Application
To The National Science Foundation
For Grant Support
Of Scientific Research

1. **Institution**: University of Massachusetts - Amherst
2. **Personnel**: Frank Sottile, Principal Investigator
3. **Title**: "Real and Combinatorial Algebraic Geometry."
4. **Period of Requested Support**: June 1, 2000 to May 31, 2003
5. **Summary**

Algebraic geometry, which is the study of solutions to systems of polynomial equations, is important for its potential applications—polynomial equations are ubiquitous in mathematics and the applied sciences. Real and effective aspects of algebraic geometry are of particular interest, as applications often demand explicit, real-number answers to mathematical questions. This project will serve to increase the applicability of algebraic geometry by studying real and effective aspects of algebraic geometry, by deepening the understanding of related combinatorial structures, and lastly by Sottile writing an application-driven algebraic geometry text for the applied scientist with J. Rosenthal and X. Wang.

A major theme of this project is the study of real enumerative geometry, which is the geometric counterpart of the important practical problem of finding real solutions to a system of polynomial equations. This project will include studying this question of real solutions for many classical problems from enumerative geometry. One focus is a very strong conjecture of Shapiro and Shapiro about geometric problems concerning subspaces of vector spaces or matrices. Another will be to embark on the topological classification of maximally inflected curves, which are rational curves, all of whose inflection points are real.

A good understanding of combinatorial structures often underlies effective methods in algebraic geometry, and this project will further illuminate some structures of the Bruhat orders on Weyl groups and their relation to the Littlewood-Richardson coefficients for flag manifolds. These Littlewood-Richardson coefficients are the structure constants for the multiplication in the cohomology ring of a flag manifold with respect to its basis of Schubert classes and they generalize the classical Littlewood-Richardson coefficients which are important in representation theory and combinatorics. Besides seeking combinatorial formulas and identities satisfied by these coefficients, this proposed research will study inequalities satisfied by the indices of nonzero Littlewood-Richardson coefficients.

Related ideas and techniques from algebraic geometry have figured prominently in recent advances in several problems including the pole placement problem in systems theory, matrix completion in linear algebra, homotopy continuation, and in computational robotics. Together with Rosenthal and Wang, Sottile will use this work to motivate a practical introduction to algebraic geometry for the applied scientist and give a unified treatment of these recent advances. This will include a useful treatment of the real numbers (as well as the more traditional complex numbers) and an introduction to ideas from symbolic and numerical computation, all sprinkled liberally with examples and applications drawn from the literature.

6. **Results from Prior NSF Support**. Not Applicable

7. **Proposed Research.**

The aim of this project is to increase our understanding of real number phenomena in algebraic geometry, to improve combinatorial tools for studying classical varieties, and to play an expository role by coauthoring an algebraic geometry text for applied scientists. In addition to the particular results I will prove, this project will increase the role of algebraic geometry in applications of mathematics. I provide more details below about specific lines of proposed research. Section I is devoted to problems concerning real number phenomena, Section II to combinatorial aspects of this program, and Section III describes the book I am writing with J. Rosenthal (Notre Dame) and X. Wang (Texas Tech).

I. **Real Enumerative Geometry.**

Enumerative geometry is the geometric counterpart of counting the common solutions to a system of polynomial equations. By this, we mean that enumerative geometry is concerned with counting geometric figures of some kind that have specified positions with respect to some general, but fixed figures. Real enumerative geometry [46] is likewise the geometric counterpart of the problem of determining the real solutions to a system of real polynomial equations, that is, count or understand the real solutions when the fixed figures are real. Since the number of real solutions depends subtly on the configuration of the conditions, the general problem is intractable. Despite this difficulty, these are important questions, as applications often demand real solutions.

An enumerative problem is *fully real* if there exist real fixed figures so that every figure having specified positions with respect to the fixed figures is real. A promising question is to ask when an enumerative problem is fully real. Using ideas from systems theory [41] and numerical homotopy continuation [61], I recently showed that a large class of enumerative problems arising from the Schubert calculus (both classical and quantum [9]) are fully real [48, 50]. This research will extend these results and explore their consequences.

I.1. **Reality and non-reality in flag manifolds**

Flag manifolds G/P (G a complex semisimple algebraic group and P a parabolic subgroup of G) have distinguished Schubert subvarieties $X_\alpha F_\bullet$. Here $F_\bullet$ is a fixed complete flag (or a Borel subgroup of G) and $X_\alpha F_\bullet$ consists of those $x \in G/P$ which have specified position α with respect to $F_\bullet$. We call α a *Schubert condition.* Given a collection $\alpha_1, \ldots, \alpha_n$ of Schubert conditions and flags $F_\bullet^1, \ldots, F_\bullet^n$ in general position, the intersection

$$X_{\alpha_1} F_\bullet^1 \cap \ldots \cap X_{\alpha_n} F_\bullet^n \qquad (1)$$

consists of those $x \in G/P$ having position α_i with respect to flag $F_\bullet^i$ for each $i = 1, \ldots, n$. When this intersection is zero-dimensional (a combinatorial condition on the α_i) we say that $\alpha_1, \ldots, \alpha_n$ are (data for) a *Schubert-type enumerative problem.* Other spaces, such as the quantum Grassmannian [50, 54] (a singular space of rational curves in the Grassmannian) have similar Schubert varieties [63], and thus an analog of (1).

In this situation, real enumerative geometry asks the following question: When the flags $F_\bullet^1, \ldots, F_\bullet^n$ are real, how many of the points in (1) are real? A particular, but nontrivial case of this question for the Grassmannian and quantum Grassmannian has been important in systems theory [13]. When the α_i are special Schubert conditions for the Grassmannian or natural codimension 1 conditions for the quantum Grassmannian, then all solutions may be real [48, 50]. The new technique introduced for these results is related to numerical homotopy continuation algorithms [31, 32] for finding the points in the intersection (1), and is based upon a key technical lemma I wish to extend to all flag manifolds and to the quantum Grassmannian.

Let me state this extension for the flag manifold. Let N be a regular nilpotent element in the Lie algebra of G and for $s \in \mathbb{C}$, set $g(s) := \exp(sN)$, a 1-parameter subgroup of G. Choose $F_\bullet$ so that for $s \neq 0$ the flag $F_\bullet(s) := g(s)F_\bullet$ is in linear general position with respect to $F_\bullet$.

Question 1. *When α and β are Schubert conditions, do we have*

$$\lim_{s\to 0}(X_\alpha F_\bullet \cap X_\beta F_\bullet(s)) \;=\; \sum_\gamma c^\gamma_{\alpha\,\beta}\, X_\gamma F_\bullet\,? \tag{2}$$

Here, the limit is taken cycle-theoretically. For the Grassmannian, the answer to Question 1 is yes [19]. It is also yes when β is a codimension 1 condition on either the quantum Grassmannian [50] or a flag manifold [53]. This limit (2) is important as it is the analog for homogeneous spaces of toric deformations, which are useful in toric geometry and form the foundation of Viro's method [43, 59].

An affirmative answer to Question 1 would follow by showing that the scheme-theoretic limit is supported on Schubert varieties $X_\gamma F_\bullet$ of the correct codimension. It would follow that the components are these Schubert varieties, with some multiplicities $c^\gamma_{\alpha\,\beta}$. From intersection theory, we know that the $c^\gamma_{\alpha\,\beta}$ are analogs of the Littlewood-Richardson coefficients for multiplication in the basis of Schubert classes of the flag manifold. This limit (2) should not be hard to establish for flag manifolds, but it should be more delicate for the singular quantum Grassmannian or its quot scheme desingularization [9].

A Schubert condition β is *multiplicity-free* if for all α and γ, the coefficient $c^\gamma_{\alpha\,\beta}$ is either 1 or 0. When the limit (2) is known for a collection of multiplicity-free conditions β and for all α, then any Schubert-type enumerative problem involving those multiplicity-free conditions is fully real [53]. (Here, N and $F_\bullet$ are real and $s \in \mathbb{R}$.) When the conditions β are not multiplicity-free, then the limit (2) is still useful for real enumerative geometry. For instance, it is used to show that certain enumerative problems on the Lagrangian Grassmannian can have *no* real solutions [53]. This limit (2) will be a useful tool to investigate the more delicate question of what are the possible numbers (between 0 and all complex solutions) of real solutions for these enumerative problems.

I.2. **Shapiros's Conjecture.**

My recent work in real enumerative geometry has been motivated by the following conjecture of Shapiro and Shapiro. Let $F_\bullet(s)$ be the flag of subspaces osculating a real rational normal curve, an equivalent description of the flags $F_\bullet(s)$ from Section I.1.

Conjecture 2 (Shapiro-Shapiro)**.** *For any Schubert-type enumerative problem on the Grassmannian $\alpha_1, \ldots, \alpha_n$ and any distinct $s_1, \ldots, s_n \in \mathbb{R}$, the intersection*

$$X_{\alpha_1}F_\bullet(s_1) \cap \ldots \cap X_{\alpha_n}F_\bullet(s_n) \tag{3}$$

consists solely of real points.

There is substantial computational evidence in support of this conjecture [51, 60]. In fact, computing large instances of this conjecture continues to be instrumental in developing and testing leading symbolic [21] and numerical [61] software for finding real solutions to polynomial systems. When the α_i are special Schubert conditions, there exist real s_i for which the intersection (3) consists solely of real points [48]. A consequence of this result, and of [51, Theorem 3.3] is that Conjecture 2 follows from the following conjecture.

Conjecture 3. *Let n be the dimension of the Grassmannian. Suppose each α_i is the unique hypersurface Schubert condition, and $s_1, \ldots, s_n \in \mathbb{R}$ are distinct. Then the intersection (3) is transverse.*

This does not hold if the s_i are complex [20]. There is substantial evidence that the intersection (3) is transverse for any Schubert conditions $\alpha_1, \ldots, \alpha_n$ and distinct real numbers $s_1, \ldots, s_n$ [51]. There are several potential approaches to Conjecture 3. These include using symbolic computation to find equations of the discriminant locus (set of s_i giving non-transversality), formulating transversality in terms of singularities of mappings, and lastly working with a generalization involving totally positive matrices (see Section 4 of [51]). This last approach may be especially fruitful as totally positive matrices have a very rich structure [3] and are manifestly real objects. Conjectures 2 and 3 have analogs for the Grassmannians of maximal isotropic subspaces in an orthogonal or symplectic vector space [53], and I will look for proofs of these conjectures valid in all cases.

I.3. **Maximally Inflected Rational Curves.**

The limit (2) was proven for the Grassmannian by Eisenbud and Harris [19] in their study of ramification of maps from $\mathbb{P}^1$ to projective space. Likewise, Conjecture 2 and the results of [48] allow us to begin the topological study of real projective rational curves [33].

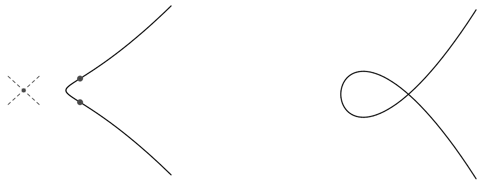
A map $\varphi : \mathbb{P}^1 \to \mathbb{P}^m$ of degree $d > m$ is *ramified* at a point $s \in \mathbb{P}^1$ if $\varphi(\mathbb{P}^1)$ is degenerate at $\varphi(s)$. That is, the first m derivatives of φ at s do not span $\mathbb{P}^n$. Associated to a ramification point is a Schubert condition α in the Grassmannian of $(d-m-1)$-planes in $\mathbb{P}^d$, and the collection of maps $\varphi : \mathbb{P}^1 \to \mathbb{P}^m$ of degree d having at least ramification α at s is identified with the Schubert variety $X_\alpha F_\bullet(s)$, where $F_\bullet(s)$ is the flag of subspaces osculating a real (parameterized) rational normal curve in $\mathbb{P}^d$ at the point s.

With V. Kharlamov [33], we define a real rational curve $\varphi : \mathbb{P}^1 \to \mathbb{P}^m$ to be *maximally inflected* if all ramification points of φ are real. Thus Conjecture 2 asserts that every rational curve $\varphi : \mathbb{P}^1 \to \mathbb{P}^m$ with all of its ramification in $\mathbb{RP}^1$ is given by real forms. The main result of [48] implies that there do exist maximally inflected curves, for certain types of ramification which includes inflection points and cusps of plane curves.

Maximally inflected curves are the rational curve analog of M-curves whose study was begun by Harnack [27] and Hilbert [29]. Harnack proved that a real smooth plane curve of genus g has at most $g+1$ ovals, and he constructed curves with this maximum number of ovals. Real plane curves with maximally many ovals are called M-curves, and their study (part of Hilbert's 16th Problem) has been a major theme of real algebraic geometry this century [64, 62]. One reason is that the class of M-curves is sufficiently restricted to enable a topological classification, yet also sufficiently rich for this classification to be interesting.

Maximally inflected curves are likewise interesting to study as there are serious topological restrictions on the image of a maximally inflected curve, there are symbolic and numeric algorithms [31, 51, 32] for computing examples, and maximally inflected curves are related to the important Conjecture 2 of Shapiro and Shapiro. I propose to study these maximally inflected curves with the goal of obtaining a topological classification.

One restriction is evident for immersed real plane cubics, which have two isomorphism classes:

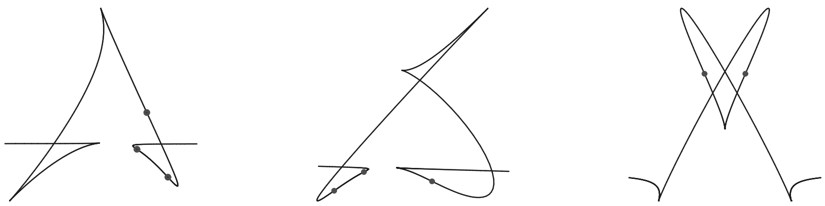


These have respective equations $y^2 = x^3 - x^2$ and $y^2 = x^3 + x^2$. Each has an inflection point at infinity ($[0 : 1 : 0]$ in projective coordinates) and two others with the same x-coordinate $4/3$. In the first, these inflection points (solid circles) are both real, while neither is real in the second. Both have a crossing at the origin, which is a pair of complex branches in the first (dotted lines), but real branches (giving a *loop*) in the second. Thus a maximally inflected plane cubic has an isolated singular point and no real crossings.

The Klein [34] and Plücker [37] formulas show that a plane curve of degree d with only simple points of inflection has at most $\frac{1}{2}(d-3)(d-2)$ of its $\frac{1}{2}(d-1)(d-2)$ crossings real. There is also a formula when the curve has cusps, and in degrees 3, 4, and 5, all possibilities occur. In degree 6 there are additional restrictions proven using symbolic methods [33].

I will pursue several lines of further inquiry. More examples need to be constructed to find other topological restrictions, perhaps involving the relative positions of isolated singular points and loops. One method is to use powerful symbolic and numerical tools developed for the Schubert calculus [32], and another would be to modify existing constructions of plane curves with singularities [44] to obtain maximally inflected curves. I also want to prove any further topological restrictions I discover. This program should also be carried out in higher dimensions. For example, rational space curves of degree d have $\frac{1}{12}(d-4)(d-3)^2(d-2)$ quadrisecant lines which should play the role of crossings for plane curves.

Maximally inflected curves are charismatic—I expect to feature them in future expository mathematical talks. Here are 3 maximally inflected plane quintics with 3 cusps and 3 inflection points. The third curve has one inflection point at infinity.



II. **Combinatorial and Effective Algebraic Geometry.**

The effective methods and computational tools discussed in Part I ultimately rely upon a good understanding of the combinatorial structures of these varieties. A major component of this project is to better understand such combinatorial structures.

The coefficients $c^{\gamma}_{\alpha\,\beta}$ which appear in (2) are the structure constants for the multiplication in the cohomology ring of the flag manifold with respect to its distinguished basis of Schubert classes. These coefficients are known only when G/P is a Grassmannian (they are the Littlewood-Richardson coefficients which are important in representation theory and combinatorics [22, 56]), when G/P is a Grassmannian of maximal isotropic subspaces of an orthogonal or symplectic vector space [57], and in a few other general cases [14, 45, 4, 8]. (See also [24].) I propose three projects to better understand these coefficients $c_{\alpha\,\beta}^{\gamma}$ and one to study related combinatorial structures.

II.1. **Geometric Proof of Littlewood-Richardson Rule.**

The Littlewood-Richardson formula is a combinatorial formula for the coefficients $c_{\alpha\,\beta}^{\gamma}$ for the Grassmannian, and it does not have a geometric proof. I seek a geometric proof of this important formula based upon the result of Eisenbud and Harris [19, Theorem 8.1]

$$\lim_{s\to 0}(X_\alpha F_{\bullet} \cap X_\beta F_{\bullet}(s)) \;=\; \sum_{\gamma} c_{\alpha\,\beta}^{\gamma}\, X_\gamma F_{\bullet}\,. \tag{4}$$

If $\mathcal{I}(s)$ is the ideal of the intersection $X_\alpha F_{\bullet} \cap X_\beta F_{\bullet}(s)$, then $\mathcal{I}(0)$ defines a scheme with components $X_\gamma F_{\bullet}$ of multiplicity $c_{\alpha\,\beta}^{\gamma}$. In Plücker coordinates, $\mathcal{I}(s)$ is generated by the Plücker relations defining $X_\alpha F_{\bullet}$ and linear equations (with parameter s), and the radical of $\mathcal{I}(0)$ is a monomial ideal. I plan to use the combinatorics of the Plücker relations to determine the multiplicities $c_{\alpha\,\beta}^{\gamma}$ directly from $\mathcal{I}(s)$ (perhaps using some Gröbner basis for $\mathcal{I}(s)$). Analyzing related deformations previously gave a new proof of the classical Pieri formula [47] and that led to numerical homotopy algorithms for the special Schubert calculus [31, 32].

The quantum Grassmannian has similar Plücker equations and Schubert varieties [63, 54] and the analog of the left-hand side of (4) is known [50]. Part of the project discussed in Section I.1 is to prove the analog of the limit (4). Given that, the method described above should generalize to the quantum Grassmannian and lead to a determination of the coefficients $c_{\alpha\,\beta}^{\gamma}$ for the quantum Grassmannian. These coefficients are related, if not equal, to the Littlewood-Richardson coefficients for the quantum cohomology ring of the Grassmannian, which presently lack any formula proving their positivity [10].

II.2. **Horn-type Inequalities.**

A more fundamental problem than that of computing the number $c_{\alpha\,\beta}^{\gamma}$ is to determine when it is nonzero. This is equivalent to determining when the triple intersection

$$X_\alpha F_{\bullet} \cap X_\beta F'_{\bullet} \cap X_{\gamma^*} F''_{\bullet} \tag{5}$$

is nonempty for all flags $F_{\bullet}, F'_{\bullet}, F''_{\bullet}$. (Here, γ^* is the Schubert condition Poincaré dual to γ.) For the classical Littlewood-Richardson coefficients, this question was recently settled by Knutson and Tao [35] using combinatorial methods building on much previous work (see [23]). A consequence is a conjecture of Horn [30], originally formulated for inequalities satisfied by eigenvalues of sums of Hermitian matrices, but valid for this problem: Briefly, the Littlewood-Richardson coefficient $c_{\alpha\,\beta}^{\gamma}$ is nonzero if and only if the indices α, β, γ satisfy a system of inequalities, where each inequality arises from a non-zero Littlewood-Richardson coefficient on a 'smaller' Grassmannian.

I seek a purely geometric proof of this recursive description. One direction is easy. If a p-plane H is in the intersection (5) and the flags $F_{\bullet}, F'_{\bullet}, F''_{\bullet}$ are in general position, then their restrictions to H give complete flags $G_{\bullet}, G'_{\bullet}, G''_{\bullet}$ in general position in H. Any nonempty triple intersection of Schubert varieties given by the flags $G_{\bullet}, G'_{\bullet}, G''_{\bullet}$ in a Grassmannian of subspaces of H gives an inequality satisfied by α, β, γ, and these are the inequalities of Horn. The other direction is that if α, β, γ satisfy every such inequality given by a nonempty intersection in a smaller Grassmannian, then the intersection (5) is nonempty. A consequence (which was already known) of the sufficiency of these inequalities is that the collection of non-zero coefficients $c^{\gamma}_{\alpha\,\beta}$ form a semigroup, and understanding its structure geometrically is one step in this line of research.

The 'easy direction' described above works for all flag manifolds and gives valid inequalities satisfied by the indices of nonzero coefficients $c^{\gamma}_{\alpha\,\beta}$, where the inequalities are imposed by non-zero coefficients $c^{\lambda}_{\mu\,\nu}$ in a smaller (SL_n) flag manifold. In particular, when the flag manifold is G/P with P maximal parabolic, then the inequalities arise in the same way as for the Grassmannian, although the actual inequalities are quite different. I plan to investigate several questions for these inequalities and the numbers $c^{\gamma}_{\alpha\,\beta}$. A first question is whether these necessary inequalities are sharp, and if not, determine what additional inequalities these coefficients satisfy. The ultimate goal is to give a complete list of sufficient inequalities for the coefficient $c^{\gamma}_{\alpha\,\beta}$ to be non-zero.

II.3. Pieri-type formulas for flag manifolds.

Special Schubert classes are Chern or Segre classes for certain tautological vector bundles which generate the cohomology of partial flag manifolds. A *Pieri-type formula* is a formula for the multiplication of an arbitrary Schubert class by one of these special Schubert classes. Such formulas are important as they determine the multiplicative structure of the cohomology rings of partial flag varieties with respect to the Schubert basis.

For the classical Grassmannian, these special Schubert classes are the universal Chern and Segre classes and the Pieri-type formula is the classical Pieri's formula. Chevalley's formula [14] is another Pieri-type formula where the special Schubert class is the first Chern class of a line bundle. More recently, I used explicit determinations of certain triple intersections of Schubert varieties to give the general Pieri-type formula for the SL_n-flag manifold [45]. With N. Bergeron, we have continued this program and given a Pieri-type formula for some special Schubert classes on symplectic and odd-orthogonal flag manifolds [8]. This also required a deep study of new combinatorial structures on the associated Weyl group and Bruhat order.

To extend these results to all special Schubert classes on all flag manifolds we first need to fully understand the Pieri-type formulas in analogs of the Grassmannian (G/P with P maximal). In these manifolds, there are two families of generating special Schubert classes. Pieri-type formulas for one family were given by Pragacz and Ratajski [38, 39, 40] using the Leibniz formula for divided differences. This method is tractable only for this family, and is difficult to generalize to the full flag manifold. The triple intersection approach I propose should work for both families, and thus give new formulas. I also expect that this method of proof will generalize to the full flag manifold, in much the same way as the triple intersection proof given in [49] laid the groundwork for the proof in [8]. There, knowledge of the Grassmannian case was a guide to formulating the necessary combinatorics.

II.4. Quasi-symmetric functions.

These Pieri-type formulas lead to a deeper study of the Bruhat order [6, 8] and of graded posets more generally. Many Littlewood-Richardson coefficients $c^{\gamma}_{\alpha\,\beta}$ for the classical flag manifold are coefficients of Schur functions in a quasi-symmetric generating function associated to intervals in the Bruhat order [7] and this function is defined for any edge-labeled graded poset [5]. It gives a Hopf morphism from the incidence Hopf algebra of these posets and recovers other quasi-symmetric functions that have appeared in combinatorics [55, 26].

With N. Bergeron, S. Mykytiuk, and S. van Willigenburg, we have recently discovered a unified mechanism to describe many quasi-symmetric functions in combinatorics, including those arising from the theory of P-partitions [26], from flag f-vectors of graded posets [17], Stanley symmetric functions [55, 11], from Stembridge's theory of enriched P-partitions [58], and from our own work [5, 7].

The ring of noncommutative symmetric functions $NC_{\bullet} := \mathbb{Z}\langle h_1, h_2, \ldots\rangle$ is the free associative algebra with generators h_i of degree i. This has an interesting Hopf algebra structure [25]. Its graded Hopf dual is the algebra $\mathcal{Q}_{\bullet}$ of quasi-symmetric functions. Given a graded poset P, let $\mathbb{Z}P$ be the graded free abelian group where the degree of $x \in P$ is its rank in P. A *Pieri operation* on P is a graded action of $NC_{\bullet}$ on $\mathbb{Z}P$ where, for $x \in P$, $h_i.x$ is a linear combination of $y \in P$ with $x \leq y$. The matrix coefficients (nonzero only if $x \leq y$) of such an action are linear maps on $NC_{\bullet}$ and hence quasi-symmetric functions. We can show that the association of a subinterval $[x, y]$ of P to this quasi-symmetric function gives a Hopf morphism from the incidence Hopf algebra of P to $\mathcal{Q}_{\bullet}$.

For many classes of posets, the image of this morphism is a Hopf subalgebra of $\mathcal{Q}_{\bullet}$ and this reflects some special structure of posets in such a class. One such Hopf subalgebra is the 'algebra of peaks' [58]. This algebra, or rather its Hopf dual is related to recent work on polytopes. In particular, posets whose image is in this algebra of peaks satisfy an analog of the Bayer-Billera [1] relations and hence there should be a **cd**-index [2] for these posets. In fact, we expect that much of the rich structure that has been discovered for flag f-vectors and the **cd**-index [18] to have analogs in this more general setting. With my coauthors, I plan to undertake a deeper study of subalgebras of the algebra of quasi-symmetric functions to better understand these relations.

III. ***Applicable Algebraic Geometry***

I am working with J. Rosenthal and X. Wang to write a book, *Applicable Algebraic Geometry*, on application-driven uses of algebraic geometry. Our goal is to provide a useful and motivated introduction to algebraic geometry for applied scientists, particularly engineers and linear algebra specialists.

The applications we discuss include pole placement in linear systems theory [41] and matrix completion [28], two areas where my coauthors have made notable advances using techniques from algebraic geometry. One strength of this book is that it will provide a coherent presentation of work in these areas that is presently dispersed throughout the literature. We also plan to cover other topics such as eigenvalue inequalities [35, 30], parallel manipulators (the Stewart platform) [36, 16], and some problems from computational vision that may be studied with symbolic computation.

Our focus on applications will keep the presentation of algebraic geometry straightforward. We plan to rely on accessible treatments of algebraic geometry (such as [15]) for proofs of some results and further topics we choose not to cover. One novel feature of our treatment of algebraic geometry will be our simultaneous and equal treatment of complex and real number phenomena. We will work as much as possible over both fields, and describe special features of each when appropriate. Our perspective will often be to ask what the complex theory tells us about the real numbers, as this has proven useful for applications. We will also present some of the standard theory of real algebraic geometry [12].

The value of algebraic geometry for the applied scientist is not just that it gives useful theoretical tools, but also that it provides practical computational tools for solving concrete problems. It is presently possible to compute numerical solutions for polynomial systems of moderate size (a formerly intractable problem), and this has immense practical significance. To familiarize our reader with these tools, we will discuss some aspects of Gröbner bases and give some computational-geometric results. We will also discuss numerical homotopy continuation.

I conclude this description with a rough idea of the organization of this book:

I)	Introduction.	Introduce major applications.	$< 20\%$
II)	Algebraic Geometry	Develop mathematical preliminaries, including mathematical systems theory.	40%
III)	Applications.	Carefully study the applications.	$> 40\%$

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9. **Biographical Sketch.**

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Education

University of Chicago, Ph.D. in Mathematics, August 1994.
Real enumerative geometry for the Grassmannian of lines in projective space
Ph.D. Supervisor: William Fulton

Cambridge University, CPGS, Mathematical Tripos, Part III (with distinction).

Michigan State University, Honors B.S., Physics, summa cum laude.

Employment

August 1998–, assistant professor, University of Massachusetts, Amherst.
(On leave until Summer 2000.)

January 1999–May 2000, van Vleck visiting assistant professor, University of Wisconsin, Madison.

August–December 1998, Postdoctoral Fellow, Mathematical Sciences Research Institute.

July 1994–June 1998, Assistant Professor of Mathematics, University of Toronto.
(Leave of absence: August 1996–July 1997)

August 1996–June 1997, Postdoctoral Fellow, Mathematical Sciences Research Institute.

Awards and Fellowships

University of Toronto Connaught Fund Committee for International Symposia Grant (for Formal Power Series and Algebraic Combinatorics 1998). March, 1997.

NSERC Individual Research Grant # OGP0170279. April 1995—June 1998.

National Science Foundation Graduate Fellow.

Winston Churchill Foundation Graduate Fellow, Churchill College, Cambridge.

Other Professional Activities

Organizing and Programme Committees, 10th International Conference on Formal Power Series and Algebraic Combinatorics, June 1998, Toronto.

Editorial Board of the Young Mathematicians Network. August 1994–August 1999.

Publications most closely related to project

Pieri's formula for flag manifolds and Schubert polynomials, Annales de l'Institut Fourier, 46 (1996), 89–110.

Numerical Schubert calculus, with B. Huber and B. Sturmfels, J. Symb. Comp., 26, (1998), 767–788.

The special Schubert calculus is real. ERA of the AMS, 5 (1999), 35–39.

Real Schubert calculus: Polynomial systems and a conjecture of Shapiro and Shapiro. Exper. Math., to appear.

Real rational curves in Grassmannians. J. AMS, to appear.

Other significant publications

Intersection theory on spherical varieties, with W. Fulton, R. MacPherson, and B. Sturmfels, J. Alg. Geom., 4, (1995), 181–193.

Enumerative geometry for the real Grassmannian of lines in projective space, Duke Math. J., 87, no. 1, (1997), 59–85.

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A monoid for the Grassmannian Bruhat order, with N. Bergeron. Europ. J. Combin., 20, (1999), 197–211.

Hopf algebras and edge-labeled posets, with N. Bergeron, J. Algebra, 216, (1999) 641–651.

Recent collaborators

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Bernd Sturmfels, University of California, Berkeley.
X.-C. Wang, Texas Tech University.
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Stephanie van Willigenburg, York University, Toronto, Canada.

Recent advisees

Not applicable

Graduate advisor and current institution

Wm. Fulton, Professor of Mathematics, University of Michigan, Ann Arbor.

10. **Justification of expense for computer.**

I am a very heavy user of computers due to the large amount of symbolic and experimental work I do. (See [42, 51, 52] and the proposal.) This involves symbolic computations in characteristic zero at the boundary of current feasibility, and these computations are neither parallelizable, nor perform well on common architectures (e.g. Sun workstations). Because of this I have often been constrained by the availability of adequate computing.

To continue my research program and complete this proposed research, I will need a very fast computer with a significant amount of RAM which is dedicated to my use. Specifically, I want to purchase a desktop workstation with an Alpha RISC processor having a speed of at least 600MHz (667MHz is the current fastest) with a large (2-8MB) Cache and a significant amount (1 GB) of RAM. I have priced currently available configurations from both Aspen Systems (`www.aspsys.com`) and Microway (`www.microway.com`), and both can offer a similar configuration for $7000 - $8000. (This will change by Summer 2000, when I plan to buy a computer.)

In my experience, neither Sun nor Intel processors give the performance of an Alpha processor for symbolic computation. Also, the current alternative top of the line configurations do not have comparable clock speeds and cache memory, and at least the Sun Sparc 10 is more expensive than the above configuration.