

Bounds for real solutions to polynomial equations

Frank Sottile, Professor of Mathematics,
Texas A&M University, College Station

Overview. This proposal requests funds to help me travel to work with Frédéric Bihan of Université de Savoie in Chambéry, France during April/May 2008. Our collaboration has been very productive. We have been working together since 2004, with 5 completed publications and one in progress. Our most significant advances arose from extended visits, in January/February 2004 and November/December 2005. Our collaboration is again at a stage where it needs another extended period of time working together.

Research Summary. In 1636 René Descartes showed that the number of positive real zeroes of a polynomial in one variable is less than its number of coefficients. For example, $x^{10} - 1$ has two terms and one positive zero, 1, while $x^{10} - 15x^8 + 85x^6 - 225x^4 + 274x^2 - 120$ has six terms and five positive zeroes, as does $6x^5 - 41x^4 + 97x^3 - 9x^2 + 41x - 6$, but only this last polynomial has degree (number of complex-number zeroes) equal to its number of positive real zeroes. Such *a priori* information about real solutions is important in applications, from chemical dynamics through celestial mechanics, and as an input to problems in engineering.

In 1980, Askold Khovanskii generalized this Descartes bound to systems of multivariate polynomial equations. He showed that a system of n polynomials in n variables which together have a total of $n+k+1$ distinct monomial terms has at most $2^{\binom{n+k}{2}}(n+1)^{n+k}$ positive real solutions. This is most relevant for polynomials with few monomials (fewnomials). For the system with $n = k = 2$,

$$(1) \quad 11y^{54}x^{108} - 11yx^{108} + 10 = 10y^{108}x^{54} - 11x^{53} + 11 = 0,$$

Khovanskii's bound is $2^{\binom{4}{2}}(2+1)^4 = 2^6 3^4 = 5184$. This is less than 11663, which is its number of complex solutions. However, the system (1) has only 5 positive real solutions. Rojas, et.al. showed that systems of this form (two trinomials) have at most 5 positive solutions, illustrating that Khovanskii's bound is far from sharp.

Bihan and I have been working to improve this bound and apply these improvements. In 2004 we established a realistic bound when $k = 1$, which led to Bihan's sharp bound of $n+1$ when $k = 1$. When we next got together (3 visits in 6 weeks while I was teaching in Paris in 2005), we had a breakthrough, which eventually led to a dramatic improvement in the fewnomial bound to $\frac{e^2+3}{4} 2^{\binom{k}{2}} n^k$. While this formula looks similar to Khovanskii's bound, it is in fact quite lower. When $n = k = 2$, it is $\frac{e^2+3}{4} \cdot 2 \cdot 2^2 \simeq 20.7$.

With Rojas, we showed that this bound is asymptotically sharp, for k fixed and n large. With Bates, realized how to use our ideas to give a bound of $\frac{e^4+3}{4} 2^{\binom{k}{2}} n^k$ for *all* real solutions to such a system (not just positive solutions). My work with Bihan suggested a new numerical algorithm to find all real solutions to such a system which only computes real solutions—previous algorithms must compute all the complex solutions. Bates and

I am writing a test implementation, and I plan to apply for a grant to hire personnel to help write a proper implementation and make our software publically available.

Bihan and I are beginning to discuss additional lines of research. One is foundational—our work gives the same bounds for polynomials whose exponents are real numbers and not just integers, but a lack of foundations for these polynomials with real exponents has hindered our work. We would like to develop these foundations. We also want to improve our bound. While the term n^k is provably asymptotically correct, we have evidence that the term $2^{\binom{k}{2}} = 2^{\frac{k^2-k}{2}}$ is simply too large (we expect n^k or even 2^k). Lastly, we want to apply our new bounds to areas where Khovanskii’s bound has been useful. We have already used our ideas to bound the topology of fewnomial hypersurfaces, but hope to find wider applications.

Timeliness of this trip. I am on teaching leave in the Winter/Spring of 2008, and so am free(r) to travel. This trip will enable me to take advantage of two workshops which are uniquely placed to enhance my collaboration with Bihan, while allowing us to work together over an extended period outside of these events. Also, it has been two years since we saw each other, and the extraordinary ideas we generated in our last meeting have largely been played out.

My proposed trip would begin with a visit to Chambéry (where Bihan works) in the week of April 14–18, 2008. Following this is a small workshop (21–26 April) involving about a 20 visitors to the Bernoulli centre in Lausanne, Switzerland (3 hours by train from Chambéry). The theme of this will be bounds in real algebraic geometry, and the main visitors include Khovanskii, as well as Moeckel and Albouy, who have applied fewnomial bounds to important applications. This workshop is embedded in an advanced study institute on Real and Tropical Geometry which runs from January–June 2008. After this workshop, either Bihan will visit Lausanne for a few days, or I will return to Chambéry (Bihan has to teach). Following this is a large international conference at the ICMS in Edinburgh (5–9 May), Scotland on the topic of “Effective Real Analytic Geometry”, where Bihan and I are invited speakers.

This trip would involve intensive periods of collaboration with Bihan interspersed with workshops on topics close to our research, and is thus a unique opportunity for collaboration, for dissemination of our results, and for exploiting the expertise of other conference participants. The multifaceted nature of this trip is similar to our last period of face-to-face collaboration, when I saw Bihan thrice while I was in residence at an advanced study institute in the Institut Henri Poincaré in Paris in November/December 2005.

Budget. This trip is complicated. The ICMS in Edinburgh has offered to cover my accommodation and possibly my European airfare (600 - 1000 Euros), while the Université de Savoie will cover my accommodations in Chambéry (approximately 600 Euros). The workshop at the Bernoulli centre only has funding for European participants. Transportation costs between College Station and Geneva, regional train travel, 2 weeks in Lausanne, two weekends between visits/meeting, an extra day in Geneva because of air schedules will run about \$3600. (Room and food in Lausanne is estimated at 1250 Swiss Francs/week, which is approximately \$1,000). Because of this, I am asking for \$2500, and will cover the remainder from my NSF grant, which will be between \$1100 and \$1600, depending upon the funds offered by the ICMS.