Form 101, Part II

Research Proposal

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My primary research interests involve algebraic geometry and algebraic combinatorics, or rather, interactions between the two subjects. I am interested in varieties such as toric varieties and homogeneous spaces, whose geometry may be understood in terms of certain associated combinatorial structures. These structures include symmetric functions, Young tableaux and Schubert polynomials, whose properties in turn are illuminated by their relation to geometry. Presently, I am working to synthesize ideas and techniques from each area to attack seemingly unrelated problems in combinatorics and geometry. These problems include real enumerative geometry and determining the structure constants for the ring of Schubert polynomials.

Real Enumerative Geometry

The basic problem of enumerative geometry is that of determining the number of geometric figures satisfying a set of independent conditions. Such questions admit an elegant solution when formulated over an algebraically closed field, such as the complex numbers. In that case there is some fixed number d of complex figures satisfying a generic choice of conditions.

When the same question is asked over the real numbers, there will typically be fewer than d real figures satisfying generic real conditions. If d is odd, complex conjugation guarantees at least one real figure. One might try to show that it is possible to find generic real conditions satisfied by some fraction of d real figures. Chiavacci [5] describes the best possible situation: An enumerative problem is *totally real* if there exist generic real conditions satisfied by d real figures. A natural place to seek totally real enumerative problems is among the best-known class of enumerative problems, those arising from the classical Schubert calculus. These enumerative problems are interpreted geometrically in terms of intersections of Schubert subvarieties of some Grassmann or flag variety.

A Schubert subvariety $X_\omega(F_\bullet)$ of a Grassmann or flag variety is determined by its *type* ω and a complete flag $F_\bullet$. It consists of all linear spaces or flags satisfying incidence conditions encoded by ω with respect to the given flag $F_\bullet$. For $\mathbf{G}(k,n)$, the Grassmannian of k-planes in $\mathbf{P}^n$, ω is a partition with $k+1$ parts, none bigger than $n-k$. For $\mathbf{F}(n)$, the variety of complete flags in $\mathbf{P}^n$, ω is a permutation in the symmetric group, S_{n+1}. The codimension of $X_\omega(F_\bullet)$ is $|\omega|$, where $|\omega|$ is either the number ω partitions, or its length as a permutation, whichever is applicable. Also, all Schubert subvarieties of the same type are rationally equivalent.

Positive results in real enumerative geometry may be of interest to a wider audience than algebraic geometers. In some applications, problems concerning matrices satisfying linear dependence relations arise, and these may be 'lifted' to enumerative problems on a Grassmannian. For example, how many 2×2 matrices ϕ satisfy $\operatorname{rank}(\phi+\phi_i)\leq 1$ for fixed (generic) real matrices ϕ_i, $1\leq i\leq 4$? The answer is two, and one may write down four real matrices ϕ_i so that the two matrices they determine are each real. This problem may be expressed as an intersection problem on the Grassmannian $\mathbf{G}(1,3)$. A class of problems of this type are those arising from the pole assignment problem [4] in systems control theory.

Chow Rings and Schubert Polynomials

Enumerative problems are typically solved (that is, the number d is determined) by computing products in the Chow (cohomology) rings of the underlying varieties. The Chow rings of these homogeneous spaces have simple descriptions (cf. Fulton, [7], ch. 14). The Chow ring of $\mathbf{G}(k,n)$ is a quotient of the ring of symmetric polynomials in the $k+1$ Chern roots of its tautological k-plane bundle. For $\mathbf{F}(n)$, it is generated by the Chern classes of the line subquotients of the tautological flag bundle, and may be realized as the quotient of a polynomial ring by the ideal generated by symmetric polynomials. Both rings also have an integral basis, called their *Schubert basis*, given by the rational equivalence classes σ_ω of Schubert subvarieties X_ω.

Two questions arise regarding these descriptions of the Chow rings. First, one may seek explicit polynomial expressions in the Chern classes for the elements of the Schubert basis. Secondly, one wishes to describe the structure constants of these rings expressing their multiplication in terms of their Schubert basis. That is, given two Schubert classes σ_ρ and σ_τ, find constants $c^\omega_{\rho\tau}$ so that

$$\sigma_\rho \cdot \sigma_\tau = \sum_\omega c^\omega_{\rho\tau}\sigma_\omega. \tag{1}$$

For Grassmann varieties, these questions have been solved using combinatorics. The Schur polynomials solve the first, and the Littlewood-Richardson rule the second. This was perhaps most satisfactorily established by Lascoux and Schützenberger using the "plactic monoid" [8].

For flag varieties, the story is not yet complete. The first question was solved by Bernstein-Gelfand-Gelfand [2] and Demazure [6], who independently introduced the Schubert polynomials. Recently, a combinatorial description of these polynomials was obtained by Bergeron [1] and Billey-Jockusch-Stanley [3]. However, the question of determining the structure constants remains open. I am studying this, using novel geometric tools developed while working on my thesis.

Weak Generic Intersection Cycles

Both the question of finding real solutions to enumerative problems and that of determining the structure constants of these Chow rings may be formulated in terms of intersections of Schubert subvarieties. I believe the same techniques may successfully be applied to both questions. Here is an indication of these techniques.

As an abelian group, the Chow ring is the group of algebraic cycles modulo rational equivalence. Its ring structure is such that if two subvarieties X and Y meet generically transversally, that is, transversally along the generic point of each component of their intersection, then the product of their rational equivalence classes equals the rational equivalence class of their intersection, $X\bigcap Y$.

To determine the constants $c^\omega_{\rho\tau}$ of (1) for either ring, I plan to exhibit a rational equivalence between a generically transverse intersection $X_\rho\bigcap X_\tau$ of Schubert subvarieties and a union of distinct Schubert subvarieties. The constants $c^\omega_{\rho\tau}$ would be the number of Schubert subvarieties of type ω in that union. Previous approaches have used other, less direct methods.

For applications to real enumerative geometry, I seek rational equivalences with additional structure, described below. Finding these will not only determine the constants $c^\omega_{\rho\tau}$, but will also address questions in real enumerative geometry.

An *intersection problem* is a list $\omega_1,\ldots,\omega_m$ of types of Schubert subvarieties. For a given intersection problem let $\mathcal{C}_\prime$ be the set of *generic intersection cycles*, that is cycles which are given by generically transverse intersections of Schubert subvarieties of types $\omega_1,\ldots,\omega_m$. This is a unirational, locally closed subvariety of the Chow variety. A cycle X is a *weak generic intersection cycle* if there exists a sequence $\mathcal{C}_\infty, \dots, \mathcal{C}_\updownarrow$ of unirational locally closed subvarieties of the Chow variety with $\mathcal{C}_\updownarrow = \{\mathcal{X}\}$, and

(∗) For $1 \le i \le l$, and $Z \in \mathcal{C}_\rangle$, there is a rational curve γ containing Z with $\gamma - \{Z\} \subset \mathcal{C}_{\rangle-\infty}$.

Such a cycle X is rationally equivalent to any generic intersection cycle, and is furthermore in the Zariski closure of the set of generic intersection cycles.

Approach

My plan is to investigate the following questions, whose significance is discussed below.

(A) Which intersection problems $\omega_1, \dots, \omega_m$ have a weak generic intersection cycle X consisting of a union of Schubert subvarieties?

(B) Given (A), over which fields K is it possible to construct the components of X, define the subvarieties $\mathcal{C}_\rangle$, and if Z in (∗) is defined over K, find a curve γ defined over K?

Solving question (A) for $m = 2$ would determine the structure constants $c^{\omega}_{\rho\tau}$ of the Chow rings. For Grassmannians, this would give a new, geometric proof of the Littlewood-Richardson rule. For flag varieties, it would resolve an outstanding question about Schubert polynomials. The Chow rings of Grassmannians are subrings of the Chow rings of flag varieties, so my immediate goal is to accomplish this for Grassmannians.

If $|\omega_1| + \cdots + |\omega_m|$ is the dimension of the ambient variety, then a generic intersection cycle is $d = \deg \prod \sigma_{\omega_i}$ points, and the intersection problem is an enumerative problem. Solving (B) for the real numbers would show that enumerative problem is totally real (cf. [10], pp. 21 and 51). Then X consists of d real points, and it is in the closure of the real points of $\mathcal{C}_l$, in the classical topology. It follows that there exist real generic intersection cycles consisting of d real points.

For any intersection problem, solving (B) for the real numbers has enumerative consequences. Let $X = X_{\tau_1} \bigcup \cdots \bigcup X_{\tau_t}$ be a real weak generic intersection cycle for the intersection problem $\omega_1, \dots, \omega_s$. Then any enumerative problem of the form $\omega_1, \dots, \omega_s, \rho_{s+1}, \dots, \rho_m$ will have a weak generic intersection cycle of the form $Z = Z_1 \bigcup \cdots \bigcup Z_t$, where each Z_i is a real generic intersection cycle for the problem $\tau_i, \rho_{s+1}, \dots, \rho_m$. This follows from lemmas involving intersecting families of cycles and maps to Chow varieties which were established to study such problems (similar to Lemmas 5.1, 5.2, and 5.3 of [10]). If d_i is the degree of the zero cycle Z_i, then Z has at least r real points, where r is the number of odd d_i's. Arguing as in the last paragraph, one sees that there exist real generic intersection cycles for the enumerative problem having at least r real points.

Progress Related to Proposal:

I. Ph.D. Thesis

Theorem 8.3 of [10] states that all enumerative problems given by the classical Schubert calculus on lines in $\mathbf{P}^n$ are totally real. My method was to solve (B) for all intersection problems on the Grassmannian of lines in $\mathbf{P}^n$. While I used a knowledge of the Chow ring to do this, I now realize that was unnecessary. These geometrical methods give a new, quite explicit determination of the multiplicative structure of that Chow ring: Given two-rowed partitions $\lambda_1, \ldots, \lambda_m$, there are constants $c^{\mu}_{\lambda_1,\ldots,\lambda_m}$ defined by the equation in the Chow ring

$$\sigma_{\lambda_1} \cdots \sigma_{\lambda_m} = \sum_{\mu} c^{\mu}_{\lambda_1,\ldots,\lambda_m} \sigma_{\mu}. \tag{2}$$

The weak generic intersection cycles produced yield a description of these numbers as the sizes of certain sets of Young tableaux (described in Corollary 7.5 of [10]), recovering the combinatorial description of iterated products of Schur functions indexed by two-rowed partitions. These geometric methods give a proof of formulas previously established by means of combinatorics.

II. A Geometric Pieri's Rule

The first phase of my project is to accomplish (A) and (B) when $m = 2$ for all Grassmannians. While this would certainly give a new, explicitly geometric determination of the structure constants of the Chow ring, much less is needed to determine its ring structure. In fact, its multiplicative structure is determined by Pieri's rule (cf. [7], ch. 14): If α is a partition with only one part of size a, and λ any partition, Pieri's rule asserts that

$$\sigma_\lambda \cdot \sigma_\alpha = \sum \sigma_{\lambda'}, \tag{3}$$

the sum over all partitions λ' with $|\lambda'| - |\lambda| = a$, and $\lambda'_1 \geq \lambda_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{k+1} \geq \lambda_{k+1}$. Pieri's rule was originally established using Poincaré duality. I establish Pieri's rule by constructing the rational equivalences asked for in (A) and (B). The weak generic intersection cycles are quite complicated to describe. However, when $a = 1$, the situation simplifies considerably:

In that case, $X_\alpha(F'_\bullet)$ is the set of k-planes meeting F'_{k+1}, the codimension $k+1$ subspace of $F'_\bullet$. (Here, flags are indexed by codimension). If λ is a partition and $F_\bullet$ a flag, choose $F'_\bullet$ so that for $0 \leq j \leq k$, the linear subspace $F_{j+\lambda_{k+1-j}}$ is a subspace of F'_j and it meets F'_{k+1} properly as a subspace of F'_j. Then $X_\lambda(F_\bullet)$ meets $X_\alpha(F'_\bullet)$ generically transversally, and

$$X_\lambda(F_\bullet) \bigcap X_\alpha(F'_\bullet) = \sum_{\lambda'} X_{\lambda'}(L_\bullet),$$

where $L_\bullet$ is the flag defined by $L_i = F_{i-1} \cap F'_{j+1}$, for $j+1+\lambda_{k-j} > i \geq j+\lambda_{k+1-j}$, and the summation is as in (3) for $a = 1$. In this case the weak generic intersection cycles are generic intersection cycles, which is not generally the case when $a > 1$.

III. A Geometric Monk's Rule

For flag varieties, the analog of Pieri's rule is Monk's rule [9], which was originally established by Monk using Poincaré duality. For each $1 \leq r \leq n$, let s_r be the transposition $(r, r{+}1)$. For any permutation $\omega \in S_{n+1}$, Monk's rule asserts that

$$\sigma_\omega \cdot \sigma_{s_r} = \sum \sigma_{\omega t}, \tag{4}$$

the sum over all transpositions $t = (ij)$ with $1 \leq i < r < j$ and $|\omega t| = |\omega| + 1$.

My proof of Monk's rule turns out to be similar to the case $a = 1$ of Pieri's rule, and much simpler than the general case of the geometric Pieri's rule. I will briefly outline it here, as it may serve to illustrate some of my techniques, albeit in a rather simplified form. Here, the weak generic intersection cycles are of the form

$$X(\omega, r, F_\bullet) = \sum X_{\omega t}(F_\bullet), \tag{5}$$

where $F_\bullet$ is a fixed flag, and the summation is the same as in (4).

The Schubert subvariety $X_{s_{k+1}}(F_\bullet')$ consists of those flags whose k-dimensional subspace meets F_{k+1}'. The flag variety $\mathbf{F}(n)$ has a projection to $\mathbf{G}(k,n)$, under which the image of a Schubert subvariety is a Schubert subvariety. Thus for any permutation ω there is a partition λ describing the image of a Schubert subvariety of type ω. Let $F_\bullet$ be any complete flag and choose $F_\bullet'$ as in the case $a=1$ of Pieri's rule. Then the Schubert subvarieties $X_{s_{k+1}}(F_\bullet')$ and $X_\omega(F_\bullet)$ meet generically transversally with the same number of components as $X(\omega, k{+}1, F_\bullet)$. Allowing $F_\bullet'$ to vary in a particular fashion gives a rational curve γ in the Chow variety meeting the point $X(\omega, k{+}1, F_\bullet)$, with $\gamma - \{X(\omega, k{+}1, F_\bullet)\}$ contained in $\mathcal{C}_I$. This accomplishes (A) and (B) over arbitrary fields in the case of Monk's rule.

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