all potentials $V(v)$ by their negatives, so that every entry in the generic characteristic matrix $M(z, \lambda, e, V) := \lambda Id_n - H(z, e, V)$ is a positive sum of monomials in z, λ, e, V (*cf.* (6)). Then there is no cancellation in the expansion of any product of entries of $M(z, \lambda, e, V)$, nor in specializing z to be $\mathbb{1} := (1, \ldots, 1) \in \mathbb{T}^d$.

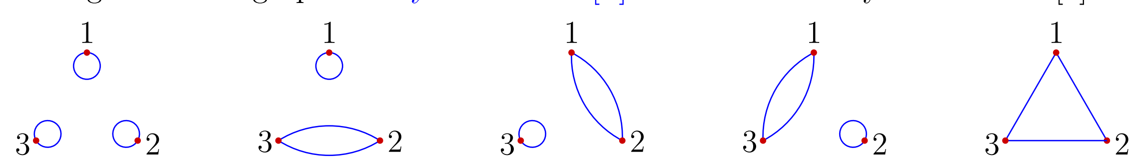
We show that the determinant of a generic symmetric matrix is cancellation-free, use that to show that $\det M(\mathbb{1}, \lambda, e, V)$ is cancellation-free, and finally deduce Proposition 2.1.

Suppose that $M = (x_{i,j})_{i,j=1}^n$ is a generic symmetric matrix in that the $x_{i,j}$ are indeterminates and $x_{i,j} = x_{j,i}$. The expression for its determinant

$$\det M \;=\; \sum_{w\in S_n} \mathrm{sgn}(w) x_{1,w(1)} x_{2,w(2)} \cdots x_{n,w(n)} = \sum_{w\in S_n} \mathrm{sgn}(w) M_w \ ,$$

may have repeated monomial terms. Collecting like terms gives a well-known, compact expression. We derive it for completeness.

For $w \in S_n$, let $|w|$ be the graph on $[n] = \{1, \ldots, n\}$ with one edge $i \sim w(i)$ for each $i \in [n]$. Every vertex of $|w|$ has degree two. Fixed points of w are loops and 2-cycles correspond to parallel edges. Such a graph is a *cycle cover of* $[n]$. There are five cycle covers of [3]:



Note that the triangle occurs twice, once for each 3-cycle in S_3.

Let c be a cycle cover of $[n]$ and let m_i be its number of components of size i; this is the number of i-cycles in w when $|w| = c$. Define $\mathrm{sgn}(c)$ to be $(-1)^m$, where m is the number of components of even size in c and set $n_c := 2^{m_3+m_4+\cdots}$. We also set M_c to be the product of $x_{i,j}$, where $i \sim j$ is an edge of c.

Proposition 2.2. *For any cycle cover c of $[n]$,*

$$n_c \;=\; \#\{w \in S_n \colon |w| = c\}\,. \tag{13}$$

For $w \in S_n$, $\operatorname{sgn}(w) = \operatorname{sgn}(|w|)$. If M is a symmetric matrix, then $M_w = M_{|w|}$, and

$$\det M \;=\; \sum_c \operatorname{sgn}(c)\, n_c\, M_c\,,$$

the sum over all cycle covers of $[n]$.

Proof. For a cycle $\zeta = (i_1, i_2, \ldots, i_r)$, set $x_\zeta := x_{i_1,i_2}x_{i_2,i_3}\cdots x_{i_r,i_1}$. As $x_{i,j} = x_{j,i}$, we have that $x_\zeta = x_{\zeta^{-1}}$. Every permutation $w \in S_n$ is uniquely a product of disjoint cycles

$$w \;=\; \zeta_1 \cdot \zeta_2 \cdots \zeta_k\,.$$

Note that $M_w = x_{\zeta_1}x_{\zeta_2}\cdots x_{\zeta_k}$. If $u = \zeta_1^{\pm}\zeta_2^{\pm}\cdots\zeta_k^{\pm}$ (any choice of $\pm$), then $|u| = |w|$, $\operatorname{sgn}(u) = \operatorname{sgn}(w)$, and $M_u = M_w$. Moreover, if $M_u = M_w$, then u has this form.

Since $\zeta = \zeta^{-1}$ if and only if the cycle has length less than 3, we deduce (13). The remaining statements follow from the observations that $\operatorname{sgn}(w)$ and M_w only depend upon $|w|$. ☐

Let $M(\mathbb{1}, \lambda, e, V)$ be the specialization of the generic characteristic matrix at the point $z = \mathbb{1}$. Note that M is symmetric. Specializing the entries of the generic symmetric matrix to those of $M(\mathbb{1}, \lambda, e, V)$, we have the formula of Proposition 2.2,

$$\det M(\mathbb{1}, \lambda, e, V) \;=\; \sum_c \operatorname{sgn}(c)\, n_c\, M_c\,, \tag{14}$$

the sum over all cycle covers of $[n]$. (Here, $M_c = M_c(\mathbb{1}, \lambda, e, V)$.)

Lemma 2.3. *For any cycle cover c, if $M_c \neq 0$, then any monomial in M_c determines c.*

Proof. Suppose that $w \in S_n$ has $|w| = c$, and the polynomial $M_c = M_w$ is nonzero. Writing $M(\mathbb{1}, \lambda, e, V) = (f_{i,j}(\lambda, e, V))_{i,j=1}^n$, we have

$$M_w \;=\; f_{1,w(1)}(\lambda, e, V)\, f_{2,w(2)}(\lambda, e, V)\, \cdots\, f_{n,w(n)}(\lambda, e, V)\,.$$

By Remark 1.3, each term in $f_{i,j}(\lambda, e, V)$ (except λ when $i = j$) determines the unordered pair $\{i, j\}$. Thus, each monomial in M_w determines all unordered pairs $\{i, w(i)\}$ when $i \neq w(i)$. But this determines $c = |w|$. ☐

Corollary 2.4. $\det M(\mathbb{1}, \lambda, e, V)$ *is cancellation-free.*

Proof. Every entry of $M(\mathbb{1}, \lambda, e, V)$ has positive coefficients, so there are no cancellations in the expansion of any non-zero product M_w. The result follows from the formula (14) and Proposition 2.2 as $M_w = M_{|w|}$. ☐

Proof of Proposition 2.1. Each term of $M(z, \lambda, e, V)_w$ has underlying monomial $\alpha(\lambda, e, V)z^{a_w}$ for some monomial $\alpha(\lambda, e, V)$ of $M(\mathbb{1}, \lambda, e, V)_w$ and exponent $a_w \in \mathbb{Z}^d$. By Lemma 2.3, the monomial $\alpha(\lambda, e, V)$ only occurs in $M(\mathbb{1}, \lambda, e, V)_u$ for $|u| = |w|$. Thus, the terms in $\det M(z, \lambda, e, V)$ involving the monomial $\alpha(\lambda, e, V)$ have the form

$$\operatorname{sgn}(w)\, \alpha(\lambda, e, V) \sum_{|u|=|w|} z^{a_u}\,. \tag{15}$$ ☐

By (15) something stronger holds. Any monomial in $M(z, \lambda, e, V)_w$ has the same sign in any other $M(z, \lambda, e, V)_u$, and these are the same sign as that monomial in $\det M(z, \lambda, e, V)$.

2.2. **Initial matrix.** We continue with the notation of Section 2.1. The *Newton polytope $\mathcal{N}(\Gamma)$ of the graph* Γ is the Newton polytope of the generic dispersion polynomial Φ, as a polynomial in z,λ. This was defined in [18, Sect. 4.1], where it was shown that the Newton polytope of the dispersion polynomial of any operator on Γ is a subset of $\mathcal{N}(\Gamma)$.

Let $\eta\in\mathbb{Z}^{d+1}$ and F be the face of $\mathcal{N}(\Gamma)$ that it exposes. For $w\in S_n$, let M_w be the corresponding term in $\det M$. Then *w contributes to* $\mathrm{in}_\eta\Phi$ if M_w has terms whose monomials lie along the face F. We define the *initial matrix* $\mathrm{in}_\eta M=\mathrm{in}_\eta M(z,\lambda,e,V)$ by its entries

$$\left(\mathrm{in}_\eta M\right)_{i,j} \;:=\; \begin{cases} \mathrm{in}_\eta(M_{i,j}) & \text{if } j=w(i) \text{ for some } w\in S_n \text{ contributing to } \mathrm{in}_\eta\Phi \\ \qquad 0 & \text{otherwise}\end{cases}\,. \tag{16}$$

Remark 2.5. Note that the initial matrix $\mathrm{in}_\eta M$ is *not* necessarily the matrix of η-initial forms of entries of M, but rather a submatrix of it. This is illustrated in Example 5.1. ◇

Lemma 2.6. *The initial matrix* $\mathrm{in}_\eta M$ *only depends on the face of* $\mathcal{N}(\Gamma)$ *exposed by* η*, and*

$$\det \mathrm{in}_\eta M(z,\lambda,e,V) \;=\; \mathrm{in}_\eta\Phi\,. \tag{17}$$

Proof. Let $\eta\in\mathbb{Z}^{d+1}$ expose the face F of $\mathcal{N}(\Phi)$ and let $N:=\mathrm{in}_\eta M(z,\lambda,e,V)$ be the initial matrix. We first prove (17) and then deduce its independence from η.

For a nonzero polynomial f in z,λ, let $\eta(f)$ be the minimum value that the linear form η takes on the exponents of f and $+\infty$ if $f=0$. Then $\mathrm{in}_\eta f$ is the sum of terms whose exponent vectors α minimize η, in that $\eta\cdot\alpha=\eta(f)$. Note how these behave under multiplication, $\eta(f\cdot g)=\eta(f)+\eta(g)$ and $\mathrm{in}_\eta(f\cdot g)=\mathrm{in}_\eta f\cdot\mathrm{in}_\eta g$. Consequently, for $w\in S_n$,

$$\mathrm{in}_\eta(M_w) \;=\; \prod_{i=1}^n \mathrm{in}_\eta(M_{i,w(i)}) \qquad\text{and}\qquad \eta(M_w) \;=\; \sum_{i=1}^n \eta(M_{i,w(i)})\,.$$

As $\det M=\Phi$ is cancellation-free and $\eta(\Phi)$ is the minimum of η on $\mathcal{N}(\Phi)$, and each entry $N_{i,j}$ of the initial matrix $N=\mathrm{in}_\eta M$ is either zero or is equal to $\mathrm{in}_\eta(M_{i,j})$, we have

$$\eta(\Phi) \;\le\; \eta(M_w) \;\le\; \eta(N_w)\,, \tag{18}$$

with equality if and only if w contributes to $\mathrm{in}_\eta\Phi$. In that case, observe that

$$\mathrm{in}_\eta(M_{i,w(i)}) \;=\; N_{i,w(i)} \;=\; \mathrm{in}_\eta(N_{i,w(i)}) \qquad \text{for } i\in[n]\,.$$

We prove (17) by showing that for $\tau\in S_n$ with $N_\tau\neq 0$, we have

$$\eta(\Phi) \;=\; \sum_{i=1}^n \eta(N_{i,\tau(i)}) \;=\; \eta(N_\tau)\,.$$

Suppose that $\tau\in S_n$ and $N_\tau\neq 0$. In particular, for each $i\in[n]$, $0\neq N_{i,\tau(i)}=\mathrm{in}_\eta(M_{i,\tau(i)})$. By the definition of $N=\mathrm{in}_\eta M$, for each $i=1,\dots,n$, there is a permutation w_i such that $w_i(i)=\tau(i)$ and w_i contributes to $\mathrm{in}_\eta\Phi$.

For each permutation $w\in S_n$, let Π_w be the corresponding permutation matrix and set

$$\Pi \;:=\; \sum_{i=1}^n \Pi_{w_i} \;-\; \Pi_\tau\,,$$

which is a matrix with nonnegative integer entries, and whose row and column sums are each $n-1$. By Birkhoff's Theorem [6] (a consequence of Hall's Theorem on matchings), there are permutations $\rho_1, \ldots, \rho_{n-1}$ such that

$$\Pi \;=\; \sum_{i=1}^{n-1} \Pi_{\rho_i}\,. \tag{19}$$

Consider the additive map $\widehat{\eta}$ on the set of nonnegative matrices $X = (x_{i,j})_{i,j=1}^{n}$ defined by

$$\widehat{\eta}\,:\; X \;\longmapsto\; \sum_{i,j} x_{i,j}\cdot\eta(N_{i,j}) \;\in\; \mathbb{R}_{\geq 0}\cup\{\infty\}\,.$$

Then $\widehat{\eta}(\Pi_w) = \eta(N_w)$ for any permutation $w \in S_n$. This, together with the definition of η, the inequality (18), and expressions for Π give

$$\begin{aligned} n\cdot\eta(\Phi) \;&\leq\; \eta(N_\tau) \;+\; \sum_{i=1}^{n-1}\eta(N_{\rho_i}) \;=\; \widehat{\eta}(\Pi_\tau) \;+\; \sum_{i=1}^{n-1}\widehat{\eta}(\Pi_{\rho_i}) \;=\; \widehat{\eta}(\Pi_\tau) \;+\; \widehat{\eta}\Big(\sum_{i=1}^{n-1}\Pi_{\rho_i}\Big) \\ &=\; \widehat{\eta}(\Pi_\tau+\Pi) \;=\; \widehat{\eta}\Big(\sum_{i=1}^{n}\Pi_{w_i}\Big) \;=\; \sum_{i=1}^{n}\widehat{\eta}(\Pi_{w_i}) \;=\; \sum_{i=1}^{n}\eta(N_{w_i}) \;=\; n\cdot\eta(\Phi)\,. \end{aligned}$$

This implies in particular that $\eta(\Phi)$ is equal to $\eta(N_\tau)$ and completes the proof of (17). By (17), $\det \mathrm{in}_\eta M(z,\lambda,e,V)$ depends on the face F of $\mathcal{N}(\Gamma)$ exposed by η. By Remark 2.5, the initial matrix $\mathrm{in}_\eta M(z,\lambda,e,V)$ also only depends on F.

Finally, note that the initial matrix $\mathrm{in}_\eta M$ depends only on the face exposed by η and is otherwise independent of $\eta\in\mathbb{Z}^{d+1}$. This follows from (16), as the only way $\mathrm{in}_\eta M$ and $\mathrm{in}_{\eta'} M$ could differ is if two nonzero entries differ, but this contradicts Corollary 2.4. □

2.3. **The initial graph.** The arguments of Sections 2.1 and 2.2, while ostensibly algebraic, have a purely combinatorial interpretation in terms of the graph Γ. Thus, $\mathrm{in}_\eta\Phi$ and ultimately the correction terms N_{vert} and N_{disc} of Theorem 3.6 are purely combinatorial and reflect structural properties of Γ.

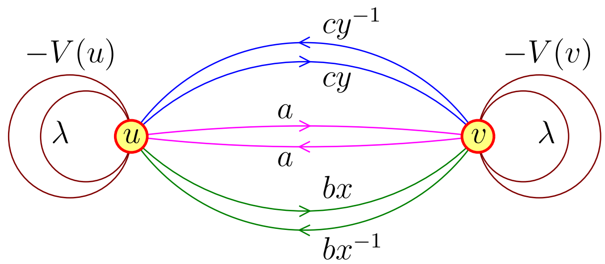
We construct a labeled, directed multigraph $\widehat{\Gamma}$ from Γ whose vertices are W and whose (weighted) adjacency matrix is $M(z,\lambda,e,V)$. The dispersion polynomial Φ is a cancellation-free sum of monomials indexed by *directed* cycle covers of $\widehat{\Gamma}$. For $\eta\in\mathbb{Z}^{d+1}$, we define a subgraph $\mathrm{in}_\eta\widehat{\Gamma}$ of $\widehat{\Gamma}$ whose adjacency matrix is $\mathrm{in}_\eta M$. Structural properties of this initial graph $\mathrm{in}_\eta\widehat{\Gamma}$ determine the correction terms N_{vert} and N_{disc} of Theorem 3.6.

Definition 2.7. Given a $\mathbb{Z}^d$-periodic labeled graph Γ with fundamental domain W, let $\widehat{\Gamma}$ be the graph with vertex set W, and labeled, directed edges as follows.

- (*i*) For $v\in W$, $\widehat{\Gamma}$ has two loops at v, one labeled λ and the other labeled $-V(v)$.
- (*ii*) For each $u,v\in W$ and $\alpha\in\mathbb{Z}^d$, if $v\sim\alpha{+}u$ is an edge of Γ, then there is a directed edge $v\leftarrow u$ in $\widehat{\Gamma}$ with label $e_{(v,\alpha+u)}z^\alpha$.

Observe that $\widehat{\Gamma}$ has loops and parallel edges. Each vertex $v\in W$ has degree $4+2d_v$ in $\widehat{\Gamma}$, where d_v is the degree of v in Γ. ◇

Example 2.8. For the hexagonal lattice Γ of Figure 2, $\widehat{\Gamma}$ is the following graph.



$\diamond$

Remark 2.9. The graph $\widehat{\Gamma}$ is independent of the choice of W. Replacing W by the set of $\mathbb{Z}^d$-orbits of vertices in Definition 2.7 gives the same graph, *mutatis mutandis*. In fact, $\widehat{\Gamma}$ is the quotient by $\mathbb{Z}^d$ of a directed version of Γ and Γ may be recovered from $\widehat{\Gamma}$ by removing the loops labeled λ, reversing signs, and then taking the underlying undirected graph of an appropriate cover in the sense of Sunada [44]. (We leave the details to the reader.) $\diamond$

Definition 2.10. The adjacency matrix of a finite, directed, labeled multigraph G with vertex set $\mathcal{V}$ is the $\mathcal{V}\times\mathcal{V}$ matrix Ad_G whose entry in position $(v,u)\in\mathcal{V}\times\mathcal{V}$ is

$$\big(\mathrm{Ad}_G\big)_{v,u} \;=\; \sum_{v\leftarrow u} e_{v\leftarrow u}\,,$$

where $e_{v\leftarrow u}$ is the label of the edge $v\leftarrow u$ in G. $\diamond$

The following observation is immediate from these definitions.

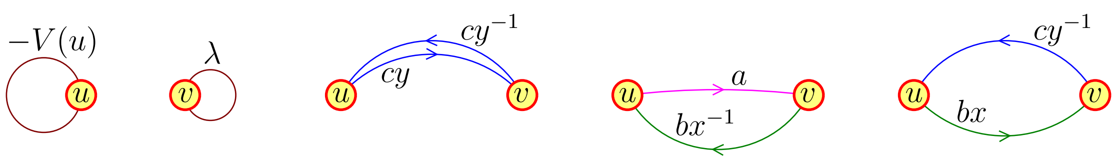
Proposition 2.11. *Let Γ be a $\mathbb{Z}^d$-periodic graph with Floquet matrix $H(z)$. Then*

$$\mathrm{Ad}_{\widehat{\Gamma}} \;=\; \lambda I_W \;-\; H(z)\,,$$

the characteristic matrix of Γ.

Thus, each labeled edge of $\widehat{\Gamma}$ corresponds to a monomial in an entry of the characteristic matrix $M(z,\lambda,e,V)$. A monomial term in the expansion of $\det(M(z,\lambda,e,V))$ corresponds to a permutation and a choice of monomials, one from each of the appropriate entries of $M(z,\lambda,e,V)$. This collection of edges corresponds to a (directed) cycle cover ζ of the graph $\widehat{\Gamma}$ and the monomial is the product $\mathrm{wt}(\zeta)$ of the labels of the directed edges of ζ.

Example 2.12. The graph $\widehat{\Gamma}$ of Example 2.8 has 13 cycle covers. We display four of them, together with their monomials.



$-\lambda V(u)$ $\qquad$ c^2 $\qquad$ abx^{-1} $\qquad$ $bcxy^{-1}$

These 13 cycle covers, with their monomials, correspond to the 13 terms in (8). $\diamond$

The following observation is immediate from these definitions.

Proposition 2.13. *Let Γ be a $\mathbb{Z}^d$-periodic graph with Floquet matrix $H(z)$. Then we have*

$$\Phi \;=\; \det(\lambda I_W - H(z)) \;=\; \det M(z,\lambda,e,V) \;=\; \sum \mathrm{wt}(\zeta)\ ,$$

the sum over all cycle covers ζ of $\widehat{\Gamma}$.

Let $\eta \in \mathbb{Z}^{d+1}$. For a cycle cover ζ of $\widehat{\Gamma}$, write $\eta(\zeta)$ for the integer $\eta(\mathrm{wt}(\zeta))$ and let $\eta(\widehat{\Gamma})$ be the minimum of $\eta(\zeta)$ over all cycle covers ζ of $\widehat{\Gamma}$.

Definition 2.14. For $\eta \in \mathbb{Z}^{d+1}$, the *initial graph* $\mathrm{in}_\eta\widehat{\Gamma}$ is a directed subgraph of $\widehat{\Gamma}$ with vertex set W. A labeled edge $v \leftarrow u$ of $\widehat{\Gamma}$ is a labeled edge of $\mathrm{in}_\eta\widehat{\Gamma}$ if and only if there is a cycle cover ζ of $\widehat{\Gamma}$ that contains the edge $v \leftarrow u$ and we have $\eta(\zeta) = \eta(\widehat{\Gamma})$. Every edge of ζ lies in $\mathrm{in}_\eta\widehat{\Gamma}$, so that ζ is a cycle cover of $\mathrm{in}_\eta\widehat{\Gamma}$ and every edge of $\mathrm{in}_\eta\widehat{\Gamma}$ occurs in some cycle cover. ⋄

A consequence of these definitions, Lemma 2.6, and Proposition 2.13 is the following.

Theorem 2.15. *Let Γ be a $\mathbb{Z}^d$-periodic graph with characteristic matrix $M(z,\lambda,e,V)$ and dispersion polynomial Φ. For any $\eta \in \mathbb{Z}^{d+1}$, we have*

$$\mathrm{in}_\eta M \;=\; \mathrm{Ad}_{\mathrm{in}_\eta\widehat{\Gamma}} \qquad \textit{and} \qquad \mathrm{in}_\eta \Phi \;=\; \sum \mathrm{wt}(\zeta)\,,$$

the sum over all cycle covers ζ of the initial graph $\mathrm{in}_\eta\widehat{\Gamma}$.

Vertical faces of $\mathcal{N}(\Phi)$ are detected by initial graphs.

Theorem 2.16. *A vector η exposes a vertical face of $\mathcal{N}(\Phi)$ if and only if the initial graph $\mathrm{in}_\eta\widehat{\Gamma}$ has a vertex v with loops labeled λ and $-V(v)$. Such a vector η is horizontal.*

Proof. For the forward implication, note that the only monomials in the matrix M whose exponents do not lie in $\mathbb{Z}^d \times \{0\}$ are the indeterminates λ of each diagonal entry. As η exposes a vertical face, it is horizontal. Consequently, if the face exposed by η is vertical, $\mathrm{in}_\eta\widehat{\Gamma}$ has a loop at some vertex v labeled λ. Let ζ be a cycle cover of $\mathrm{in}_\eta\widehat{\Gamma}$ containing this loop and let ζ' be the union of the other cycles in ζ. Let ζ'' be ζ' together with the loop at v labeled $-V(v)$; This is a cycle cover of $\widehat{\Gamma}$. Since η is horizontal, $\eta(\zeta) = \eta(\zeta') = \eta(\zeta'')$, which implies that the loop at v labeled $-V(v)$ also lies in $\mathrm{in}_\eta\widehat{\Gamma}$.

Now suppose that v is a vertex of $\mathrm{in}_\eta\widehat{\Gamma}$ with two loops labeled λ and $-V(v)$. The η is necessarily horizontal. By the same arguments as in the previous paragraph, then there are cycle covers ζ and ζ'' of $\mathrm{in}_\eta\widehat{\Gamma}$ that differ only in these two loops at v. Writing ζ' for the (common) rest of ζ, ζ'', we have

$$\mathrm{wt}(\zeta) + \mathrm{wt}(\zeta'') \;=\; (\lambda - V(v))\mathrm{wt}(\zeta')\,.$$

By the first factor, the face of $\mathcal{N}(\Phi)$ exposed by η contains the vertical vector $(\mathbf{0}, 1)$. □

Example 2.17. The periodic graph Γ on the left in Figure 6 has four (orbits of) edges and two vertices in its fundamental domain. Let u, v be the vertices and a, b, c, d the edge labels as indicated in the middle diagram in Figure 6.

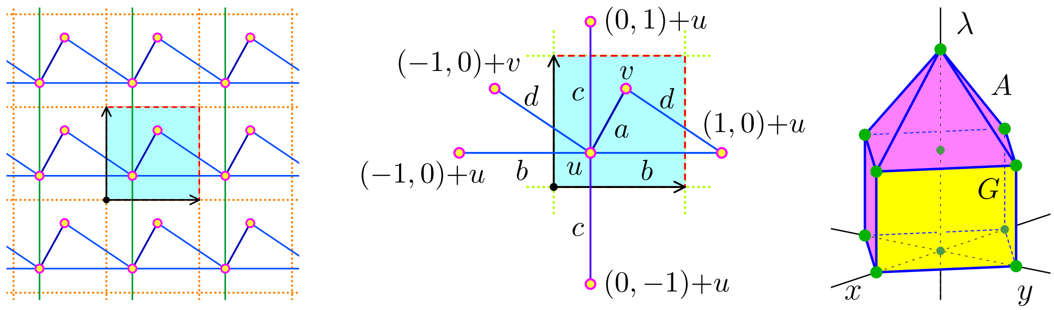


FIGURE 6. A $\mathbb{Z}^2$-periodic graph, its edge labels, and Newton polytope.

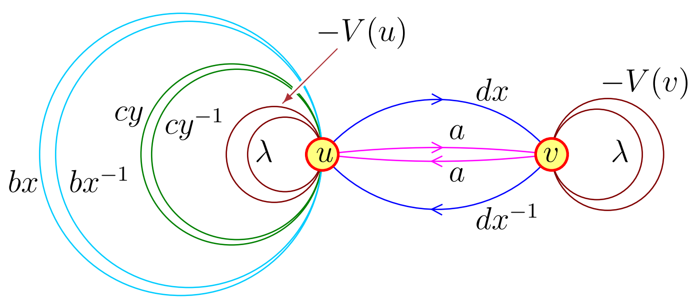
The characteristic matrix $M = \lambda I_2 - H(x,y)$ is

$$M \;:=\; \begin{pmatrix} \lambda - V(u) + \underline{bx} + bx^{-1} + \underline{cy} + cy^{-1} & \underline{a} + dx^{-1} \\ a + \underline{dx} & \underline{\lambda - V(v)} \end{pmatrix}.$$

The dispersion polynomial $\Phi(x,y,\lambda) = \det(\lambda I - H)$ is

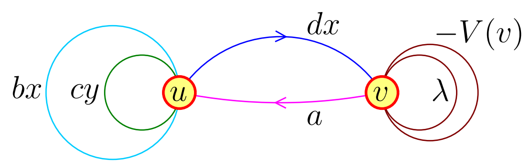
$$\begin{aligned} (20)\quad & \lambda^2 + \lambda(bx + cy - V(u) - V(v) + bx^{-1} + cy^{-1}) \\ & - bV(v)x - adx - cV(v)y - a^2 - d^2 + V(u)V(v) - (bV(v) + ad)x^{-1} - cV(v)y^{-1}\,. \end{aligned}$$

Its Newton polytope A is shown on the right in Figure 6, and here is the graph $\widehat{\Gamma}$.



The face G of A is exposed by $\eta = (-1,-1,0)$. We show $\mathrm{in}_\eta M$ and the initial graph $\mathrm{in}_\eta\widehat{\Gamma}$.

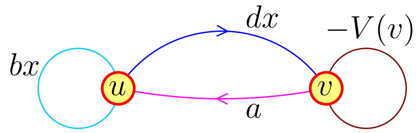
$$(21)\qquad \begin{pmatrix} bx + cy & a \\ dx & \lambda - V(v) \end{pmatrix}$$



Then $\mathrm{in}_\eta\Phi = (\lambda - V(v))(bx+cy) - adx = \mathrm{Ad}_{\mathrm{in}_\eta\widehat{\Gamma}}$, whose support is the face G of the polytope in Figure 6. $\diamond$

A connected component G of $\mathrm{in}_\eta\widehat{\Gamma}$ is *monomial* if every cycle cover of G gives the same monomial in (z,λ). Continuing Example 2.17, if $\eta = (-1,0,1)$, then $\mathrm{in}_\eta A = (1,0,0)$ is the vertex marked 'x'. We show both $\mathrm{in}_\eta M$ and $\mathrm{in}_\eta\widehat{\Gamma}$, which is monomial.

$$\begin{pmatrix} bx & a \\ dx & V(v) \end{pmatrix}$$



Definition 2.18. If $\mathrm{in}_\eta\widehat{\Gamma}$ has at least two connected components that are not monomial, then Γ is *asymptotically disconnected* in the direction η. As with $\mathrm{in}_\eta M$, this depends upon the face of $\mathcal{N}(\Phi)$ exposed by η. $\diamond$

We state an easy consequence of Definition 2.18

Lemma 2.19. *Suppose that Γ is asymptotically disconnected in the direction of η. Then $\mathrm{in}_\eta\Phi$ has a factorization of the form $\gamma g_1 \cdots g_r$, where γ arises from the monomial components of $\mathrm{in}_\eta\Gamma$ and each g_i from a non-monomial component.*

In Section 5, we explore the consequences of asymptotic disconnectedness and determine its contribution to the correction term N_{disc} of Theorem 3.6.

3. Toric varieties

The asymptotic behavior of an algebraic hypersurface $Var(\Phi)$ is reflected in the geometry of the Newton polytope $\mathcal{N}(\Phi)$ and encapsulated by a compactification of $Var(\Phi)$ in an associated projective toric variety. For critical points in Bloch varieties, the volume of the Newton polytope is an upper bound with correction terms arising from asymptotic critical points. We formulate and prove our main result, Theorem 3.6, based on results of Section 4 and 5.

We give local descriptions of the Bloch variety and critical point equations needed for those results. This includes a discussion of singular solutions in Theorem 3.14. A treatment of projective toric varieties is in [26, Ch. 5] with further results found in [9, Ch. 2], [24], [42, Ch. 3], and [43]. For projective space, [10, Ch. 8] suffices.

3.1. **Toric varieties and critical points.** To simplify notation, the indeterminates in this section will be $t_1, \ldots, t_n$. Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set of integer vectors, which are exponent vectors for monomials in t. We will use $\mathcal{A}$ as an index set. For example, $\mathbb{C}^{\mathcal{A}}$ is the set of functions from $\mathcal{A}$ to $\mathbb{C}$, a finite-dimensional $\mathbb{C}$-vector space. It has coordinates $(x_a \mid a \in \mathcal{A})$. Write $\mathbb{P}^{\mathcal{A}}$ for the corresponding projective space which has homogeneous coordinates $[x_a \mid a \in \mathcal{A}]$. This equals $(\mathbb{C}^{\mathcal{A}} \smallsetminus \{0\})/\Delta\mathbb{C}^\times$, the quotient of $\mathbb{C}^{\mathcal{A}} \smallsetminus \{0\}$ by the action of scalar multiplication given by the scalar matrices $\Delta\mathbb{C}^\times$. It is a compact complex manifold of dimension $|\mathcal{A}|-1$. The set of points $x \in \mathbb{P}^{\mathcal{A}}$ with no coordinate vanishing forms its *dense torus*, $(\mathbb{C}^\times)^{\mathcal{A}}/\Delta\mathbb{C}^\times \simeq (\mathbb{C}^\times)^{|\mathcal{A}|-1}$.

Let us consider the map $\varphi_{\mathcal{A}} \colon (\mathbb{C}^\times)^n \to \mathbb{P}^{\mathcal{A}}$ given by

$$\varphi_{\mathcal{A}} \;:\; t \;\longmapsto\; [t^a \mid a \in \mathcal{A}]\,. \tag{22}$$

We record a few facts about $\varphi_{\mathcal{A}}$ that follow from these definitions. Let $\mathbb{Z}\mathcal{A} \subset \mathbb{Z}^n$ be the free abelian subgroup generated by the differences $\{a - b \mid a, b \in \mathcal{A}\}$. Its rank equals the dimension of the affine span of $\mathcal{A}$. Its saturation $\mathrm{Sat}(\mathcal{A})$ is $\mathbb{R}\mathcal{A} \cap \mathbb{Z}^n$, the integer vectors in the $\mathbb{R}$-span of $\mathbb{Z}\mathcal{A}$, which is a free abelian group of the same rank as $\mathbb{Z}\mathcal{A}$.

Proposition 3.1. *The map $\varphi_{\mathcal{A}}$ is a group homomorphism from $(\mathbb{C}^\times)^n$ to the dense torus in $\mathbb{P}^{\mathcal{A}}$. Its image $\mathcal{O}_{\mathcal{A}}$ is a torus of dimension equal to the rank of $\mathbb{Z}\mathcal{A}$. Its kernel is*

$$\ker(\mathcal{A}) \;=\; \{t \in (\mathbb{C}^\times)^n \mid t^\alpha = 1 \ \ \textit{for all}\ \ \alpha \in \mathbb{Z}\mathcal{A}\}\,.$$

We have the exact sequence

$$1 \;\longrightarrow\; \ker(\mathcal{A})_0 \;\longrightarrow\; \ker(\mathcal{A}) \;\longrightarrow\; \ker(\mathcal{A})_{\mathrm{tor}} \;\longrightarrow\; 1\,,$$

where $\ker(\mathcal{A})_0$ *is the connected component of the identity in* $\ker(\mathcal{A})$*,*

$$\ker(\mathcal{A})_0 \;:=\; \{t \in (\mathbb{C}^\times)^n \mid t^\beta = 1 \ \ \textit{for all}\ \ \beta \in \mathrm{Sat}(\mathcal{A})\}\,,$$

which is isomorphic to $(\mathbb{C}^\times)^{n-\mathrm{rank}(\mathbb{Z}\mathcal{A})}$*. The component group* $\ker(\mathcal{A})_{\mathrm{tor}}$ *is*

$$\begin{aligned}\ker(\mathcal{A})_{\mathrm{tor}} \;&:=\; \{z \in (\mathbb{C}^\times)^n/\ker(\mathcal{A})_0 \mid z^\alpha = 1 \ \ \textit{for all}\ \ \alpha \in \mathbb{Z}\mathcal{A}\}\\ &\simeq\; \mathrm{Hom}(\mathrm{Sat}(\mathcal{A})/\mathbb{Z}\mathcal{A}\,,\ \mathbb{C}^\times)\,,\end{aligned}$$

which is a finite group of order $[\mathrm{Sat}(\mathcal{A})\colon \mathbb{Z}\mathcal{A}]$*. The induced map* $\varphi_{\mathcal{A}}\colon (\mathbb{C}^\times)^n/\ker(\mathcal{A})_0 \to \mathcal{O}_{\mathcal{A}}$ *is a covering space of degree* $[\mathrm{Sat}(\mathcal{A})\colon \mathbb{Z}\mathcal{A}]$.

Let us write $X_{\mathcal{A}}$ for the closure of the image of $\varphi_{\mathcal{A}}$. This is the *projective toric variety* associated to $\mathcal{A}$. A reason for these definitions is that they interchange nonlinearities of equations in the following sense. A linear combination of monomials,

$$f \;=\; \sum_{a\in\mathcal{A}} c_a t^a \qquad c_a \in \mathbb{C}\,,$$

is a polynomial with *support* $\mathcal{A}$. Define $\Lambda_f := \mathit{Var}(\sum c_a x_a)$, the hyperplane in $\mathbb{P}^{\mathcal{A}}$ whose defining linear form has coefficients from f. Then $\mathit{Var}(f) = \varphi_{\mathcal{A}}^{-1}(X_{\mathcal{A}} \cap \Lambda_f)$. The image under $\varphi_{\mathcal{A}}$ of the nonlinear hypersurface $\mathit{Var}(f)$ is a hyperplane section of $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^n)$.

Let us specialize these definitions to the objects in this paper. Let Φ be a dispersion polynomial for an operator on a periodic graph, let $\mathcal{A}$ be its support, and let $A := \mathcal{N}(\Phi) = \mathrm{conv}(\mathcal{A})$ be its Newton polytope. As $\mathbb{Z}^d$ acts cocompactly on Γ, $\mathrm{Sat}(\mathcal{A}) = \mathbb{Z}^{d+1}$ and $\mathbb{Z}\mathcal{A}$ has full rank $d{+}1$. (In fact they are equal in all examples that we have considered.)

We will write t for the indeterminates (z,λ) on $(\mathbb{C}^\times)^d \times \mathbb{C}$. As λ only appears with a positive exponent, the map $\varphi_{\mathcal{A}}$ (22) extends to $(\mathbb{C}^\times)^d \times \mathbb{C}$,

$$\varphi_{\mathcal{A}}\;:\; (\mathbb{C}^\times)^d \times \mathbb{C} \ni t \;\longmapsto\; [t^a \mid a \in \mathcal{A}] \in \mathbb{P}^{\mathcal{A}}\,. \tag{23}$$

We obtain the projective toric variety $X_{\mathcal{A}}$ as the closure of the image of $\varphi_{\mathcal{A}}$. Write $X^\circ_{\mathcal{A}}$ for the image $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \mathbb{C})$ and $\partial X_{\mathcal{A}} := X_{\mathcal{A}} \smallsetminus X^\circ_{\mathcal{A}}$ for its complement. Then

$$X_{\mathcal{A}} \;=\; X^\circ_{\mathcal{A}} \coprod \partial X_{\mathcal{A}}\,. \tag{24}$$

The *compactified Bloch variety* is $\overline{BV} := X_{\mathcal{A}} \cap \Lambda_\Phi$, where Λ_Φ is the hyperplane corresponding to the dispersion polynomial Φ. It is the projective closure of $\varphi_{\mathcal{A}}(BV)$. We have

$$\varphi_{\mathcal{A}}(BV) \;=\; \varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \mathbb{C})) \cap \Lambda_\Phi \;=\; \overline{BV} \cap X^\circ_{\mathcal{A}}\,.$$

Each polynomial $f \in \Psi$ in the critical point equations (10) has support a subset of $\mathcal{A}$, and therefore corresponds to a hyperplane Λ_f. Let L_Ψ be the intersection of these hyperplanes. Then the critical points of the Bloch variety are $\varphi^{-1}(X^\circ_{\mathcal{A}} \cap L_\Psi)$. We have the inequality,

$$\#\big(X^\circ_{\mathcal{A}} \cap L_\Psi\big) \ \le\ \#\big(X_{\mathcal{A}} \cap L_\Psi\big)\,, \tag{25}$$

where we count isolated points with their multiplicities.

Remark 3.2. We recall some basics about these intersection multiplicities. There are many sources; for example, this may be extracted from Sections 1.6 and 2.1 of [25] with full justifications found in the books [12, 23].

Let $Y \subset \mathbb{P}^n$ be a projective variety, and let $L \subset \mathbb{P}^n$ be a linear subspace of dimension $n-\dim(Y)$, the *codimension* of Y. We expect that $Y \cap L$ is zero-dimensional, in which case we call $Y \cap L$ a *linear section* of Y and say that the intersection is *proper*. Bertini's Theorem asserts that there is a Zariski-open subset in the Grassmannian of linear subspaces of $\mathbb{P}^n$ of dimension $n - \dim(Y)$ such that the intersection $Y \cap L$ is transverse (and thus also zero-dimensional). Moreover, the number of points in such a transverse linear section is independent of L and is called the *degree* of Y, $\deg(Y)$. This independence from L extends to all linear sections, if we count points with appropriate multiplicities.

Let $y \in Y \cap L$. The local *intersection multiplicity* $m(y; Y, L)$ is the dimension (as a complex vector space) of the local ring of the intersection $Y \cap L$ supported at the point y. This may be computed in any affine neighborhood of y using local equations for Y and L. It has the following dynamic definition. If $L(s)$ for $s \in [-\epsilon, \epsilon]$ is a continuous family of linear spaces with $\lim_{s\to 0} L(s) = L$ such that for $s \neq 0$, $Y \cap L(s)$ is transverse, then $m(y; Y, L)$ is the number of points in $Y \cap L(s)$ that converge to y as $s \to 0$. ⋄

We record the fundamental global property of these local multiplicities.

Proposition 3.3. *Suppose that $Y \cap L$ is a linear section of a projective variety Y. Then*

$$\deg(Y) \ =\ \sum_{y \in Y\cap L} m(y; Y, L)\,. \tag{26}$$

We use this to refine Proposition 1.12, and deduce a step towards Theorem 3.6.

Corollary 3.4. *Let Γ be a periodic graph, $\mathcal{A}$ the support of its generic dispersion polynomial, and $A = \mathrm{conv}(\mathcal{A})$ its Newton polytope. Suppose that $\Phi(z,\lambda)$ is the dispersion polynomial of a general operator on Γ with critical point equations Ψ. If the intersection $X_{\mathcal{A}} \cap L_\Psi$ of the toric variety $X_{\mathcal{A}}$ with the linear space L_Ψ of $\mathbb{P}^{\mathcal{A}}$ corresponding to Ψ is a linear section, then*

$$\mathrm{cpdeg}(\Gamma) \ =\ \mathrm{nvol}(A) \ -\ [\mathbb{Z}^{d+1} \colon \mathbb{Z}\mathcal{A}] \sum_{x \in \partial X_{\mathcal{A}} \cap L_\Psi} m(x; X_{\mathcal{A}}, L_\Psi)\ . \tag{27}$$

Proof. The decomposition $X_{\mathcal{A}} = X^\circ_{\mathcal{A}} \coprod \partial X_{\mathcal{A}}$ (24) and Proposition 3.3 imply that

$$\deg X_{\mathcal{A}} \ =\ \sum_{x \in X^\circ_{\mathcal{A}} \cap L_\Psi} m(x; X_{\mathcal{A}}, L_\Psi) \ +\ \sum_{x \in \partial X_{\mathcal{A}} \cap L_\Psi} m(x; X_{\mathcal{A}}, L_\Psi)\ .$$

By Kushnirenko's Theorem [31], we have $\mathrm{nvol}(A) = [\mathbb{Z}^{d+1} \colon \mathbb{Z}\mathcal{A}] \cdot \deg(X_{\mathcal{A}})$, and by Proposition 3.1, the map $\varphi_{\mathcal{A}} \colon (\mathbb{C}^\times)^d \times \mathbb{C} \to X^\circ_{\mathcal{A}}$ has degree $[\mathbb{Z}^{d+1} \colon \mathbb{Z}\mathcal{A}]$, so that

$$[\mathbb{Z}^{d+1} \colon \mathbb{Z}\mathcal{A}] \sum_{x \in X^\circ_{\mathcal{A}} \cap L_\Psi} m(x; X_{\mathcal{A}}, L_\Psi)$$

counts the critical points of the Bloch variety of $\Phi(z, \lambda)$. As $\Phi(z, \lambda)$ is general, this is the critical point degree of Γ. This implies the formula (27). $\square$

Remark 3.5. We study the structure of the toric variety $X_{\mathcal{A}}$ in the remainder of this section, including points in a linear section $\partial X_{\mathcal{A}} \cap L_\Psi$ and local multiplicities $m(x; X_{\mathcal{A}}, L_\Psi)$. The remaining two sections are devoted to understanding two types of contributions to the sum in (27). In Section 4, Corollary 4.4 gives a contribution N_{vert} from vertical faces of the Newton polytope A, and in Section 5, we identify a contribution N_{disc} coming from oblique faces whose initial graph is disconnected. By Theorem 2.16 and Lemma 2.19 the contributions arise from structural properties of the graph Γ. By Corollaries 4.4 and 5.5 we have the inequality,

$$\sum_{x \in \partial X_{\mathcal{A}} \cap L_\Psi} m(x; X_{\mathcal{A}}, L_\Psi) \ \geq\ N_{\mathrm{vert}} \ +\ N_{\mathrm{disc}}\,. \qquad \diamond$$

We deduce our main theorem from this.

Theorem 3.6. *Let Γ be a $\mathbb{Z}^d$-periodic graph with $d \leq 3$. Its critical point degree satisfies the inequality,*

$$\mathrm{cpdeg}(\Gamma) \ \leq\ \mathrm{nvol}(A) \ -\ [\mathbb{Z}^{d+1} \colon \mathbb{Z}\mathcal{A}](N_{\mathrm{vert}} \ +\ N_{\mathrm{disc}})\,.$$

The version stated in the Introduction is when $\mathbb{Z}\mathcal{A} = \mathbb{Z}^{d+1}$.

Proof. This is a consequence of Corollary 3.4 and the inequality of Remark 3.5. $\square$

3.2. **Structure of $X_{\mathcal{A}}$.** The inequality (25) is strict when $\partial X_{\mathcal{A}} \cap L_\Psi \neq \emptyset$. Points in $\partial X_{\mathcal{A}} \cap L_\Psi$ are *asymptotic critical points*. Proposition 1.12 concerns the existence of asymptotic critical points. Our results involve counting them with their multiplicities.

To study the asymptotic critical points, we will need local coordinates for $X_{\mathcal{A}}$ near them, expressions for the critical point equations in these local coordinates, and a determination of the multiplicities of points in $\partial X_{\mathcal{A}}$.

The torus $(\mathbb{C}^\times)^{d+1}$ acts on $\mathbb{P}^{\mathcal{A}}$ through the map $\varphi_{\mathcal{A}}$ (23). As $X_{\mathcal{A}}$ is the closure of an orbit, $(\mathbb{C}^\times)^{d+1}$ acts on $X_{\mathcal{A}}$. As it is a toric variety, $X_{\mathcal{A}}$ is a disjoint union of finitely many orbits of $(\mathbb{C}^\times)^{d+1}$. There is one orbit for each face F of the polytope A. Each orbit has an affine open neighborhood in $X_{\mathcal{A}}$ which is itself a toric variety, and which has a canonical ring of polynomial functions—its coordinate ring. Each such coordinate ring is an easily-described subalgebra of the ring $\mathbb{C}[t^\pm]$ of Laurent polynomials in t.

The face A corresponds to the orbit $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^{d+1})$, and $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \{0\})$ is the orbit for the base (*i.e.* $\lambda = 0$). These two orbits form the image $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \mathbb{C})$ of $\varphi_{\mathcal{A}}$, which is $X^\circ_{\mathcal{A}}$. The boundary $\partial X_{\mathcal{A}}$ is the union of the remaining orbits.

Let F be a proper face of A that is not its base. Write $\mathcal{F} := F \cap \mathcal{A}$ for the points of $\mathcal{A}$ that lie along F. As $A = \mathrm{conv}(\mathcal{A})$, we have $F = \mathrm{conv}(\mathcal{F})$, the convex hull of $\mathcal{F}$. We define two subsets of the projective space $\mathbb{P}^{\mathcal{A}}$ corresponding to F. Let $\mathbb{P}^{\mathcal{F}} \subset \mathbb{P}^{\mathcal{A}}$ be the subspace spanned by the coordinates indexed by $\mathcal{F}$ so that $\mathbb{P}^{\mathcal{F}} = \{x \in \mathbb{P}^{\mathcal{A}} \mid x_a = 0 \text{ for } a \notin \mathcal{F}\}$, and set $U_{\mathcal{F}} := \{x \in \mathbb{P}^{\mathcal{A}} \mid x_b \neq 0 \text{ for } b \in \mathcal{F}\}$. Then $\mathbb{P}^{\mathcal{F}}$ is a projective subspace of $\mathbb{P}^{\mathcal{A}}$ and $U_{\mathcal{F}}$ is an affine open subset of $\mathbb{P}^{\mathcal{A}}$.

The set $\mathcal{F}$ gives a map analogous to $\varphi_{\mathcal{A}}$,

$$\varphi_{\mathcal{F}} : (\mathbb{C}^\times)^{d+1} \ni t \longmapsto [t^a \mid a \in \mathcal{F}] \in \mathbb{P}^{\mathcal{F}} \subset \mathbb{P}^{\mathcal{A}}. \tag{28}$$

Write $\mathcal{O}_{\mathcal{F}}$ for its image $\varphi_{\mathcal{F}}((\mathbb{C}^\times)^{d+1})$, which is the orbit corresponding to the face $\mathcal{F}$. Its closure is the toric variety $X_{\mathcal{F}}$, which is a subvariety of $X_{\mathcal{A}}$. We summarize some consequences of these definitions and Proposition 3.1, which describe the large-scale structure of $X_{\mathcal{A}}$ as a variety with an action by $(\mathbb{C}^\times)^{d+1}$.

Definition-Proposition 3.7. *Let F be a non-base proper face of A. Define $V_{\mathcal{F}} := U_{\mathcal{F}} \cap X_{\mathcal{A}}$. Then $V_{\mathcal{F}}$ is an affine toric variety and an open neighborhood in $X_{\mathcal{A}}$ of the orbit $\mathcal{O}_{\mathcal{F}}$, and*

(i) $\mathbb{P}^{\mathcal{F}} \cap X_{\mathcal{A}} = X_{\mathcal{F}}$, $\mathcal{O}_{\mathcal{F}} = X_{\mathcal{F}} \cap V_{\mathcal{F}}$, and $\mathcal{O}_{\mathcal{F}} \simeq (\mathbb{C}^\times)^{\dim(F)}$.

(ii) The sets $\mathcal{O}_{\mathcal{F}}$, $X_{\mathcal{F}}$, and $V_{\mathcal{F}}$ are all $(\mathbb{C}^\times)^{d+1}$-stable subsets of $X_{\mathcal{A}}$ with $\mathcal{O}_{\mathcal{F}}$ an orbit.

(iii) $X_{\mathcal{A}}$ is the disjoint union of $X^\circ_{\mathcal{A}}$ and the orbits $\mathcal{O}_{\mathcal{F}}$ for F a non-base proper face of A.

By (iii), each asymptotic critical point lies on a unique orbit $\mathcal{O}_{\mathcal{F}}$, and may be studied in the neighborhood $V_{\mathcal{F}}$ of $\mathcal{O}_{\mathcal{F}}$. We describe the coordinate ring of $V_{\mathcal{F}}$ as a subalgebra of $\mathbb{C}[t^\pm]$, the coordinate ring of $(\mathbb{C}^\times)^{d+1}$. This will include the coordinate ring of $\mathcal{O}_{\mathcal{F}}$, and we explain what the critical point equations (10) become in both $V_{\mathcal{F}}$ and $\mathcal{O}_{\mathcal{F}}$.

The monomials in $\mathbb{C}[t^\pm]$ are identified with their exponent vectors—elements of $\mathbb{Z}^{d+1}$. Subalgebras R of $\mathbb{C}[t^\pm]$ generated by monomials correspond to affine monoids—subsets M of $\mathbb{Z}^{d+1}$ that contain 0 and are closed under addition. Here, $M = \{a \in \mathbb{Z}^{d+1} \mid t^a \in R\}$ and $R \simeq \mathbb{C}[M]$ is the monoid algebra. We describe the monoids for the coordinate rings of $V_{\mathcal{F}}$ and $\mathcal{O}_{\mathcal{F}}$ of Definition-Proposition 3.7. Define

$$M_{\mathcal{F}} := \mathbb{N}\{a - b \mid a \in \mathcal{A},\ b \in \mathcal{F}\}, \tag{29}$$

the affine monoid generated by the differences $a - b$ of integer vectors for $a \in \mathcal{A}$ and $b \in \mathcal{F}$. Let $\mathbb{Z}\mathcal{F} := M_{\mathcal{F}} \cap -M_{\mathcal{F}}$, which is the maximal subgroup of $M_{\mathcal{F}}$. It is also the integer span of differences of elements of $\mathcal{F}$,

$$\mathbb{Z}\mathcal{F} = \mathbb{N}\{a - b \mid a, b \in \mathcal{F}\} = \mathbb{Z}\{a - b \mid a, b \in \mathcal{F}\}.$$

Proposition 3.8. *The coordinate ring of $V_{\mathcal{F}}$ is the monoid algebra $\mathbb{C}[M_{\mathcal{F}}]$ and the coordinate ring of $\mathcal{O}_{\mathcal{F}}$ is $\mathbb{C}[\mathbb{Z}\mathcal{F}]$.*

Proof. On the affine open set $U_{\mathcal{F}}$ no coordinate x_b for $b \in \mathcal{F}$ vanishes, thus $\{x_a/x_b \mid a \in \mathcal{A} \text{ and } b \in \mathcal{F}\}$ are regular functions on $U_{\mathcal{F}}$, and they generate its coordinate ring. Restricting them to $V_{\mathcal{F}}$ gives $\{t^{a-b} \mid a \in \mathcal{A} \text{ and } b \in \mathcal{F}\}$, which generates $M_{\mathcal{F}}$, showing that $\mathbb{C}[M_{\mathcal{F}}]$ is the coordinate ring of $V_{\mathcal{F}}$.

Similarly, on the dense torus of $\mathbb{P}^{\mathcal{F}}$, $\{x_a/x_b \mid a, b \in \mathcal{F}\}$ are regular functions that generate its coordinate ring. Their restrictions $\{t^{a-b} \mid a, b \in \mathcal{F}\}$ to $\mathcal{O}_{\mathcal{F}}$ generate $\mathbb{Z}\mathcal{F}$, showing that $\mathbb{C}[\mathbb{Z}\mathcal{F}]$ is the coordinate ring of $\mathcal{O}_{\mathcal{F}}$. $\square$

Example 3.9. In Figure 7, we label some faces of the polytope A of Figure 6. Let E be the indicated face (the vertical edge containing $\mathcal{E} := \{(1,0,0),(1,0,1)\}$). Translating A by

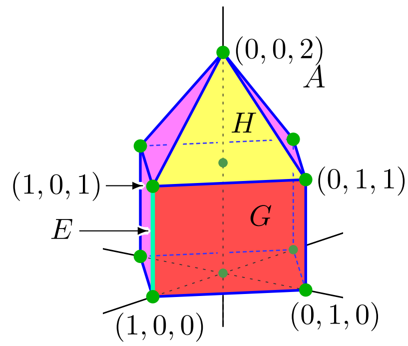


FIGURE 7. A polytope A with three indicated faces.

each endpoint of E moves E to a line segment along the vertical axis containing the origin, showing that $\mathbb{Z}\mathcal{E}$ is the set of integer points along the vertical axis. We also have that

$$M_{\mathcal{E}} \;=\; \{(x,y,\lambda)\in\mathbb{Z}^3 \mid -x-y\geq 0 \text{ and } -x+y\geq 0\}\,.$$

Note that $M_{\mathcal{E}} = \mathbb{Z}\mathcal{E}\oplus\tau_{\mathcal{E}}$, where

$$\tau_{\mathcal{E}} \;:=\; \{(x,y,0)\in\mathbb{Z}^3 \mid -x-y\geq 0 \text{ and } -x+y\geq 0\}\,. \tag{30}$$

Now consider G, the facet exposed by $(-1,-1,0)$ and set $\mathcal{G} := G\cap\mathcal{A}$. Then

$$M_{\mathcal{G}} \;=\; \{(x,y,\lambda)\mid -x-y\geq 0\} \qquad\text{and}\qquad \mathbb{Z}\mathcal{G} \;=\; \{(x,y,\lambda)\mid x+y=0\} \;\simeq\; \mathbb{Z}^2\,,$$

and we have $M_{\mathcal{G}} = \mathbb{Z}\mathcal{G}\oplus\mathbb{N}\cdot(-1,0,0)\simeq\mathbb{Z}^2\oplus\mathbb{N}$.

Similarly, the facet H is exposed by $(-1,-1,-1)$. Setting $\mathcal{H} := H\cap\mathcal{A}$, we have

$$M_{\mathcal{H}} \;=\; \{(x,y,\lambda)\mid -x-y-\lambda\geq 0\} \quad\text{and}\quad \mathbb{Z}\mathcal{H} \;=\; \{(x,y,\lambda)\mid x+y+\lambda=0\} \;\simeq\; \mathbb{Z}^2\,,$$

and we have $M_{\mathcal{H}} = \mathbb{Z}\mathcal{H}\oplus\mathbb{N}\cdot(0,0,-1)\simeq\mathbb{Z}^2\oplus\mathbb{N}$. $\diamond$

In all three cases considered in Example 3.9, we have that

$$M_{\mathcal{F}} \;=\; \mathbb{Z}\mathcal{F}\oplus\tau_{\mathcal{F}}\,.$$

A direct sum decomposition does not hold in general [9, § 2.2].

An *ideal* of a monoid M is a subset $I\subset M$ such that $M+I\subset I$. It is *prime* when its complement $I^c := M\smallsetminus I$ is a submonoid. If $I\subset M$ is prime, then $\langle I\rangle := \langle t^a \mid a\in I\rangle$ is a prime ideal of $\mathbb{C}[M]$ with $\mathbb{C}[M]/\langle I\rangle$ canonically isomorphic to the monoid algebra $\mathbb{C}[I^c]$.

Lemma 3.10. *For any face F of P, we have canonical maps of monoid algebras,*

$$\mathbb{C}[\mathbb{Z}\mathcal{F}] \xhookrightarrow{\;\iota\;} \mathbb{C}[M_{\mathcal{F}}] \xrightarrow{\;p\;} \mathbb{C}[\mathbb{Z}\mathcal{F}] \tag{31}$$

which induce canonical maps of affine toric varieties,

$$\mathcal{O}_{\mathcal{F}} \xhookrightarrow{\;p^*\;} V_{\mathcal{F}} \xrightarrow{\;\iota^*\;} \mathcal{O}_{\mathcal{F}}\,.$$

Proof. Let $I := M_{\mathcal{F}}\smallsetminus\mathbb{Z}\mathcal{F}$. To see that I is an ideal, suppose that there are $a\in I$ and $b\in M_{\mathcal{F}}$ such that $a+b\notin I$, so that $a+b\in\mathbb{Z}\mathcal{F}$. As $\mathbb{Z}\mathcal{F}$ is a group, $-(a+b)\in\mathbb{Z}\mathcal{F}$, and thus $-a = -(a+b)+b$ of $\mathbb{Z}^{d+1}$ lies in $M_{\mathcal{F}}$. But then $a\in M_{\mathcal{F}}\cap -M_{\mathcal{F}} = \mathbb{Z}\mathcal{F}$, a contradiction.

As $M_{\mathcal{F}} \smallsetminus I = \mathbb{Z}\mathcal{F}$ is a submonoid of $M_{\mathcal{F}}$, I is prime. The map ι of (31) is induced by the inclusion $\mathbb{Z}\mathcal{F} \subset M_{\mathcal{F}}$ and the map p is the quotient map by the ideal $\langle I \rangle$. □

Let f be a polynomial with support $\mathcal{A}$ and let F be a face of A exposed by η. Recall that the initial form $\mathrm{in}_\eta f$ of f is the sum of its terms whose exponent vectors lie in F.

Lemma 3.11. *Let f be a polynomial with support $\mathcal{A}$ and $B = \mathit{Var}(f) \subset (\mathbb{C}^\times)^{d+1}$ be the subvariety of the torus it defines. Write $\overline{B}$ for the closure of $\varphi(B)$ in the projective toric variety $X_{\mathcal{A}}$. Let F be face of A exposed by η and $b \in \mathcal{F}$ a monomial on F, we have*

(i) $t^{-b} f \in \mathbb{C}[M_{\mathcal{F}}]$ *and* $\mathit{Var}(t^{-b} f) \subset V_{\mathcal{F}}$ *equals* $\overline{B} \cap V_{\mathcal{F}}$.

(ii) $t^{-b}\mathrm{in}_\eta f = p(t^{-b} f) \in \mathbb{C}[\mathbb{Z}\mathcal{F}]$ *and* $\mathit{Var}(t^{-b}\mathrm{in}_\eta f) \subset \mathcal{O}_{\mathcal{F}}$ *equals* $\overline{B} \cap \mathcal{O}_{\mathcal{F}}$, *where* $p \colon \mathbb{C}[M_{\mathcal{F}}] \twoheadrightarrow \mathbb{C}[\mathbb{Z}\mathcal{F}]$ *is the map of* (31).

Proof. The principal ideal $\langle f \rangle$ generated by f in $\mathbb{C}[\mathbb{Z}^{d+1}]$ defines $B = \mathit{Var}(f)$ in $(\mathbb{C}^\times)^{d+1}$. Its closure $\overline{B} \cap V_{\mathcal{F}}$ in $V_{\mathcal{F}}$ is defined by $\langle f \rangle \cap \mathbb{C}[M_{\mathcal{F}}]$.

Let αt^c with $\alpha \neq 0$ and $c \in \mathcal{F}$ be a term of f. Then $t^{-c} f$ has a constant term α and $t^{-c} f \in \mathbb{C}[M_{\mathcal{F}}]$ by the definition of $M_{\mathcal{F}}$, as f has support $\mathcal{A}$ and $c \in \mathcal{F}$. If $a \in M_{\mathcal{F}}$, $t^a \in \mathbb{C}[M_{\mathcal{F}}]$ so that $t^a t^{-c} f \in \mathbb{C}[M_{\mathcal{F}}]$. If $t^a t^{-c} f \in \mathbb{C}[M_{\mathcal{F}}]$ for some $a \in \mathbb{Z}^{d+1}$, then αt^a is a term of $t^a t^{-c} f$ so that $a \in M_{\mathcal{F}}$. This shows that $\langle f \rangle \cap \mathbb{C}[M_{\mathcal{F}}]$ is the principal ideal of $\mathbb{C}[M_{\mathcal{F}}]$ generated by $t^{-c} f$.

If $b \in \mathcal{F}$, then $t^{-b} f = t^{c-b} t^{-c} f$, which generates the same ideal in $\mathbb{C}[M_{\mathcal{F}}]$ as $t^{-c} f$, as t^{b-c} is invertible in $\mathbb{C}[M_{\mathcal{F}}]$. This proves (i). For (ii), apply the map p to $t^{-b} f$. □

We relate this to the critical point equations. Recall that Φ is the dispersion polynomial (7) with support $\mathcal{A}$ and Newton polytope A and that Ψ is the critical point equations (10). For $b \in \mathbb{Z}^{d+1}$, let $t^{-b}\Psi := \{t^{-b} f \mid f \in \Psi\}$, and the same for any initial system $\mathrm{in}_\eta \Psi$.

Given a face F of A exposed by η, we call $\mathrm{in}_\eta BV := \overline{BV} \cap \mathcal{O}_{\mathcal{F}}$ the *initial Bloch variety*.

Corollary 3.12. *Let F be a face of A exposed by a vector η. The intersection of the compactified Bloch variety with $V_{\mathcal{F}}$ is defined by $t^{-b}\Phi$ for any $b \in \mathcal{F}$. Its component along the orbit $\mathcal{O}_{\mathcal{F}}$, the initial Bloch variety, is defined by $t^{-b}\mathrm{in}_\eta \Phi$. The set of critical points in $V_{\mathcal{F}}$ is defined by $t^{-b}\Psi$ for any $b \in \mathcal{F}$ and the asymptotic critical points lying along $\mathcal{O}_{\mathcal{F}}$ are defined by $t^{-b}\mathrm{in}_\eta \Psi$ for any $b \in \mathcal{F}$.*

Corollary 3.12 describes the equations for asymptotic critical points. We turn our attention to the multiplicities of such points.

Example 3.13. Consider affine toric varieties associated to faces from Example 3.9. Let E be the edge in Figure 7 and $\tau_{\mathcal{E}}$ the cone (30). Figure 8 shows $\tau_{\mathcal{E}}$ and the corresponding affine toric variety $Y_{\mathcal{E}}$, which is the closure of the image of $(\mathbb{C}^\times)^2$ under the map

$$(\mathbb{C}^\times)^2 \ \ni \ (x, y) \ \longmapsto \ (x^{-1}y\,,\ x^{-1}\,,\ x^{-1}y^{-1}) \ \in \ \mathbb{C}^3\,.$$

Writing (a, b, c) for the coordinates of $\mathbb{C}^3$, we have that $Y_{\mathcal{E}} = \mathit{Var}(ac - b^2)$, which is a quadratic cone in $\mathbb{C}^3$. Figure 8 shows the real points of $Y_{\mathcal{E}}$. This has an isolated singularity of multiplicity two at the origin. Indeed,

$$\big(\mathbb{C}[a, b, c]/\langle ac - b^2 \rangle\big)/\langle a, c \rangle \ \simeq \ \mathbb{C}[b]/\langle b^2 \rangle\,,$$

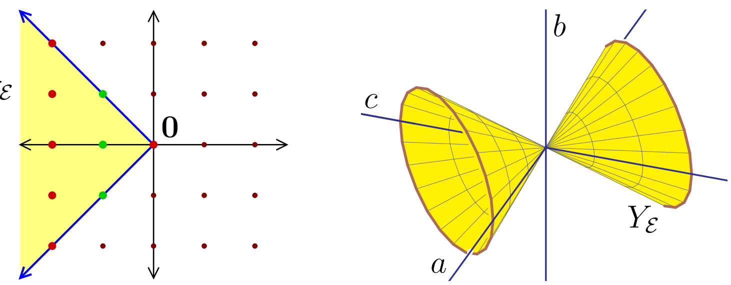


FIGURE 8. Cone $\tau_{\mathcal{E}}$ and associated affine toric variety $Y_{\mathcal{E}}$.

which has dimension 2 as a complex vector space. As $\mathbb{Z}\mathcal{F} \simeq \mathbb{Z}$, we have that $V_{\mathcal{E}} \simeq \mathbb{C}^{\times} \times Y_{\mathcal{E}}$ is singular along $\mathbb{C}^{\times} \times \{(0,0,0)\}$ of constant multiplicity two.

For the facets G and H, we have

$$\mathbb{Z}^3 \;=\; \mathbb{Z}(\mathcal{G}) \oplus \mathbb{Z}\cdot(-1,0,0) \;=\; \mathbb{Z}(\mathcal{H}) \oplus \mathbb{Z}\cdot(0,0,-1)\,,$$

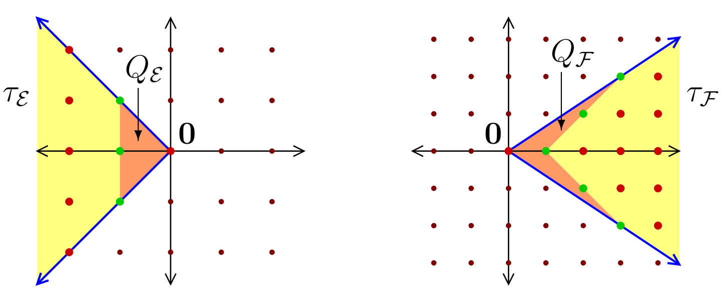
and thus $V_{\mathcal{G}} \simeq V_{\mathcal{H}} \simeq (\mathbb{C}^{\times})^2 \times \mathbb{C}$. Both affine toric varieties are smooth. $\diamond$

Let F be a face of A. Let $\mathrm{Sat}(\mathcal{F})$ be the saturation of $\mathbb{Z}\mathcal{F}$ in $\mathbb{Z}\mathcal{A}$. The composition of the inclusion $M_{\mathcal{F}} \subset \mathbb{Z}\mathcal{A}$ with the projection $\mathbb{Z}\mathcal{A} \twoheadrightarrow \mathbb{Z}\mathcal{A}/\mathrm{Sat}(\mathcal{F})$ induces a map of monoids,

$$M_{\mathcal{F}} \xrightarrow{\;\pi_{\mathcal{F}}\;} \mathbb{Z}\mathcal{A}/\mathrm{Sat}(\mathcal{F}) \;\simeq\; \mathbb{Z}^{d+1-\dim(F)} \;=\; \mathbb{Z}^{\mathrm{codim}(F)}\,.$$

Write $\tau_{\mathcal{F}} \subset \mathbb{Z}^{\mathrm{codim}(F)}$ for the image monoid, $\pi_{\mathcal{F}}(M_{\mathcal{F}})$. Let $\mathrm{conv}(\tau_{\mathcal{F}}) \subset \mathbb{R}^{\mathrm{codim}(F)}$ be the convex cone it generates and $\mathrm{conv}(\tau_{\mathcal{F}}^{\circ}) \subset \mathrm{conv}(\tau_{\mathcal{F}})$ be the convex hull of $\tau_{\mathcal{F}} \smallsetminus \{\mathbf{0}\}$. Finally, define $Q_{\mathcal{F}} := \mathrm{conv}(\tau_{\mathcal{F}}) \smallsetminus \mathrm{conv}(\tau_{\mathcal{F}}^{\circ})$ to be their difference, which is a non-convex lattice polytope. Define $\mu_{\mathcal{F}} := \mathrm{nvol}(Q_{\mathcal{F}})$.

Consider these for $\tau_{\mathcal{F}} = \mathbb{N}\{(1,0),(2,1),(3,2),(2,-1),(3,-2)\}$ and the cone $\tau_{\mathcal{E}}$ of Figure 8,



.

Thus, $\mu_{\mathcal{F}} = \mathrm{nvol}(Q_{\mathcal{F}}) = 4$, while $\mu_{\mathcal{E}} = \mathrm{nvol}(Q_{\mathcal{E}}) = 2$, the multiplicity of the origin in $Y_{\mathcal{E}}$. This quantity $\mu_{\mathcal{F}}$ is called a subdiagram volume and written $u(S/\Gamma)$ in [26, §5.3], where $S = M_{\mathcal{F}}$ and $\Gamma = \mathrm{Sat}(\mathcal{F})$.

Suppose that a linear section $X_{\mathcal{A}} \cap L$ of the projective toric variety $X_{\mathcal{A}}$ contains a point x that lies in an orbit $\mathcal{O}_{\mathcal{F}}$ in $\partial X_{\mathcal{A}}$. We determine the intersection multiplicity $m(x, X_{\mathcal{A}}, L)$ of the linear section at x when the intersection $X_{\mathcal{F}} \cap (\mathbb{P}^{\mathcal{F}} \cap L)$ in $\mathbb{P}^{\mathcal{F}}$ is also proper. Define $L_{\mathcal{F}} := \mathbb{P}^{\mathcal{F}} \cap L$. As $X_{\mathcal{F}} \cap L_{\mathcal{F}}$ is proper we have $\dim(L_{\mathcal{F}}) + \dim(X_{\mathcal{F}}) = \dim(\mathbb{P}^{\mathcal{F}})$.

Theorem 3.14. *Let L be a linear subspace of $\mathbb{P}^{\mathcal{A}}$ with $X_{\mathcal{A}} \cap L$ and $X_{\mathcal{F}} \cap L_{\mathcal{F}}$ proper. Then* (*i*) $\deg(X_{\mathcal{F}}) \;=\; \sum_{x\in X_{\mathcal{F}}\cap L} m(x;X_{\mathcal{F}},L_{\mathcal{F}})$, *and*

(*ii*) *for* $x \in \mathcal{O}_{\mathcal{F}} \cap L$,

$$m(x;X_{\mathcal{A}},L) \;\geq\; \mu_{\mathcal{F}}\cdot[\mathrm{Sat}(\mathcal{F}):\mathbb{Z}\mathcal{F}]\cdot m(x;X_{\mathcal{F}},L_{\mathcal{F}})\,,$$

with equality if L *is general given that* $\mathbb{P}^{\mathcal{F}}\cap L = L_{\mathcal{F}}$.

Proof. Statement (*i*) is Proposition 3.3 for $X_{\mathcal{F}}\cap L_{\mathcal{F}}$. For (*ii*), $\mu_{\mathcal{F}}\cdot[\mathrm{Sat}(\mathcal{F}):\mathbb{Z}\mathcal{F}]$ is the multiplicity of the toric variety $X_{\mathcal{A}}$ along points of the orbit $\mathcal{O}_{\mathcal{F}}$ [26, Ch. 5, Thm. 3.16]. This implies the result when $m(x;X_{\mathcal{F}},L_{\mathcal{F}}) = 1$. The result follows by perturbing $L_{\mathcal{F}}$ and the dynamic interpretation of intersection multiplicities from Remark 3.2. □

4. Vertical faces

When the Newton polytope A of a periodic graph has a vertical face F, there will be $\mathrm{nvol}(F)$ asymptotic critical points on $X_{\mathcal{F}}$. If F has a vertical face E, then at least $\mathrm{nvol}(E)$ of these will lie on $X_{\mathcal{E}}$. As observed in Example 3.13, $X_{\mathcal{A}}$ may be singular along $X_{\mathcal{E}}$, leading to additional contributions. Complicating this, E may lie on more than one vertical face. Before we do the accounting, consider the following example.

Example 4.1. Recall the graph Γ and polytope of Example 2.17 with dispersion polynomial (20). The polytope has normalized volume 16, and a symbolic computation with parameters shows that the Bloch variety has eight critical points, which is the lower bound of Corollary 1.8. These occur at the corner points $(\pm1,\pm1)$, for general values of the parameters. The corresponding critical values λ are

$$\pm b \pm c + \frac{V(u)+V(v)}{2} \;\pm\; \sqrt{(a+d)^2 \;+\; \left(\pm b\pm c+\tfrac{V(u)-V(v)}{2}\right)^2}\ ,$$

where the $\pm$ first coordinate of the corner point is the sign of b, and the second is the sign of c. Thus, $\mathrm{cpdeg}(\Gamma)=8$ and the vertical faces contribute eight to the count of 16 critical points.

We verify that. The Newton polytope has four vertical facets and four vertical edges. The vertical facet G exposed by $(-1,-1,0)$ has initial form

$$\lambda(bx+cy) \;+adx - V(v)(bx+cy) \;=\; x\big(\lambda(b+cyx^{-1}) \;+ad - V(v)(b+cyx^{-1})\big)\,,$$

which has no critical points on the orbit $\mathcal{O}_{\mathcal{G}} \simeq (\mathbb{C}^\times)^2$ corresponding to this facet. (The coordinates along $\mathcal{O}_{\mathcal{G}}$ are λ and yx^{-1}.) Thus, the critical points on $X_{\mathcal{G}}$ occur along its boundary. The same holds for the other vertical facets.

There are four vertical edges, each of length 1. Removing units, their facial forms are

$$\lambda b + ad - bV(v) \qquad\text{or}\qquad \lambda - V(v)\,.$$

Each has one solution on the orbit corresponding to its edge ($\mathbb{C}^\times$ with coordinate λ).

As each vertical facet G has two vertical edges, the toric subvariety $X_{\mathcal{G}}$ corresponding to G has two solutions to the critical point equations, one for each edge. This gives four solutions to the critical point equations along vertical faces, one-half of the expected eight.

To account for this discrepancy, recall that the toric variety $X_{\mathcal{A}}$ is singular of multiplicity $\mu_{\mathcal{E}} = 2$ along each orbit $\mathcal{O}_{\mathcal{E}}$ corresponding to a vertical edge E. Thus, the critical point equations have $8 = 4 \cdot 2$ solutions counted with multiplicity on $\partial X_{\mathcal{A}}$. ◇

Let $L \subset \mathbb{P}^{\mathcal{A}}$ be the linear space corresponding to the critical point equations (10) and assume that $X_{\mathcal{A}} \cap L$ is proper so that it consists of $\deg(X_{\mathcal{A}})$ points.

Lemma 4.2. *For each vertical face F of $A = \operatorname{conv}(\mathcal{A})$, the hypotheses of Theorem 3.14 hold and $X_{\mathcal{F}} \cap L$ consists of $\deg(X_{\mathcal{F}})$ points.*

Proof. Let $L_{\mathcal{F}} := \mathbb{P}^{\mathcal{F}} \cap L$, which is defined by the linear forms corresponding to members of the critical point equations and $\{x_a = 0 \mid a \in \mathcal{A} \smallsetminus \mathcal{F}\}$. By Corollary 1.11, those linear forms satisfy $\operatorname{codim}(F)$-many linear equations. Consequently, $L_{\mathcal{F}}$ has codimension in $\mathbb{P}^{\mathcal{F}}$ at most $d{+}1{-}\operatorname{codim}(F) = \dim(F)$. As this is the dimension of $X_{\mathcal{F}}$, $\emptyset \neq X_{\mathcal{F}} \cap L_{\mathcal{F}}$. Since

$$X_{\mathcal{F}} \cap L_{\mathcal{F}} \ \subset\ X_{\mathcal{F}} \cap L \ \subset\ X_{\mathcal{A}} \cap L\,,$$

$X_{\mathcal{F}} \cap L_{\mathcal{F}}$ has dimension zero and is therefore a proper intersection. □

Definition 4.3. If the only vertical faces of A are facets (see, e.g. Figure 9), then we set

$$N_{\text{vert}} \ :=\ \sum_{F \text{ vertical}} \mu_{\mathcal{F}} \cdot [\operatorname{Sat}(\mathcal{F}) : \mathbb{Z}\mathcal{F}] \cdot \deg(X_{\mathcal{F}})\ ,$$

which is combinatorial. For such a face F, $\mathcal{O}_{\mathcal{F}} \subsetneq X_{\mathcal{F}}$, and this is the only orbit corresponding to a vertical face in $X_{\mathcal{F}}$. Then Lemma 4.2 implies that

$$N_{\text{vert}} \ \geq\ \sum_{F \text{ vertical}} \ \sum_{x \in \mathcal{O}_{\mathcal{F}} \cap L} \mu_{\mathcal{F}} \cdot [\operatorname{Sat}(\mathcal{F}) : \mathbb{Z}\mathcal{F}] \cdot m(x; X_{\mathcal{F}}, L_{\mathcal{F}})\ . \tag{32}$$

If there exist vertical faces of A of codimension more than 1, then we set

$$N_{\text{vert}} \ :=\ \sum_{F \text{ vertical}} \ \sum_{x \in \mathcal{O}_{\mathcal{F}} \cap L} \mu_{\mathcal{F}} \cdot [\operatorname{Sat}(\mathcal{F}) : \mathbb{Z}\mathcal{F}] \cdot m(x; X_{\mathcal{F}}, L_{\mathcal{F}})\ .$$

Notice that inequality 32 holds in either case. ◇

Let $\partial_{\text{vert}} X_{\mathcal{A}}$ be the union of all toric subvarieties $X_{\mathcal{F}}$ for F a proper vertical face of A.

Corollary 4.4. *We have the following inequality,*

$$\sum_{x \in \partial_{\text{vert}} X_{\mathcal{A}}} m(x; X_{\mathcal{A}}, L) \ \geq\ N_{\text{vert}}\,.$$

Proof. Collecting the terms in the sum by the orbit they lie on gives

$$\sum_{x \in \partial_{\text{vert}} X_{\mathcal{A}}} m(x; X_{\mathcal{A}}, L) \ =\ \sum_{F \text{ vertical}} \ \sum_{x \in \mathcal{O}_{\mathcal{F}} \cap L} m(x; X_{\mathcal{A}}, L)\,.$$

The lemma follows from the inequality of Theorem 3.14(ii) and definition (4.3). □

Remark 4.5. The definition of N_{vert} is not entirely combinatorial, as it is challenging to determine the sum in (32). Nevertheless, this may be done in many cases, such as in Examples 4.1 and 5.6. ◇

5. Initial graph disconnected

In Section 2 we studied initial graphs and in Definition 2.18 we explained what it means for a periodic graph Γ to be asymptotically disconnected. Lemma 2.19 noted the consequence of this for the initial form $\mathrm{in}_\eta\Phi$ of the dispersion polynomial Φ. Here, we quantify how this contributes to the enumeration of asymptotic critical points. We begin with an example.

Example 5.1. The periodic graph on the left in Figure 9 has six (orbits of) edges and three ver-

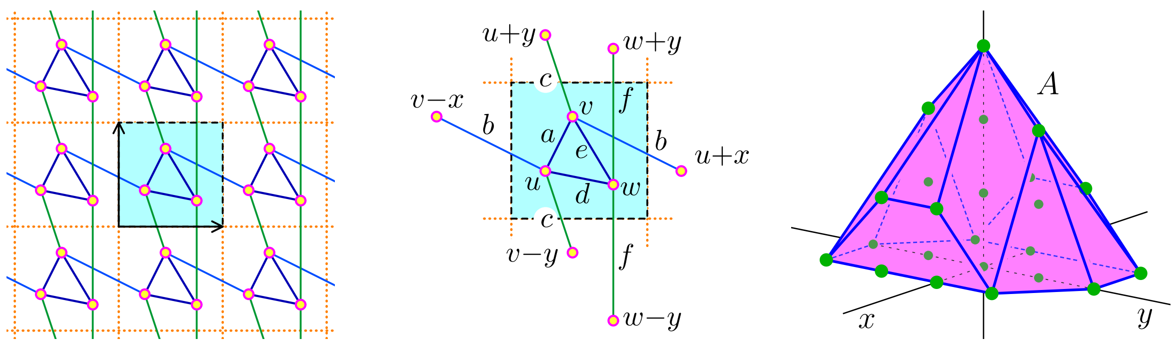


FIGURE 9. A $\mathbb{Z}^2$-periodic graph, its edge labels, and Newton polytope.

tices in the fundamental domain. Let u, v, w be the values of the potential at the eponymous vertices, and a, b, c, d, e, f be edge weights, as indicated in the middle diagram in Figure 9. We have also simplified notation, writing x for $(1, 0)$ and y for $(0, 1)$.

Here is its characteristic matrix (in the ordered basis corresponding to u, v, w),

$$M \;:=\; \begin{pmatrix} \underline{\lambda} - u & \underline{a} + bx^{-1} + cy^{-1} & d \\ a + \underline{bx} + cy & \underline{\lambda} - v & e \\ d & e & \underline{\lambda} - w + \underline{fy} + fy^{-1}) \end{pmatrix} . \tag{33}$$

The Newton polytope has two vertical faces (the trapezoids in Figure 10) and four faces whose initial graphs are disconnected. All have asymptotic critical points. The face F is

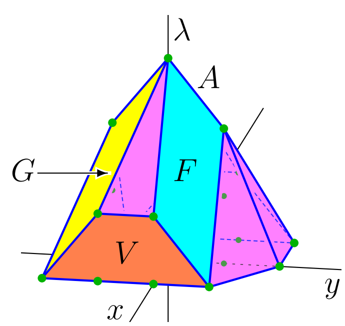
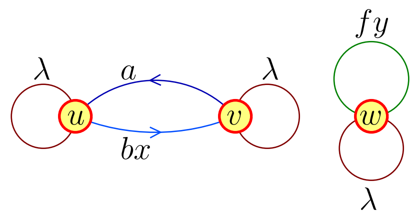


FIGURE 10. Facets V, F, and G with asymptotic critical points.

exposed by the vector $\eta = (-2, -1, -1)$. It consists of those points of the Newton polytope with minimum weight -3 with respect to η. The underlined entries in the characteristic matrix (33) form the initial matrix. The edge weights d and e are the initial forms of their entries, but do not contribute to the initial matrix, which has block form,

$$\mathrm{in}_\eta M \;=\; \begin{pmatrix} \lambda & a & 0 \\ bx & \lambda & 0 \\ 0 & 0 & \lambda + fy \end{pmatrix}, \tag{34}$$

with no block a monomial. This is apparent from the initial graph $\mathrm{in}_\eta\widehat{\Gamma}$,



By Theorem 2.15,

$$\begin{aligned} \mathrm{in}_\eta\Phi \;=\; \det \mathrm{in}_\eta M \;&=\; (\lambda^2 - abx)(\lambda + fy) \;=\; \lambda^3 + fy\lambda^2 - abx\lambda - abfxy \\ &=\; \lambda^3(1 - abx\lambda^{-2})(1 + fy\lambda^{-1})\,. \end{aligned}$$

The four monomials $\{\lambda^3, y\lambda^2, x\lambda, xy\}$ correspond to the vertices of the facet F and form the set $\mathcal{F}$. The span $\mathbb{Z}\mathcal{F}$ of their differences is

$$\mathbb{Z}\mathcal{F} \;=\; \mathbb{Z}(1, 0, -2) \oplus \mathbb{Z}(0, 1, -1)\,.$$

Thus $\mathcal{O}_\mathcal{F} \simeq (\mathbb{C}^\times)^2$ with coordinates $x\lambda^{-2}$ and $y\lambda^{-1}$. Also, $X_\mathcal{F} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, reflecting that F is a parallelogram that is the Minkowski sum of two primitive line segments. Thus, the initial Bloch variety $\mathrm{in}_\eta BV = \mathit{Var}(\mathrm{in}_\eta\Phi)$ is the union of two coordinate lines on $X_\mathcal{F}$ that intersect, and has one singular point, p_F. As $\mathbb{Z}^3 = \mathbb{Z}\mathcal{F} + \mathbb{Z}(0, 0, -1)$, $\mathbb{Z}\mathcal{F}$ equals its saturation and $X_\mathcal{A}$ is smooth along $\mathcal{O}_\mathcal{F}$, which implies that the multiplicities $\mu_\mathcal{F}$ and $[\mathrm{Sat}(\mathcal{F}) : \mathbb{Z}\mathcal{F}]$ in Definition 5.3 are both 1. $\diamond$

Let us now consider the general case. Suppose that Γ is asymptotically disconnected in the direction of $\eta \in \mathbb{Z}^{d+1}$. Let F be the face of $\mathcal{N}(\Phi)$ exposed by η and set $\mathcal{F} := F \cap A$. Let

$$\mathrm{in}_\eta\Phi \;=\; \mu g_1 \cdot g_2 \cdots g_r$$

be the factorization of the initial form given by Lemma 2.19. Then

$$\mathrm{in}_\eta BV \;=\; \mathit{Var}(g_1) \cup \mathit{Var}(g_2) \cup \cdots \cup \mathit{Var}(g_r)\,.$$

As the factors g_i arise from disjoint components of $\mathrm{in}_\eta\widehat{\Gamma}$, the parameters E, V in one factor g_i are distinct from the parameters in other factors.

Before we explain the consequence for asymptotic critical points, we need a definition. Suppose that P and Q are lattice polytopes in $\mathbb{Z}^2$—polygons or degenerate line segments. Their *mixed area*, $\mathrm{MA}(P, Q)$, is

$$\mathrm{MA}(P, Q) \;:=\; \mathrm{area}(P + Q) - \mathrm{area}(P) - \mathrm{area}(Q)\,,$$

where area is the usual area in $\mathbb{R}^2$ in which the unit square has unit area. Figure 11 shows

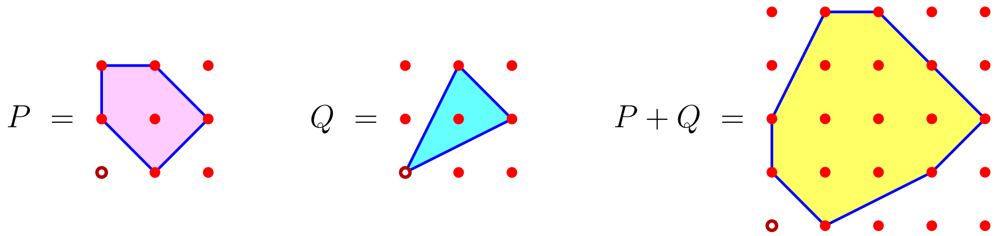


FIGURE 11. As area$(P) = 5/2$, area$(Q) = 3/2$, and area$(P+Q) = 10$, the mixed area is $\mathrm{MA}(P,Q) = 10 - 5/2 - 3/2 = 6$.

an example. A one-dimensional lattice polytope F is a line segment and its *lattice length* $\ell(F)$ satisfies

$$\ell(F) + 1 \;=\; \#(F \cap \mathbb{Z}\mathcal{F})\,,$$

the 1-dimensional volume of F measured with respect to the 1-dimensional lattice $\mathbb{Z}\mathcal{F}$.

Lemma 5.2. *Suppose that the V, E are general and Γ is asymptotically disconnected in the direction of $\eta \in \mathbb{Z}^{d+1}$ with F the face of $\mathcal{N}(\Phi)$ exposed by η. We have the trichotomy.*

(i) *If $\dim F = 1$, then $\mathrm{in}_\eta BV$ consists of $\ell(F)$ points.*

(ii) *If $\dim F > 2$, then for any $1 \le i < j \le r$ either $Var(g_i) \cap Var(g_j) = \emptyset$ or the intersection $Var(g_i) \cap Var(g_j)$ is a positive-dimensional set of singular points on $\mathrm{in}_\eta BV$, which are all asymptotic critical points.*

(iii) *If $\dim F = 2$, then the initial Bloch variety $\mathrm{in}_\eta BV$ has*

$$\sum_{1 \le i < j \le r} \mathrm{MA}(\mathcal{N}_i, \mathcal{N}_j) \tag{35}$$

singular points, counted with multiplicity as a subset of $\mathcal{O}_\mathcal{F}$. All are asymptotic critical points of BV in the direction η. Here, $\mathcal{N}_i \subset \mathbb{Z}\mathcal{F}$ is the Newton polytope of the factor g_i.

Proof. Statement (i) holds as $\mathrm{in}_\eta \Phi$ is a univariate polynomial of degree $\ell(F)$, in the coordinates of $\mathcal{O}_\mathcal{F}$.

For (ii), $\mathcal{O}_\mathcal{F}$ is at least 3-dimensional. As the factors g_i are irreducible and distinct, the intersection $Var(g_i) \cap Var(g_j)$ is either empty or has dimension $n - 2 \ge 1$. If nonempty, then the intersection consists of singular points of $\mathrm{in}_\eta BV$. Statement (ii) holds by Proposition 1.12(ii).

For (iii), if $i \ne j$, then $Var(g_i) \cap Var(g_j)$ consists of singular points, each is an asymptotic singular point of BV in the direction of η and hence is an asymptotic critical point. If g_i, g_j are general, then this consists of $\mathrm{MA}(\mathcal{N}_i, \mathcal{N}_j)$ points, counted with multiplicity. Generality is ensured, as they depend on independent parameters. Generality also implies that all triple intersections are empty. This proves (35). □

Definition 5.3. Let F be a two-dimensional oblique face of $\mathcal{N}(\Phi)$ such that Γ is asymptotically disconnected in the direction of η, where η exposes F. Define

$$N_{\mathrm{disc}}(F) \;:=\; \mu_\mathcal{F}[\mathrm{Sat}(\mathcal{F}) : \mathbb{Z}\mathcal{F}] \sum_{1 \le i < j \le r} \mathrm{MA}(\mathcal{N}_i, \mathcal{N}_j)\,,$$

the product of the sum (35) and the multiplicity of the orbit $\mathcal{O}_{\mathcal{F}}$ in $X_{\mathcal{A}}$.

For all other faces F set $N_{\rm disc}(F) = 0$ and let $N_{\rm disc}$ be the sum of $N_{\rm disc}(F)$ over all faces. $\diamond$

Lemma 5.4. *With these definitions, and where L_Ψ is the linear space corresponding to the critical point equations Ψ, we have*

$$\sum_{x\in sing(\mathrm{in}_\eta BV)} m(x; X_{\mathcal{A}}, L_\Psi) \ \geq\ N_{\rm disc}(F)\,. \tag{36}$$

Proof. A singular point $x \in Var(g_i) \cap Var(g_j)$ has multiplicity $m(x, X_{\mathcal{A}}, L_\Psi)$ as a critical point that is at least the product of its multiplicity as a point of $Var(g_i) \cap Var(g_j)$ and the multiplicity $\mu_{\mathcal{F}}[\mathrm{Sat}(\mathcal{F}) : \mathbb{Z}\mathcal{F}]$ of the orbit $\mathcal{O}_{\mathcal{F}}$ in $X_{\mathcal{A}}$ (This is Theorem 3.14(ii)). Since such points are a subset of the singular points of $\mathrm{in}_\eta BV$, we deduce (36). $\square$

Let L_Ψ be the linear space corresponding to the critical point equations Ψ and define $\partial_{\rm obl}X_{\mathcal{A}}$ to be the union of all toric subvarieties $X_{\mathcal{F}}$ for F an oblique face of A.

Corollary 5.5. *We have the following inequality,*

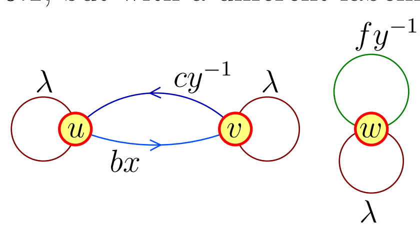
$$\sum_{x\in\partial_{\rm obl}X_{\mathcal{A}}} m(x; X_{\mathcal{A}}, L_\Psi) \ \geq\ N_{\rm disc}\,.$$

Proof. Collecting the terms in the sum by the orbit they lie on gives

$$\sum_{x\in\partial_{\rm obl}X_{\mathcal{A}}} m(x; X_{\mathcal{A}}, L_\Psi) \ =\ \sum_{F \text{ oblique}}\ \sum_{x\in\mathcal{O}_{\mathcal{F}}\cap L_\Psi} M(x; X_{\mathcal{A}}, L_\Psi)\,.$$

The lemma follows from the inequality of Theorem 3.14(ii) and Lemma 5.4. $\square$

Example 5.6. Let us now consider the facet G of the polytope in Figures 9 and 10. This is exposed by the vector $\eta = \langle -1, 1, -1\rangle$, which takes value -3 on G. The initial graph has the same structure as in Example 5.1, but with a different labeling,



.

Here is the facial form,

$$(\lambda^2\ -\ bcxy^{-1})(\lambda + fy^{-1})\,. \tag{37}$$

As Example 5.1, the monomials $\{\lambda^3, y^{-1}\lambda^2, xy^{-1}\lambda, xy^{-2}\}$ correspond to the vertices of the facet G and form the set $\mathcal{G}$. The span $\mathbb{Z}\mathcal{G}$ of their differences is

$$\mathbb{Z}\mathcal{G}\ =\ \mathbb{Z}(1,-1,-2)\oplus\mathbb{Z}(0,-1,-1)\ =\ \mathbb{Z}(1,0,-1)\oplus\mathbb{Z}(0,-1,-1)\,.$$

Thus $\mathcal{O}_{\mathcal{G}} \simeq (\mathbb{C}^\times)^2$ with coordinates $x\lambda^{-1}$ and $y^{-1}\lambda^{-1}$. Also, $X_{\mathcal{G}} \simeq \mathbb{P}^1\times\mathbb{P}^1$, reflecting that G is a parallelogram that is the Minkowski sum of two primitive line segments. Thus the initial Bloch variety $\mathrm{in}_\eta BV = Var(\mathrm{in}_\eta\Phi)$ is the union of two coordinate lines on $X_{\mathcal{G}}$ that intersect, and has one singular point, p_G. As $\mathbb{Z}^3 = \mathbb{Z}\mathcal{G} + \mathbb{Z}(0,0,-1)$, $\mathbb{Z}\mathcal{G}$ equals its saturation and $X_{\mathcal{A}}$ is smooth along $\mathcal{O}_{\mathcal{G}}$, which implies that the multiplicities $\mu_{\mathcal{G}}$ and $[\mathrm{Sat}(\mathcal{G}) : \mathbb{Z}\mathcal{F}]$ in Definition 5.3 are both 1.

The point p_G has multiplicity two on the compactified Bloch variety. We study it in $V_{\mathcal{G}}$. If we multiply each entry of the characteristic matrix by λ^{-1}, then each entry lies in

$$\mathbb{C}[M_{\mathcal{G}}] \;=\; \mathbb{C}[x\lambda^{-1}, x^{-1}\lambda, y^{-1}\lambda^{-1}, y\lambda][\lambda^{-1}] \;=\; \mathbb{C}[\zeta, \zeta^{-1}, \xi, \xi^{-1}][\rho]\,,$$

where $\zeta = x\lambda^{-1}$, $\xi = y^{-1}\lambda^{-1}$, and $\rho = \lambda^{-1}$. This ring is graded by the degree of the indeterminate ρ. It is the coordinate ring of $V_{\mathcal{G}}$ and $\rho = 0$ defines the orbit $\mathcal{O}_{\mathcal{G}}$. This shows that $V_{\mathcal{G}} \simeq (\mathbb{C}^{\times})^2 \times \mathbb{C}$ Here is the characteristic matrix (in the coordinates of $\mathbb{C}[M_{\mathcal{G}}]$),

$$\begin{pmatrix} 1 - u\rho & a\rho + b\zeta^{-1}\rho^2 + c\xi & d\rho \\ a\rho + b\zeta + c\xi^{-1}\rho^2 & 1 - v\rho & e\rho \\ d\rho & e\rho & 1 - w\rho + f\xi^{-1}\rho^2 + f\xi \end{pmatrix}. \tag{38}$$

Its determinant Φ defines the Bloch variety in $V_{\mathcal{G}}$. We write Φ as a polynomial in ρ (suppressing terms of degree greater than 2),

$$\begin{aligned}(1 - b\zeta\xi)(1 + f\xi) \;\;-\;\; &\rho\big(w(1 - b\zeta\xi) + (1 + f\xi)(u + v + ac\xi + ab\zeta)\big) \\ +\;\; &\rho^2(uw + vw - e^2 - d^2 + b\zeta(de - cf + aw) + c\xi(aw + de) + f\xi^{-1}) \\ &\qquad\qquad +\;\; \rho^2(1 + f\xi)(uv - a^2 - b^2 - c^2) \;+\; \rho^3(\cdots\end{aligned} \tag{39}$$

The terms of degree zero in ρ, $(1 + f\xi)(1 - b\zeta\xi)$, give the initial form in these coordinates.

We use this expression (39) to study the point p_G. Observe that in $V_{\mathcal{G}}$, the point p_G is defined by the vanishing of ρ, $\tau := 1 - b\zeta\xi$, and $\sigma := 1 + f\xi$. These provide local coordinates for $V_{\mathcal{G}}$ at the point p_G. Similarly, the terms w, $f := (u+v+ac\xi+ab\zeta)$, the coefficient g of ρ^2 in the second line, and $h = uv - a^2 - b^2 - c^2$ are all non-zero at p_G. (This uses the assumption that the graph parameters $a, \dots, f, u, v, w$ are general.) Let us rewrite the expression (39) for Φ in these local coordinates

$$\Phi \;=\; \tau\sigma \;-\; w\tau\rho \;-\; f\sigma\rho \;+\; g\rho^2 \;+ h\sigma\rho^2 \;+\; \rho^3(\cdots \tag{40}$$

This vanishes at p_G, recovering that p_G lies on the compactified Bloch variety , $\overline{BV}$. As there are no terms in Φ that are linear in the local coordinates ρ, σ, τ for p_G, we see that p_G is a singular point on BV, and thus its multiplicity as a critical point is at least two. The Hessian matrix of Φ at the point p_G (with coordinates in the order (τ, σ, ρ)) is

$$\begin{pmatrix} 0 & 1 & -w \\ 1 & 1 & -f(p_G) \\ -w & -f(p_G) & g(p_G) \end{pmatrix}.$$

This has full rank, so the multiplicity of p_G is exactly two.

For the face F, a similar computation shows that the critical point p_F studied in Example 5.1 is a smooth point of the Bloch variety, and hence its multiplicity as a critical point is 1. Indeed, the expression in local coordinates at p_F similar to (40) has the term $bdex\,\rho$, which is linear in ρ as $bdex$ is nonzero at p_F. The difference in the cases for the faces F and G comes from the structure of the Floquet matrix and ultimately of the graph. Unlike the analysis leading to the (dis)connectivity of the initial graph, we do not yet understand this singularity/smoothness in terms of the graph.

Nevertheless, we do understand the critical point degree of the graph of Figure 9. The Newton polytope has normalized volume 54, and a computation at numerical values of the parameters shows that the Bloch variety has 40 critical points. We show that the contribution of asymptotic critical points is at least 14.

There are asymptotic critical points arising from the two vertical faces, and each of the four parallelogram faces, F and G, and their reflections in the λ-axis.

(1) Each vertical face has normalized area four, so $N_{\mathrm{vert}} \geq 8$.
(2) The face F and its reflection each have $N_{\mathrm{disc}} = 1$, which accounts for at least two more.
(3) The face G and its reflection each contribute two, for at least four more.

As $8+2+4=14$, we see that the critical point degree is at most $54-14=40$. With the computation, this shows that the critical point degree of the graph is exactly 40. $\diamond$

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Matthew Faust, Department of Mathematics, Michigan State University, East Lansing, MI 48832, USA
Email address: mfaust@msu.edu
URL: https://mattfaust.github.io/

Jonah Robinson, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
Email address: jonahrob@umich.edu
URL: https://jonah-robinson.github.io/

Frank Sottile, Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
Email address: sottile@tamu.edu
URL: https://franksottile.github.io