Combinatorial and Real Algebraic Geometry: Project Description

This proposal is to support my scientific work and that of my advisees in the areas of combinatorial and real algebraic geometry, including related work in applications of algebraic geometry. This will help support members of my research team which will consist of between six and ten junior scientists. This support includes a stipend for a graduate student and support for an undergraduate student to work with me on research, as well as some professional travel for those working on the research from this proposal.

More broadly, this proposal will support my organizational work in the mathematical community, which includes special sessions, conferences, summer schools, and thematic semesters. It will also support my outreach work such as organizing the Texas A&M Math Circle, work at circles and math clubs throughout the US, and my sustained interactions with the Nigerian mathematics community.

1. Intellectual Merits

My current research interests involve algebraic varieties with significant combinatorial structures, such as toric varieties and flag varieties, as well as combinatorial aspects of algebraic geometry, such as tropical varieties and the other 'tropical objects' (amoebae and coamoebea). This focus is in part because it is such objects with structure that we can say the most about, but also because it is precisely these objects which arise in applications. Having long worked in real algebraic geometry, I find it fruitful to investigate questions of reality in algebraic geometry, not only real properties (such as convexity) but also real structures and in particular real solutions to systems of equations. Here as well, understanding real number phenomena is not only intellectually challenging (geometry over $\mathbb{R}$ is typically harder and less uniformly elegant than over $\mathbb{C}$), but it is also important as applications often demand real number solutions.

This work is organized into the three main themes of real toric varieties, tropical geometry, and the Schubert calculus. Each theme builds on current and recent work and includes not only projects whose shape appears evident, but also projects that are less clear, and as always, there are a few more projects than can be done in 36 months, considering the likelihood of new directions arising by then. However, if we knew exactly what would work and which ideas would be the most fruitful, it would not be called research.

1.1. Real Toric Varieties.

Toric varieties are important in algebraic geometry because they are a large class of well-understood varieties that form a laboratory for testing general theories *and* because they often arise in mathematics as key examples or special cases. They are important in applications for essentially the same two reasons: They—particularly real toric varieties—often arise in applications as central objects or important special cases; from there our deep understanding of their structure is brought to bear on the original problem. (This is a main theme of [96].) The real or the positive points of toric varieties are often where the interest lies, because often only these points are physically meaningful.

This area of my research intends to extend the role of and possibilities for real toric varieties in applications. It involves theoretical, including foundational, work on real toric varieties, and addresses practical questions about real points of toric varieties. This is organized into two distinct subthemes, each involving a different notion of real toric variety.

1.1.1. *Background.* A toric variety X [25, 39] has canonical sets of real ($X_{\mathbb{R}}$) and nonnegative ($X_{\geq}$) points. These are CW complexes with the closure of $X_{\geq}$ a ball (think nonconvex polytope with some faces removed), and $X_{\mathbb{R}}$ is obtained by glueing $2^{\dim(X)}$ copies of $X_{\geq}$ along some faces. Davis and Januszkiewicz [28] gave a generalization (small covers) that is part of the field of *toric topology* [22] which is *not* an area of my proposed research.

Geometric modeling gives two other generalizations of real toric varieties that I *am* studying: Real analytic spaces parameterized by monomials whose exponents are real numbers (*irrational toric varieties*) and *real arithmetic toric varieties* which are complex toric varieties whose conjugation is twisted by an involution.

1.1.2. *Toric geometry inspired by geometric modeling.* Geometric modeling uses parameterized curves and surfaces to represent objects on a computer for computer-aided design and manufacture. The basic objects, Bézier curves and surface patches, are images of positive parts of certain toric varieties under linear projections, and methods developed for toric varieties give insight into these and more general *toric patches* [67]. These same basic objects appear as certain log-linear models in algebraic statistics [31], and the flow of ideas across these disciplines has been fruitful. This perspective was developed with García-Puente [27, 41, 42]. With Craciun [27], we adapted a method to prove uniqueness of equilibria in chemical reaction networks [26] to give a geometric-combinatorial criterion implying injectivity of toric patches, and we interpreted toric degenerations corresponding to regular triangulations in terms of geometric modeling. With Zhu [42], we applied a description of the space of toric degenerations [61] of a toric variety $X_{\mathcal{A}}$ to explain the possible limiting positions of a toric Bézier patch and identify these with the secondary polytope of the point configuration $\mathcal{A} \subset \mathbb{Z}^n$ giving the patch. This involved the nonnegative part of a complex toric variety.

Irrational toric varieties. Some of these results extend immediately to *irrational toric varieties*, which give rise to *irrational toric patches* and coincide with general log-linear models in algebraic statistics. Let $\mathcal{A} \subset \mathbb{R}^n$ be a finite set of exponent vectors for monomials in positive variables $x \in \mathbb{R}^n_{>}$. The monomials give a map from $\mathbb{R}^n_{>} \to \mathbb{R}^{\mathcal{A}}_{>}$, and the closure of the image in the positive orthant is an irrational affine toric variety $X_{\mathcal{A}}$ (we index coordinates in $\mathbb{R}^{\mathcal{A}}$ with the elements of $\mathcal{A}$). Intersecting the cone over $X_{\mathcal{A}}$ with the standard simplex $\triangle^{\mathcal{A}}$ gives a closed real analytic subset $Y_{\mathcal{A}}$ of the simplex, which is also an irrational toric variety. It is homeomorphic to the convex hull $\mathrm{conv}(\mathcal{A})$ of the set $\mathcal{A}$ of exponent vectors, and is the analog of the positive part of a (not necessarily normal) projective toric variety.

There are many connections between integer polytopes and toric varieties. These are established by applying tools from algebraic geometry to the complex toric variety X_P associated to an integer polytope P. A guiding principle in this line of my research is to relax the condition that P have integer vertices, and then extend known connections between integer polytopes and toric varieties to connections between arbitrary polytopes $\mathrm{conv}(\mathcal{A})$ and their associated irrational toric varieties $Y_{\mathcal{A}}$. This work requires completely new methods that have no analog in the classical setting, as the tools from algebraic geometry simply do not apply. Establishing such connections in this more general setting not only tells us more about irrational toric varieties (and thus toric patches and log linear models in algebraic statistics), but it gives new proofs and a new understanding of the original classical connection.

Injectivity and positive solutions to systems of polynomials. The result in [27] is formulated in terms of irrational toric patches, and it implies injectivity only in the interior of a patch. Zhu and I [99] gave geometric-combinatorial conditions implying that an irrational surface patch is injective up to and including its boundary. Extending this to higher-dimensional irrational patches appears to require subtle, technical work involving geometric combinatorics. Here there is no difference between classical and irrational patches.

The injectivity result in [27] has been reworked and reapplied back to equilibria of chemical reaction networks [101]. From this it is clear that results giving lower bounds on the number of real solutions to sparse polynomial systems [92, 93] will be useful for chemical reaction networks, if those bounds can be extended to positive solutions. One value is that nontrivial bounds are existence proofs of real solutions. This also needs to be extended to mixed systems (the polynomials have different Newton polytopes), which remains an important open problem in the area. Soprunova and I have a new approach involving sign conditions and winding numbers that reproves the main result in [92], but for positive solutions. In principle the main ideas should generalize to mixed systems, and we plan to work on this.

Foundations of irrational toric varieties. The torus in the ambient projective space acts on a complex projective toric variety $X_{\mathcal{A}}$. Torus translations of $X_{\mathcal{A}}$ and their limiting schemes form a subset of the Hilbert scheme that is the projective toric variety of the secondary polytope of $\mathcal{A}$ [61]. With García-Puente and Zhu [42], we used this to study the space of all patches and their limiting positions with a given set of control points, identifying this space with the secondary polytope.

More recently, Postinghel, Villamizar, and I [85] partially extended this to irrational toric varieties, showing that the combinatorics of the limiting positions of $Y_{\mathcal{A}}$ is encoded by the *secondary fan*, which is the normal fan of the secondary polytope. This was technically challenging and involved establishing several subtle properties about the interaction of sequences with the secondary fan. Our inability to identify the set of limiting positions with the secondary polytope (better, with the corresponding irrational toric variety) is because the theory of irrational toric varieties is currently incomplete.

There are different, but related constructions of classical toric varieties. Affine toric varieties are constructed from sets $\mathcal{A}$ of integer points, projective toric varieties from integer polytopes via line bundles, and abstract toric varieties constructed functorially from rational fans, which is also a geometric invariant theory quotient. For irrational toric varieties, only the first construction is developed. Providing the remaining pieces of this as yet undeveloped theory is an important goal. To an arbitrary fan Σ in $\mathbb{R}^n$ and a continuous semigroup Γ (such as the nonnegative real numbers $\mathbb{R}_{\geq}$), I seek a functorial construction (in Σ and Γ) of a real analytic set $X_\Sigma(\Gamma)$, so that when Σ is the normal fan of the convex hull of a finite set $\mathcal{A} \subset \mathbb{R}^n$ there is a natural homeomorphism between $Y_{\mathcal{A}}$ and $X_\Sigma(\mathbb{R}_{\geq})$ coming from a natural map $X_\Sigma(\mathbb{R}_{\geq}) \to \triangle^{\mathcal{A}}$. This should arise as a categorical quotient of a simplex under a torus action and have other desirable properties. Some of this is clear how to begin, while other parts require more research.

Fibre polytopes and beyond. The first test of/goal for this theory will be if it is powerful enough to complete the result of [85] and identify the space of degenerations of $Y_{\mathcal{A}}$ with the secondary polytope of $\mathcal{A}$. This picture of toric degenerations of a toric subvariety of projective space extends to toric degenerations of a toric subvariety $X_{\mathcal{A}}$ of a toric variety $X_{\mathcal{B}}$, where $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^m$. Such subvarieties come from linear maps $\mathbb{R}^m \to \mathbb{R}^n$ which send $\mathcal{B}$ to $\mathcal{A}$. This can also be viewed as a quotient of the toric variety $X_{\mathcal{B}}$ by the subtorus giving $X_{\mathcal{A}}$. When $\mathcal{A}$ and $\mathcal{B}$ are sets of integer vectors, there is a classical theory which identifies a space of degenerations, called the *Chow quotient*, with the toric variety of the fiber polytope of the map $\mathrm{conv}(\mathcal{B}) \to \mathrm{conv}(\mathcal{A})$ [60]. With a theory of irrational toric varieties from fans, I hope to establish the analogous result for irrational toric varieties, identifying the space of toric degenerations of an irrational toric subvariety $X_{\mathcal{A}}$ of an irrational toric variety $X_{\mathcal{B}}$ with the corresponding fiber polytope.

The theory of toric varieties provides many specific connections between the algebraic geometry of toric varieties and the geometric combinations of rational fans, integer polytopes, and sets $\mathcal{A}$ of integer vectors. A goal of my research on irrational toric varieties is to develop this dictionary for more general fans, polytopes, and point configurations.

1.1.3. *Arithmetic toric varieties.* Both the 2-sphere and hyperboloid of one sheet are quadratic surfaces with complexification isomorphic to the toric variety $\mathbb{CP}^1 \times \mathbb{CP}^1$. The hyperboloid has the standard complex conjugation (real structure), $\overline{(v,w)} = (\overline{v}, \overline{w})$, which is twisted for the sphere, $\overline{(v,w)} = (\overline{w}, \overline{v})$, and both conjugations intertwine the torus $(\mathbb{C}^\times)^2$ action on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Delauney studied such nonstandard real toric varieties [29] and Krasauskas proposed them as exotic patches in geometric modeling [66]. With Elizondo, Lima-Filho, and Teitler, we studied twisted forms of toric varieties for any field, calling them *arithmetic toric varieties* [34]. This extended earlier work [108]. Together with a contemporaneous work [56] and a deeper extension [32], these, particularly real arithmetic toric varieties, are now classified.

This classification allows the development of further topics that I intend to work on. One topic is foundational, seeking arithmetic analogs of aspects of classical toric varieties, and the others use these new objects as a laboratory for studying real-number phenomena.

Arithmetic affine toric varieties. Abstract (normal) toric varieties correspond to fans Σ, which are collections of finitely generated saturated subsemigroups of a finitely generated free abelian group N. A fan Σ and a field k together give a split toric variety $X_\Sigma(k)$ with torus $\mathbb{T}_N := \mathrm{spec}(k[M])$ where $M := \mathrm{Hom}(N, Z)$ is the character group of $\mathbb{T}_N$. In an arithmetic toric variety, the Galois action on $X_\Sigma(\overline{k})$ is twisted by a representation of the Galois group of $\overline{k}/k$ in the group of toric automorphisms of X_Σ, and this action satisfies descent.

A classical affine toric variety $X_{\mathcal{A}}$ is the closure of the image of $\mathbb{T}_N(k)$ in the affine space $k^{\mathcal{A}}$ under the map $\varphi_{\mathcal{A}} \colon \mathbb{T}_N(k) \to k^{\mathcal{A}}$ given by a finite set $\mathcal{A} \subset M$ of characters. Such a parametrization may be twisted by an action of the Galois group on M that preserves $\mathcal{A}$, and we obtain an *arithmetic affine toric variety* (descent is automatically satisfied). For example, the trigonometric moment curve,

$$\theta \ \longmapsto\ (1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos n\theta, \sin n\theta)\,,$$

comes from the rational normal curve

$$t \ \longmapsto\ (t^{-n}, \ldots, t^{-2}, t^{-1}, 1, t, t^2, \ldots, t^n) \qquad (x_i = t^i,\ i = -n, \ldots, n)$$

with twisted conjugation $\overline{(x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n)} := (\overline{x_n}, \ldots, \overline{x_1}, \overline{x_0}, \overline{x_{-1}}, \ldots, \overline{x_{-n}})$.

The ideal of an affine toric variety (a *toric ideal*) has an elegant description [103] as the linear span of binomials $x^u - x^v$ where $\mathcal{A}u = \mathcal{A}v$ ($\mathcal{A}$ is a map $\mathbb{N}^{\mathcal{A}} \to M$). I intend to develop a theory of ideals of arithmetic toric varieties. This is nontrivial because taking invariants is not an exact functor, so the *arithmetic toric ideal* (ideal of the invariants) is not necessarily the invariants of the toric ideal. There are other topics in toric geometry, such as toric degenerations, whose arithmetic analogs and consequences I also want to understand.

Convex geometry of arithmetic affine toric varieties. Like the trigonometric moment curve or sphere, the set of real points of an affine arithmetic toric variety is often compact. I plan to study the convex geometry of arithmetic toric varieties as laid out in [89]. Of particular interest is the algebraic (Zariski closure of the) boundary of these convex hulls, as well as their facial structure. The zeroth step is understanding arithmetic toric ideals.

This question is motivated by work of Vinzant [107]. The convex hull of the trigonometric moment curve is well-understood (e.g. §5 of [89]). It is a spectrahedron and every face is *exposed* (there is a linear function whose minimum on the convex hull is attained exactly on that face). In contrast, Vinzant showed that the convex hull of the odd moment curve,

$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \ldots, \cos(2k{-}1)\theta, \sin(2k{-}1)\theta)\,,$$

has many faces that are not exposed, and is therefore not a spectrahedron. One difference between these two curves is that the underlying toric variety of the trigonometric moment curve is normal, while that for the odd moment curve is not normal.

I want to understand which arithmetic affine toric varieties have convex hulls that are spectrahedra (these are the possible domains for optimization via a semidefinite programming). A motivating question is how do combinatorial properties of the set $\mathcal{A}$ of exponent vectors, as well as the Galois action on the toric variety $X_{\mathcal{A}}$ and its geometry, affect the existence of nonexposed faces in its convex hull? There are theoretical reasons that normality may be necessary—nonnormal toric varieties are linear projections of normal toric varieties, and projections of convex bodies often have non exposed faces.

Equivariant cohomology of arithmetic affine toric varieties. A complex toric variety X is equipped with an action of the torus $\mathbb{T} = (\mathbb{C}^\times)^{\dim X}$, and its $\mathbb{T}$-equivariant cohomology is an important invariant. The additional data of a real structure leads to finer invariants. One of the finest is bigraded Bredon cohomology [20], which is subtle to compute and has important connections to algebraic cycles and motivic cohomology [30]. My colleague Lima-Filho and I plan to compute such finer equivariant cohomology theories for arithmetic toric varieties. He brings to this project experience computing Bredon cohomology for real quadrics and I bring my familiarity with the combinatorics of toric varieties. We expect this to lead to new tools for and a better understanding of Bredon cohomology.

1.2. **Tropical topology and the structure of amoebae.** The group of non-zero complex numbers $\mathbb{C}^\times$ decomposes as a product $\mathbb{R} \times \mathbb{U}$ ($\mathbb{U} := \mathbb{R}/2\pi\mathbb{Z}$ is the circle group) under the map $(r,\theta) \mapsto e^{r+\sqrt{-1}\theta}$. Given a subvariety V of the algebraic torus $(\mathbb{C}^\times)^n$, its *amoeba* $\mathcal{A}(V)$ and *coamoeba*, *co*$\mathcal{A}(V)$, are its images under the projections to $\mathbb{R}^n$ and $\mathbb{U}^n$, respectively.

Work of Bergman [14] and Bieri-Groves [15] showed that the limiting directions of the amoeba of V (its *logarithmic limit set*, $\mathcal{L}^\infty(V)$) is a polyhedral complex in the sphere S^{n-1}. Speyer and Sturmfels [100], recognized the cone over $\mathcal{L}^\infty(V)$ as the *tropical variety* $\mathcal{T}(V)$ of V, which is a rational polyhedral fan, noncanonically.

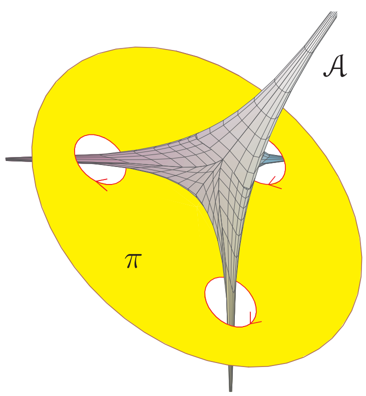
Kapranov [33] gave another link between tropical varieties and amoebae. Let $\mathbb{K}$ be a valued local field with residue field $\mathbb{C}$ and value group $\mathbb{R}$. The valuation is a homomorphism $\nu\colon \mathbb{K}^\times \twoheadrightarrow \mathbb{R}$, $\nu(ab) = \nu(a) + \nu(b)$, for which $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$ with equality if $\nu(a) \neq \nu(b)$ and where $\nu(0) := \infty$. Then $R := \nu^{-1}[0,\infty]$ is its valuation ring with maximal ideal $\mathfrak{m} := \nu^{-1}(0,\infty]$ where $R/\mathfrak{m} = \mathbb{C}$. The image in $\mathbb{R}^n$ of a subvariety $V \subset (\mathbb{K}^\times)^n$ under the coordinatewise valuation map is its *non-archimedean amoeba* or *tropical variety*.

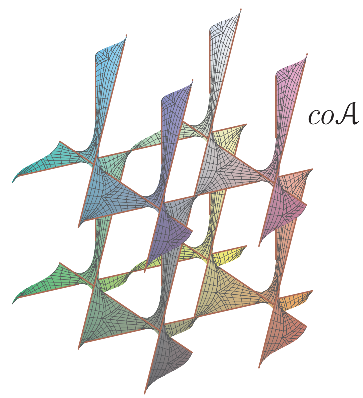
I have become interested in certain structural and topological properties of these tropical objects: amoebae, coamoebae, and tropical varieties. This involves work with former TAMU postdoc Mounir Nisse and current TAMU postdoc Timo de Wolff.

1.2.1. *Tropical topology.* Just as amoebae, coamoebae, and tropical varieties are shadows of varieties in $(\mathbb{C}^\times)^n$, their complements are related to the complement of the original variety. As these tropical objects have structure controlled (in part) by geometric combinatorics, we hope to begin to understand the topology of complements of a variety by first determining the topology of the complements of the associated tropical objects. My goal is to develop these tropical objects as tools to understand the topology of subvarieties of tori and their complements.

Higher convexity. A fundamental property of an amoeba of a hypersurface is that every component of its complement is convex [43]. Nisse proved the same result for hypersurface coamoebae [77] (lifted to the universal cover $\mathbb{R}^n$ of $\mathbb{U}^n$). Henriques generalized convexity using homology: an open subset $X \subset \mathbb{R}^n$ is *k-convex* if for every affine plane π of dimension $k{+}1$, the natural map on reduced homology, $\iota_k \colon \widetilde{H}_k(X \cap \pi) \to \widetilde{H}_k(X)$, is injective. Convex and connected is 0-convex. Henriques conjectured that the complement of an amoeba of codimension $k{+}1$ is k-convex. He proved a weak version of this conjecture—that the map ι_k sends no positive cycle to zero [52]. Bushueva and Tsikh showed that complements of amoebae of complete intersections have this higher convexity [23]. Higher convexity of general amoebae remains open. We illustrate this with pictures of the amoeba (on the left) and coamoeba (displayed in eight fundamental domains) of the same line in $(\mathbb{C}^\times)^3$.

(1)





Both are 1-convex; for the amoeba this is illustrated by the cycles on the affine plane π.

Nisse and I used our description of the phase limit set of a coamoeba [81] (the analog of $\mathcal{L}^\infty(V)$) to prove that the coamoeba of a variety $V \subset (\mathbb{C}^\times)^n$ of codimension $k{+}1$ is k-convex [82]. Jonsson showed that tropical varieties are limits of amoebae [57], and we hope to use this to extend the results of Henriques and of Bushueva-Tsikh to tropical varieties: that the map ι_k sends no positive cycle to zero and is injective for compete intersections.

Abstractly, a tropical variety is a polyhedral complex of pure codimension in $\mathbb{R}^n$ that is weighted and balanced. With Nisse, we can show that complements of abstract tropical hypersurfaces and curves are 0-convex and $(n{-}2)$-convex, respectively. We hope to use the local properties of tropical varieties to establish higher convexity of their complements. We also see a possibility for using higher convexity of complements of tropical varieties and Jonsson's limit theorem to prove the higher convexity of complements of general amoebae.

Higher solidity and beyond. Another topological property of complements of some amoebae is solidity: a hypersurface amoeba is *0-solid* if its complement in $\mathbb{R}^n$ has no bounded components. Amoebae of $\mathcal{A}$-discriminants [83] and of hypersurfaces defined by maximally sparse polynomials [76] are solid. For any variety $V \subset (\mathbb{C}^\times)^n$ there is a natural map $\phi_k \colon \widetilde{H}_k(S^{n-1} \smallsetminus \mathcal{L}^\infty(V)) \to \widetilde{H}_k(\mathbb{R}^n \smallsetminus \mathcal{A}(V))$ from the reduced homology of the complement of the logarithmic limit set to the reduced homology of the complement of the amoeba. This is injective when V is a hypersurface and $k = 0$ [38], by the order map from complex analysis. Nisse and I conjecture that this is always an injection when V is a variety of codimension $k+1$. We can show that ϕ_{n-2} is an injection when V is a curve.

We say that a variety V of codimension $k+1$ is *k-solid* if this natural map ϕ_k is an isomorphism. We can show that amoebae of lines in $(\mathbb{C}^\times)^n$ are $n-2$ solid and we conjecture that amoebae of linear subspaces are solid in this sense. We are quite close; there is a diagram of natural maps forming a hexagon, of which ϕ_k is one edge and four of the others are isomorphisms, but have not yet been able to show that the hexagon commutes.

Our injectivity conjecture may be the first step towards a generalization of the order map. The injective order map sends a component of the complement of a hypersurface amoeba $\mathcal{A}(V)$ to a monomial in the defining equation of V that is, from $\pi_0(\mathbb{R}^n \smallsetminus \mathcal{A}(V))$ to $\mathrm{Hom}((\mathbb{C}^\times)^n, \mathbb{C}^\times)$. It is the derivative of the Ronkin function on the complement. We may extend this to a map from $H_0(\mathbb{R}^n \smallsetminus \mathcal{A}(V))$ to $H^1((\mathbb{C}^\times)^n)$. Next would be to define a map from $H_k(\mathbb{R}^n \smallsetminus \mathcal{A}(V))$ to $H^{k+1}((\mathbb{C}^\times)^n)$, when V has codimension $k+1$, which would be a "cohomological order map". The last step in this only-partially-baked-program would be to generalize the Ronkin map (a higher Ronkin map?) so that its derivative is this cohomological order map.

1.2.2. *Coamoebae of linear spaces.* Nillson and Passare [75] proved a beautiful result about the structure of the coamoeba of a reduced $\mathcal{A}$-discriminant (which arises in $\mathcal{A}$-hypergeometric functions and sparse polynomial systems), when it is a curve. In this case, they showed that the coamoeba of the reduced $\mathcal{A}$-discriminant is the complement of the zonotope given by the vector configuration $\mathcal{B}$ Gale dual to the vector configuration $\mathcal{A}$. We may see this in the coamoeba in (2), where the hexagon in the complement of the coamoeba is the zonotope generated by $\mathcal{B} := \{(1,0), (0,1), (-1,-1)\}$. Passare and I used the Horn-Kapranov parametrization [59] of the reduced $\mathcal{A}$-discriminant to give a conceptual proof of this result [84]. By the Horn-Kapranov parametrization, the coamoeba of the reduced $\mathcal{A}$-discriminant is the image of the coamoeba of a real affine line under the linear map given by $\mathcal{B}$. The structure of the coamoeba of that line and its interaction with $\mathcal{B}$ give the result.

Nisse and I seek to generalize this work. A first step is to describe the coamoeba of a real affine space in $(\mathbb{C}^\times)^n$, and we have a promising partial description. Next, we need to understand its image under the linear map given by the vector configuration $\mathcal{B}$. This is challenging because it is hard to visualize the geometry (visualization was key to formulating and proving the result in [84]). An additional motivation for this work is that one approach to the monodromy of $\mathcal{A}$-hypergeometric functions begins with a description of the coamoeba of the reduced $\mathcal{A}$-discriminant.

1.2.3. *Structure of amoebae and coamoebae.* Mathematicians have tried to understand amoebae algorithmically. For example, what are natural representations for amoebae and coamoebae, and how do we compute them? Theobald [104] pioneered the computation and visualization of amoebae. Later, Purbhoo [87] gave a version of the Nullstellensatz for amoebae, proving that for every $x \notin \mathcal{A}(V)$, there is a polynomial f vanishing on V with $x \notin \mathcal{A}(f)$, the amoeba of the hypersurface defined by f. A *basis* for an amoeba $\mathcal{A}(V)$ is a set S of polynomials such that $\mathcal{A}(V)$ is the intersection of the $\mathcal{A}(f)$ for $f \in S$. Purbhoo's result implies that the ideal of V is a basis for $\mathcal{A}(V)$ and the question arose if a finite subset would suffice. While finiteness was not believed to hold in general, it was only settled recently.

Building on work of Schroeter and de Wolff [90], Nisse proved finiteness when V is zero-dimensional, giving bounds on the number of degree of the hypersurfaces needed [79]. He also showed that if V is not a hypersurface and not zero-dimensional, then $\mathcal{A}(V)$ is not the intersection of finitely many hypersurface amoebae [78]. His proof is difficult and does not give a conceptual understanding of the obstruction to finiteness.

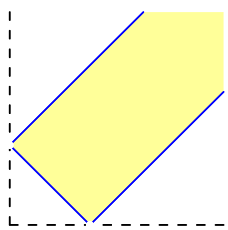
Semialgebraic descriptions. I believe that the difficulty here, both in the results and in their limitations, is that we have not been asking the right questions. A complex algebraic variety $V \subset (\mathbb{C}^\times)^n$ is a real algebraic subvariety of $\mathbb{R}^n_> \times \mathbb{U}^n$, where $\mathbb{R}_>$ is the positive real numbers and $\mathbb{U} \subset \mathbb{C}$ is the circle group, and we identify $\mathbb{C}^\times$ with $\mathbb{R}_> \times \mathbb{U}$ under the map $(r,\theta) \mapsto r\cdot\theta$. By the Tarski-Seidenberg Theorem, the projections of V to $\mathbb{R}^n_>$ (its *algebraic amoeba*, $al\mathcal{A}(V)$) and to $\mathbb{U}^n$ (its coamoeba) are semialgebraic sets. Thus, both the algebraic amoeba and coamoeba should be described as semialgebraic sets. For example, when $V \subset (\mathbb{C}^\times)^2$ is the line $x+y+1=0$, these are (we give the closure of the coamoeba):

(2)

$al\mathcal{A}(V)$

$r+s \geq 1$

$|r-s| \leq 1$

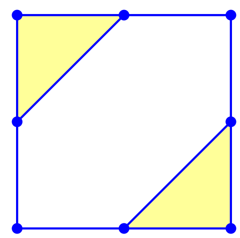


$\overline{co\mathcal{A}(V)}$

$(r,s) \in [-\pi,\pi]^2$

$0 \leq r+\pi \leq s$

or $0 \geq s+\pi \geq r$



We are only at the beginning of this project, so it is hard to predict how it will progress. However, we do have sufficient background and interest in this question.

Boundary of amoebae. While I have the opinion that previous algorithmic work on amoebae may not have addressed the right questions, I want to give a new proof of Nisse's result that a positive dimensional amoebae that is not a hypersurface is not the intersection of finitely many hypersurface amoebae. Embedded in Nisse's proof is the observation that the boundary of a hypersurface amoeba is positively curved, which he uses very indirectly to obtain a contradiction to finiteness for non-hypersurface amoeba. His arguments are difficult, and I seek a direct proof that explains the result.

With de Wolff, we will study the curvature of the boundary of amoebae and coamoebae, at smooth points. The curvature of both the amoeba and coamoeba in (1) is negative at almost all points. It is reasonable to expect that this holds almost always. Even when it does not, there may be some tractable local geometric structure. This may be consequence of (or equivalent to) higher convexity of the complements. The motivation for this last belief is that convexity of a closed set is equivalent to local convexity at all points, and thus smooth boundary points have positive curvature. Establishing such a local property should imply Nisse's theorem, for the boundary of a finite intersection of hypersurface amoebae, if positive dimensional, would have some positivity in its curvature at smooth points.

1.3. **Schubert Calculus.** Schubert calculus has come to mean the geometry, algebra, and combinatorics of Schubert varieties in homogeneous spaces. While intensely studied, fundamental positivity questions remain open, and I believe there are still elegant formulae to discover. Enumerative geometry problems coming from intersecting Schubert varieties, *Schubert problems*, are a rich, well-understood, and well-structured class of problems. Schubert problems, like toric varieties, form a laboratory for investigating new phenomena in enumerative geometry. My current research in Schubert calculus falls into two categories, one involving the combinatorics of the Schubert basis of cohomology, and the other investigating Galois groups of Schubert problems.

1.3.1. *Combinatorics of the Schubert basis.* The cohomology ring of a flag manifold has a distinguished *Schubert basis* coming from its Schubert varieties. A major open problem is to find formulae for the *Schubert structure constants* expressing the multiplication in this basis. These constants are generalizations of Littlewood-Richardson coefficients, and for many cohomology theories, they are known to be positive in an appropriate sense through geometric arguments. Even for ordinary cohomology, until recently there was no combinatorial proof of positivity or formula for any of these coefficients, besides the classical Littlewood-Richardson formula for Grassmannians, and a few special cases [62, 74, 94].

Positivity of Schubert vs. Schur coefficients. Assaf, Bergeron, and I recently solved a 20-year old problem, giving a combinatorial proof of positivity for *Schubert vs. Schur coefficients* [2], which are structure constants for the product with a class pulled back from a Grassmannian. In [11, 13] a quasisymmetric generating function for chains in intervals of the *Grassmannian-Bruhat order* on the symmetric group was defined using the Pieri formula [86, 94]. This quasisymmetric function is symmetric and the Schubert vs. Schur coefficients appear in its expansion into Schur functions. In 2011, we three realized that the detailed study of chains in this suborder [12] (from 1999) implied many of Assaf's axioms for a weak dual equivalence [3, 4] for these sets of chains. Subsequent improvements in Assaf's theory coupled with 1.4 GHz-weeks of computer computation and theoretical analysis showed that there is a unique structure of a weak dual equivalence on these sets of chains, which implies positivity for Schubert vs. Schur coefficients.

This approach, suitably modified, should prove positivity of Schubert vs. Schur coefficients for the symplectic (type C) flag manifold. Those coefficients come from the product of a Schubert class with one pulled back from the Lagrangian Grassmannian, and were studied in [10]. This program requires developing type C versions of the ingredients behind [2].

The first tiny steps have been accomplished. Our theory of peak enumeration [9] and the Pieri formula of [10] give an Eulerian quasisymmetric function associated to intervals in an order (the *Lagrangian-Bruhat order*) on the Weyl group of type C whose expansion in Schur P-functions gives the type C Schubert vs. Schur coefficients. Separately, Assaf [5] and Billey, et al. [16] defined type C dual equivalence graphs. A next step is a detailed (but likely straightforward) study of chains in the Lagrangian-Bruhat order, generalizing [12].

Giving a useful type C version of weak dual equivalence will require sustained, creative effort as that was a difficult and subtle part of [4]. I suspect that this effort will require or result in a simplification or reformulation of the notion of weak dual equivalence. Bringing these parts together may be theoretically straightforward but computationally challenging, as it should require several orders of magnitude more computation than [2].

Explicit formulae in Schubert calculus. Finding explicit formulae in Schubert calculus is a longstanding interest. With Morrison, we have found a Murnaghan-Nakayama formula in the quantum cohomology rings of the Grassmannian and the flag manifold—this is a formula for multiplying by the analog of a power sum symmetric function. Our formula is a pleasing signed sum of Schubert classes, completely analogous to the classical Murnaghan-Nakayama formula. We believe that we can lift it to a similarly beautiful formula in the Fomin-Kirrilov quadratic algebra [35] using the hook formula of Mészáros, et al. [73]. Explicit computations in the Grothendieck ring are promising and suggest the possibility of extending this to the K-theory of the flag manifold.

Bergeron and I [11] discovered remarkable formulae that come from specializing variables in a Schubert polynomial, equivalently, from pulling back along an embedding of a smaller flag manifold into a larger one, and later Lenart, Robinson, and I generalized these formulae to K-theory [69]. In the meantime, Billey and Braden generalized these embeddings to all Lie types [18]; they are the geometric counterpart of permutation patterns [17].

With Dr. Praise Adeyemo of the University of Ibadan in Nigeria, we extended the specialization formulas in [11, 69] to all types using the maps of Billey and Braden [1]. This was done on my summer 2014 trip to Nigeria, with Adeyemo learning some additional background in the months preceding my visit. His University will send him to Texas A&M for a semester in 2015 to continue our collaboration; we plan to extend our results to equivariant and more general cohomology theories. When we met in 2012 and I learned of his background in Schubert calculus, I proposed this project not only because I was interested in the topic, but also as a way to establish scientific connections with the Nigerian Mathematical community by mentoring a promising faculty member at one of Nigeria's leading universities.

1.3.2. *Galois groups of Schubert problems.* Problems in enumerative geometry, like field extensions or polynomials, have Galois groups encoding their intrinsic structure. This has been understood since at least 1870 [58], and their equivalence to monodromy groups is even earlier [53]. It was expected that the Galois groups of enumerative problems are nearly always the full symmetric group, with a few sporadic exceptions. This expectation persisted I think because work of Harris [44] showed that several were full symmetric and because the difficulty in determining these groups led to a dearth of examples.

This belief was turned on its head by Vakil, who gave combinatorial criteria that could be used to show that the Galois group of a Schubert problem was nearly symmetric in that it contained the alternating group [105, 106]. We say that a Schubert problem whose Galois group contains the alternating group *at least alternating*, and one that does not is *deficient*. Deficient problems possess interesting internal structure. Vakil wrote a Maple script and used it to investigate all Schubert problems on Grassmannians of k-planes in n-space ($G(k,n)$) for $k(n-k) \leq 18$, and a few more with $k = 2, 3$. He found several candidate deficient problems on $G(4,8)$, one of which Derksen proved was deficient, and then Vakil generalized Derksen's example to an infinite family of deficient Schubert problems, with at least one in every Grassmannian $G(k,n)$ with $k, n-k \geq 4$.

Leykin and I directly computed Galois groups of many simple (all but two Schubert conditions are hypersurfaces) Schubert problems on Grassmannians, and all were the full symmetric group [70]. More recently, I have begun to systematically study these Galois groups, and a picture is emerging of a potential classification of deficient Schubert problems. Its most potent feature are two dichotomies: (1) Either a Schubert problem is at least alternating, and therefore acts highly transitively on the solutions, or its action is imprimitive and it fails to be 2-transitive. Thus the inverse Galois problem fails for Schubert problems. Also, (2) all Schubert problems in a Grassmannian $G(k,n)$ with $\min\{k, n-k\} < 4$ have 2-transitive Galois groups and every $G(k,n)$ with $k, n-k \geq 4$ has a deficient Schubert problem.

Transitivity and Vakil's criteria. Vakil gave two criteria to show that a Schubert problem is at least alternating. One is purely combinatorial and the other uses knowledge of 2-transitivity. With undergraduate Brooks and graduate student Martín del Campo, we implemented Vakil's first criterion in `python` and found that all Schubert problems on $G(2,n)$

were at least alternating for $n \leq 35$. This empowered us to find a very difficult, but elementary elementary proof that every Schubert problem on a Grassmannian $G(2, n)$ is at least alternating [21]. That was the first example of a large and intricate class of enumerative problems whose Galois groups were essentially determined.

Postdoc White and I are studying 2-transitivity. This is equivalent to the irreducibility of a particular subset of an incidence variety constructed from the Schubert problem and from Schubert varieties. Studying these incidence varieties and this subset involves a mixture of combinatorics and geometric reasoning. Our first paper showed that all Schubert problems on $G(k, n)$ for $\min\{k, n-k\} < 4$ were 2-transitive [98]. Together with Vakil's second criterion and some combinatorics, this gives an easier proof that Schubert problems on $G(2, n)$ are at least alternating. We are working on a sequel, and have shown that all simple Schubert problems are 2-transitive, as are all that involve at most one partition with more than one part. We hope to prove that all simple Schubert problems are at least alternating, which is nearly the conjecture of [70] that their Galois groups are all full symmetric. Our eventual goal is to show that all Schubert problems on $G(3, n)$ are at least alternating, which will require a detailed analysis of incidence varieties constructed from Vakil's *checkerboard varieties*.

Computational study of Galois groups. This theoretical work to understand Galois groups of Schubert problems would be impossible without computer calculations which gave a fairly small number of Schubert problems warranting further study for deficiency. Vakil implemented a massively recursive algorithm based on his first criterion in Maple. He used it to find the first example of a deficient Schubert problem in $G(4, 8)$ and he found no deficient Schubert problems in any Grassmannians $G(2, n)$ or $G(3, n)$. This motivated my work with Brooks, et al. [21], and with White [98].

Brooks and now undergraduate Moore implemented Vakil's algorithm in `python` with a vastly improved efficiency: It finds the approximately 350 candidate deficient Schubert problems out of 3,500 on $G(4, 9)$ in about 5 Giga Hertz hours, while Vakil's code completes the same task in over a Giga Hertz month. Undergraduate student Maril is reimplementing it in `haskell`, a computer language whose structure is optimized for massively recursive algorithms. We plan to use his software on a supercomputer to investigate hundreds of millions of Schubert problems on small Grassmannians for possible deficiency and further study. These data will be crucial for developing and testing the conjectural classification of deficient problems.

Symbolic computation gives a method to experimentally determine the Galois group of any Schubert problem of moderate size, including proving that it is full symmetric. An instance of a particular Schubert problem is given by a choice of general flags. If the flags are defined over the rational numbers, then the solutions are modeled by the solutions to a system of polynomial equations with integer coefficients. Reducing modulo a prime number p and computing an eliminant gives a polynomial $f(x) \in \mathbb{F}_p[x]$. When square-free, the degrees of its factors give the cycle type of the Frobenius automorphism acting on the solutions in $\overline{\mathbb{F}_p}$. This lifts to a *Frobenius element* in the Galois group of the Schubert problem with the same cycle type, and the Chebotarev density theorem asserts that these are uniformly distributed for p large enough (In our calculations, 'large enough' often means $p > 5$).

Computing sufficiently many Frobenius elements identifies a candidate Galois group. If the candidate is the full symmetric group, we are done, and if not then the candidate's structure informs geometric arguments to show that it is the actual Galois group. This *Frobenius method* is feasible for Schubert problems with $\lesssim 500$ solutions and a formulation with fewer than 20 variables. It is also not confined to Schubert problems and may be applied to nearly any enumerative problem of moderate size. We intend to use this Frobenius method, either alone or with our `haskell` implementation of Vakil's algorithm, to determine the Galois groups of all computable Schubert problems to inform our classification. The scope of this computation will require a supercomputer and it will be modeled on our reality experiments [40, 48, 49].

A model for this investigation is my group's project for this academic year. Using our list of approximately 350 candidate deficient Schubert problems on $G(4,9)$, we apply the Frobenius method to each to identify its Galois group. Previously, this was partially done on an ad hoc basis, and we are writing a library to automate this (and future) computation. Already at this early stage, we have classified nearly all Schubert problems on $G(4,9)$. The deficient Schubert problems all fall into easily identifiable families of problems whose internal structure is now well-understood. None has more than 10 solutions (in contrast, there are Schubert problems on $G(4,9)$ with more than a million solutions), and the observed Galois groups are either wreath products such as $(S_5 \times S_5) \rtimes S_2$ or $(S_2)^3 \rtimes S_3$, or S_4 acting on the six pairs in $\{1,2,3,4\}$. All but one of these families generalizes a deficient Schubert problem in $G(4,8)$. For every deficient Schubert problem, the structure of its Galois group as a permutation group reflects intrinsic structure of the geometric problem. Also, for nearly every deficient Schubert problem, there is observed structure in its possible number of real solutions. This emerging picture in $G(4,9)$ (which we are writing up) suggests that a classification of deficient Schubert problems in Grassmannians is possible and possibly in reach. Working towards this classification is a goal of my research.

2. Broader Impacts

Just as Man does not live by bread alone, the health of Mathematics requires that we do more than conduct research. A scientist has a duty both to advance scientific knowledge and to maintain the infrastructure of science. For me, this second task includes training future scientists, organizing workshops, summer schools, and thematic semesters, and conducting outreach. I am requesting support to help me to conduct the research described in §1 and to conduct activities which have impact beyond that core intellectual inquiry.

2.1. Training and Mentoring.

I am in active research collaboration with over 28 mathematicians, including ten with Ph.D.s who are pre-tenure, three graduate students, and two undergraduate students. I use collaboration with junior scientists as a tool to train them in all aspects of the profession.

In recent years my research team at Texas A&M has had up to eleven members: undergraduate students, graduate students, and postdocs. In Spring 2015, there will be at least nine, including four graduate students (one on a 12-month fellowship from China) and four postdocs. On top of one-on-one collaborations, I run collaborative research projects. These include large computer investigations as in [40, 48, 49] or as described in § 1.3.2. These complex experiments involve the integration and development of software tools that enable them to run for months on supercomputers. They lead to traditional mathematical research [21, 50, 51, 98] and are a vehicle for training and mentoring. The new members learn the essential mathematics we are investigating, everyone gains experience with teamwork and scientific discovery through advanced computing, and the older members learn mentoring skills.

My advising is both one-on-one and in group settings. My collaborations at TAMU have regular meetings at which everyone describes their work and we discuss our research objectives and the mathematical background. I hold a weekly group lunch for those I advise during which we discuss professional matters—everything from classes to teaching to research to applying for jobs/graduate school to professional travel to preparing papers and presentations. When I have a visitor, they come along to share their experiences. This helps my advisees learn the conduct and culture of our profession.

I help everyone get invitations to give talks and support their travel when necessary. They have regular opportunities to speak in the seminars I attend and in the Graduate Student Organization Seminar. The Algebraic Geometry Seminar is usually co-organized by a postdoc, which gives them useful organizational experience. Before members of my group give talks or posters, we have a group meeting at which they practice their presentation, followed by a group discussion. This directly helps the speakers and helps members of my group develop critical thinking about professional presentations.

I work directly with the students and postdocs on professional matters. I help the postdocs write grants; many postdocs who have worked with me at Texas A&M have been a PI or co-PI on a funded grant. We discuss writing. I send/direct referee requests at least once to each graduate student and postdoc, and then work with them on their report to help them learn about the review process. I give them copies of some of the many excellent books written on professional development and writing [8, 24, 64, 63, 65, 102, 109], and we discuss these. I am committed to turning out well-rounded professionals.

Building on my experiences as an early leader in the Young Mathematicians' Network, as an applicant, and as an advisor, I have organized panel discussions at Texas A&M for our students and postdocs applying for their next jobs. I involve those who I supervise as panelists when appropriate.

2.1.1. *Support for Training and Mentoring.* This proposal asks for funds to support these activities. I am asking for partial support (stipend+tuition) for one graduate student, which I expect will be used by different students in different semesters. Currently there are three graduates student working with me; with several others expressing interest. This support will enable some students to have more time for collaboration outside my team and for travel, and to attend thematic programs, including the program on combinatorial algebraic geometry at the Fields Institute in Fall 2016. I also am asking for funds to support one undergraduate part time (8hrs./week) during the academic semesters and full time over the summers.

Part of the funds I request for travel are to help these junior members of my team travel to conferences, including the MathFest and Joint Mathematical Meetings for undergraduates, and important international conferences for postdocs and graduate students.

2.2. **Conference organization.** Mathematics is a human subject, and it is absolutely necessary to gather people to share their ideas. Important things happen when you get the right people together for a weekend or a week. I typically organize several conferences each year. In 2014–5 these include a workshop at the Simons Institute for the Theory of Computing (October), Texas Algebraic Geometry Symposium (April), and I am on the programme committee for FPSAC 2015 in Korea (July). Some that are in the planning/proposal stage include a summer school in Nigeria in 2016, a semester on Matroids and Algebraic Geometry at the Newton Institute (our proposal has not yet been approved, but it has been revised and resubmitted) and another for a semester at the ICERM in Providence, RI (this is in the preproposal stage).

2.3. **Outreach.** There is a great need, both at home and abroad, for the development of mathematics outside of the high-level research community in which we work. I am committed to sustained mathematical activity outside of our research community.

I have been giving talks to math clubs and volunteering at outreach events since my first academic job in 1994. In 2011 I joined TAMU faculty Yasskin and Sprintson to organize a middle school math circle. I recruit graduate students and postdocs to help with the circle with the goal of influencing some to make mathematical outreach a lifelong activity. This is also one of my mentoring activities. I expect this weekly commitment to continue for many years, and am asking for $500 each year for supplies, to bring visitors from other circles, and to support my (or perhaps TAMU graduate students) traveling to other circles within Texas. Whenever I travel professionally, I try to schedule an outreach activity during my visit.

I am in the middle of a multi-year project of mathematical outreach in Nigeria. Nigeria is the giant of Africa; with over 160 million people it has great potential, but also great problems. It has over one hundred Universities, many Ph.D.-granting mathematics departments, a National Mathematics Center in Abuja, and a few mathematicians with international reputations. On my first trip in June 2012, I was struck by the enthusiasm of the people I met, both students and professors, for serious interaction with me as a representative of the international mathematical community. I was also struck by the lack of contact of many with current trends in mathematics. I talked to many people, made many presentations to students, and started to make contacts within the Nigerian Mathematical Community.

A UNESCO supported summer school I was organizing for June 2013 was cancelled due to security concerns, but in January 2014 I hosted a visitor (Prof. M.O. Ibrahim of the University of Ilorin, who is President-Elect of the Mathematical Association of Nigeria), and returned to Nigeria for two weeks in June 2014 to work with collaborator Dr. P. Adeyemo of Ibadan and also to visit Ibrahim at Ilorin. Adeyemo will visit me for a semester in 2015, and we are looking to organize a summer school in Nigeria, possibly in 2016. I plan to continue this engagement with Nigerian Mathematics with regular visits and hope to involve others in this outreach (already Rudy Yoshida of Kentucky has expressed interest), organizing meetings or giving advanced courses, collaborating with people I meet, or training others who will return to Nigeria and help to build their infrastructure for mathematics. For example, I have recruited a graduate student from Ibadan whom I met on my visits there. A portion of my international travel will be for this activity.

3. Results from Prior Support

In the past five years, I have been a PI on the following two grants:

Award No.	Amount	Support Period	Title
DMS-1001615	$235,395	8/1/10—7/31/14	Applications and Combinatorics in Algebraic Geometry
DMS-0915211	$435,757	8/1/09—7/31/12	Numerical Real Algebraic Geometry

I give highlights of the resulting work.

Intellectual merit. These grants supported my work in several partially overlapping areas of mathematics, and in this period I completed 34 papers, including 10 in the past 12 months (to October 2014). This included work in combinatorial Hopf algebras [36, 37, 68], tropical geometry [80, 81, 82, 84], real toric varieties [34, 42, 85, 93], applications of algebraic geometry [42, 55, 88, 89, 93, 95, 96, 99], numerical algebraic geometry [6, 7, 45, 46, 47, 97], and the Schubert calculus: real [40, 48, 49, 50, 51, 54, 72, 95], combinatorial [1, 2, 68], and Galois groups of Schubert problems [21, 72, 98].

Some highlights of this work supported by the grant DMS-1001615 include the book *Real Solutions to Equations in Geometry* [95], which dealt with upper and lower bounds on the numbers of real solutions to systems of equations. With Nisse and Passare [81, 80, 82, 84], we developed fundamental properties of coamoebae, particularly their relation to tropical geometry. With Assaf and Bergeron [2], we solved a 20-year old problem of positivity in the Schubert Calculus, and with Sanyal and Sturmfels [89], we wrote what has become a standard reference in Convex Algebraic Geometry, discussing many fundamental questions in that field in the context of the highly symmetric orbitopes.

The grant DMS-0915211 supported work on Galois groups of Schubert problems and numerical algebraic geometry. This included the initial systematic study of these Galois groups [21, 72, 98]. With undergraduate Brooks and graduate student Martín del Campo, we nearly determined the Galois groups for all Schubert problems involving 2-planes in n-space [21]. Another focus was the development of numerical methods to solve Schubert problems on Grassmannians, which included a novel formulation of Schubert problems as square (number of equations = number of variables) systems of equations [45] and the design of a numerical homotopy algorithm for solving any Schubert problem on a Grassmannian [97] that is based on Vakil's geometric Littlewood-Richardson rule [105, 106]. Another highlight was the first implementation in software of Smale's α-theory [91] for certifying numerical solutions to square systems of polynomial equations [46], which has been used by many researchers, Bozóki, et al. in their solution of Gardner's seven cylinders problem [19, 71].

Broader Impacts. These grants supported postdocs Jonathan Hauenstein and Jacob White, and the travel of postdocs Aaron Lauve, Zach Teitler, and Mounir Nisse. Four graduate students who were partially supported received their Ph.D.s in this period, and there were four undergraduate students who were also supported. This supported me while I was the chair of the SIAM activity group on algebraic geometry, and helped me to organize or serve on the programme committees of over a dozen workshops, conferences, minisymposia and summer schools. It has also facilitated my work on mathematical outreach as a founder of the Texas A&M Math Circle in 2011 and my outreach work at math circles, math clubs, REUs and other venues in in the US as well as goodwill trips to Nigeria in 2012 and 2014, each consisting of two week-long visits to different Universities at which I gave short graduate courses, talks to undergraduate students, colloquia, and public lectures, including a math circle.

References

1. Praise Adeyemo and Frank Sottile, *Cohomological consequences of the pattern map*, arXiv.org/1408.4085, 2014.
2. Sami Assaf, Nantel Bergeron, and Frank Sottile, *A combinatorial proof that Schubert vs. Schur coefficients are nonnegative*, arXiV.org/1405.2603, 2014.
3. Sami H. Assaf, *Dual equivalence and a Schur positivity*, arXiv:1107.0090 .
4. ———, *Dual equivalence graphs and a combinatorial proof of LLT and Macdonald positivity*, arXiv:1005.3759.
5. ———, *Shifted dual equivalence and Schur P-positivity*, arXiv:1402.2570.
6. Daniel J. Bates, Jonathan Hauenstein, Matthew Niemerg, and Frank Sottile, *Software for the Gale transform of fewnomial systems and a Descartes rule for fewnomials*, 2014.
7. Daniel J. Bates and Frank Sottile, *Khovanskii-Rolle continuation for real solutions*, Found. Comput. Math. **11** (2011), no. 5, 563–587.
8. Curtis Bennett and Annalisa Crandel (eds.), *Starting our careers*, AMS, 1999.
9. N. Bergeron, S. Mykytiuk, F. Sottile, and S. van Willigenburg, *Shifted quasi-symmetric functions and the Hopf algebra of peak functions*, Discrete Math. **256** (2002), 57–66.
10. N. Bergeron and F. Sottile, *A Pieri-type formula for isotropic flag manifolds*, Trans. Amer. Math. Soc. **354** (2002), no. 7, 2659–2705.
11. Nantel Bergeron and Frank Sottile, *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds*, Duke Math. J. **95** (1998), no. 2, 373–423.
12. ———, *A monoid for the Grassmannian Bruhat order*, European J. Combin. **20** (1999), no. 3, 197–211.
13. ———, *Skew Schubert functions and the Pieri formula for flag manifolds*, Trans. Amer. Math. Soc. **354** (2002), no. 2, 651–673 (electronic).
14. George M. Bergman, *The logarithmic limit-set of an algebraic variety*, Trans. Amer. Math. Soc. **157** (1971), 459–469.
15. Robert Bieri and J. R. J. Groves, *The geometry of the set of characters induced by valuations*, J. Reine Angew. Math. **347** (1984), 168–195.
16. S. Billey, Z. Hamaker, A. Roberts, and B. Young, *Coxeter-knuth graphs and a signed little map for type b reduced words*, arXiv:1405.4891.
17. Sara Billey and Alexander Postnikov, *Smoothness of Schubert varieties via patterns in root subsystems*, Adv. in Appl. Math. **34** (2005), no. 3, 447–466.
18. Sara C. Billey and Tom Braden, *Lower bounds for Kazhdan-Lusztig polynomials from patterns*, Transform. Groups **8** (2003), no. 4, 321–332.
19. Sándor Bozóki, Tsung-Lin Lee, and Lajos Rónyai, *Seven mutually touching infinite cylinders*, arXiv:1308.5164, 2013.
20. Glen E. Bredon, *Equivariant cohomology theories*, Bull. Amer. Math. Soc. **73** (1967), 266–268.
21. Christopher J. Brooks, Abraham Martín del Campo, and Frank Sottile, *Galois groups of Schubert problems of lines are at least alternating*, *Transactions of the AMS*, to appear, 2014.
22. Victor M. Buchstaber and Nigel Ray, *An invitation to toric topology: vertex four of a remarkable tetrahedron*, Toric topology, Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 1–27.
23. N. A. Bushueva and A. K. Tsikh, *On amoebas of algebraic sets of higher codimension*, Proc. Steklov Inst. Math. **279** (2012), 52–63.
24. John B. Conway, *On being a department head: A personal view*, AMS, 1996.
25. David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.
26. Gheorghe Craciun and Martin Feinberg, *Multiple equilibria in complex chemical reaction networks. I. The injectivity property*, SIAM J. Appl. Math. **65** (2005), no. 5, 1526–1546 (electronic).
27. Gheorghe Craciun, Luis García-Puente, and Frank Sottile, *Some geometrical aspects of control points for toric patches*, Mathematical Methods for Curves and Surfaces, Lecture Notes in Computer Science, vol. 5862, Springer, Berlin, Heidelberg, New York, 2010, pp. 111–135.
28. M.W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451.

29. Claire Delaunay, *Real structures on compact toric varieties*, Ph.D. thesis, Institut de Recherche Mathématique Avancée, Université Louis Pasteur, Strasbourg, 2004, Prépublication de l'IRMA, 2004/18.
30. Pedro F. dos Santos and Paulo Lima-Filho, *Bigraded equivariant cohomology of real quadrics*, Adv. Math. **221** (2009), no. 4, 1247–1280.
31. Mathias Drton, Bernd Sturmfels, and Seth Sullivant, *Lectures on algebraic statistics*, Oberwolfach Seminars, vol. 39, Birkhäuser Verlag, Basel, 2009.
32. Alexander Duncan, *Twisted forms of toric varieties*, 2014, `arXiv:1408.4039`.
33. Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind, *Non-Archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157.
34. E. Javier Elizondo, Paulo Lima-Filho, Frank Sottile, and Zach Teitler, *Arithmetic toric varieties*, Math. Nachr. **287** (2014), no. 2-3, 216–241.
35. Sergey Fomin and Anatol N. Kirillov, *Quadratic algebras, Dunkl elements, and Schubert calculus*, Advances in geometry, Progr. Math., vol. 172, Birkhäuser Boston, Boston, MA, 1999, pp. 147–182.
36. Stefan Forcey, Aaron Lauve, and Frank Sottile, *Hopf structures on the multiplihedra*, SIAM J. Discrete Math. **24** (2010), no. 4, 1250–1271.
37. ———, *Cofree compositions of coalgebras*, Ann. Comb. **17** (2013), no. 1, 105–130.
38. Mikael Forsberg, Mikael Passare, and August Tsikh, *Laurent determinants and arrangements of hyperplane amoebas*, Adv. Math. **151** (2000), no. 1, 45–70.
39. Wm. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993.
40. Luis García-Puente, Nickolas Hein, Christopher Hillar, Abraham Martín del Campo, James Ruffo, Frank Sottile, and Zach Teitler, *The secant conjecture in the real Schubert calculus*, Exp. Math. **21** (2012), no. 3, 252–265.
41. Luis García-Puente and Frank Sottile, *Linear precision for parametric patches*, Adv. Comput. Math. **33** (2010), no. 2, 191–214.
42. Luis García-Puente, Frank Sottile, and Chungang Zhu, *Toric degenerations of Bézier patches*, ACM Transactions on Graphics **30** (2011), Issue 5, Article 110.
43. I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser Boston Inc., Boston, MA, 1994.
44. J. Harris, *Galois groups of enumerative problems*, Duke Math. J. **46** (1979), 685–724.
45. Jonathan Hauenstein, Nickolas Hein, and Frank Sottile, *A primal-dual formulation for certifiable computations in Schubert calculus*, `arXiv.org/1406.0864`, 2014.
46. Jonathan D. Hauenstein and Frank Sottile, *Algorithm 921: alphaCertified: certifying solutions to polynomial systems*, ACM Trans. Math. Software **38** (2012), no. 4, Art. ID 28, 20.
47. ———, *Newton Polytopes and Witness Sets*, Math. Comput. Sci. **8** (2014), no. 2, 235–251.
48. Nickolas Hein, Christopher Hillar, Abraham Martín del Campo, Frank Sottile, and Zach Teitler, *The monotone secant conjecture in the real Schubert calculus*, `arXiv.org/1109.3436`, 2014.
49. Nickolas Hein, Christopher Hillar, and Frank Sottile, *Lower bounds in real schubert calculus*, São Paulo Journal of Mathematics **7** (2013), no. 1, 33–58.
50. Nickolas Hein, Frank Sottile, and Igor Zelenko, *A congruence modulo four in real schubert calculus*, *Journal für die reine und angewandte Mathematik*, to appear, 2013.
51. ———, *A congruence modulo four for real schubert calculus with isotropic flags*, `arXiv.org/1405.7761`, 2014.
52. A. Henriques, *An analogue of convexity for complements of amoebas of varieties of higher codimension*, Advances in Geometry **4** (2004), no. 1, 61–73.
53. Charles Hermite, *Sur les fonctions algébriques*, CR Acad. Sci.(Paris) **32** (1851), 458–461.
54. Christopher Hillar, Luis García-Puente, Abraham Martín del Campo, James Ruffo, Zach Teitler, Stephen L. Johnson, and Frank Sottile, *Experimentation at the frontiers of reality in Schubert calculus*, Gems in experimental mathematics, Contemp. Math., vol. 517, Amer. Math. Soc., Providence, RI, 2010, pp. 365–380.
55. Christopher J. Hillar and Frank Sottile, *Complex static skew-symmetric output feedback control*, SIAM J. Control Optim. **51** (2013), no. 4, 3011–3026.

56. Mathieu Huruguen, *Toric varieties and spherical embeddings over an arbitrary field*, J. Algebra **342** (2011), 212–234.
57. Mattias Jonsson, *Degenerations of amoebae and Berkovich spaces*, 2014, ArXiV.org/1406.1430.
58. C. Jordan, *Traité des substitutions*, Gauthier-Villars, Paris, 1870.
59. M. M. Kapranov, *A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map*, Math. Ann. **290** (1991), no. 2, 277–285.
60. M. M. Kapranov, B. Sturmfels, and A. V. Zelevinsky, *Quotients of toric varieties*, Math. Ann. **290** (1991), no. 4, 643–655.
61. ———, *Chow polytopes and general resultants*, Duke Math. J. **67** (1992), no. 1, 189–218.
62. Mikhail Kogan, *RC-graphs and a generalized Littlewood-Richardson rule*, Internat. Math. Res. Notices (2001), no. 15, 765–782.
63. Steven Krantz, *A primer of mathematical writing*, AMS, 1996.
64. ———, *How to teach mathematics, second edition*, AMS, 1999.
65. ———, *A mathematician's survival guide*, AMS, 2004.
66. Rimvydas Krasauskas, *Shape of toric surfaces*, Proceedings of the Spring Conference on Computer Graphics SCCG 2001, IEEE, 2001, pp. 55–62.
67. ———, *Toric surface patches*, Adv. Comput. Math. **17** (2002), no. 1-2, 89–133, Advances in geometrical algorithms and representations.
68. Thomas Lam, Aaron Lauve, and Frank Sottile, *Skew Littlewood-Richardson rules from Hopf algebras*, Int. Math. Res. Not. IMRN (2011), no. 6, 1205–1219.
69. Cristian Lenart, Shawn Robinson, and Frank Sottile, *Grothendieck polynomials via permutation patterns and chains in the Bruhat order*, Amer. J. Math. **128** (2006), no. 4, 805–848.
70. A. Leykin and F. Sottile, *Galois groups of Schubert problems via homotopy computation*, Math. Comp. **78** (2009), no. 267, 1749–1765.
71. Dana Mackenzie, *A tale of touching tubes*, www.sciencenews.org/article/tale-touching-tubes, 2014.
72. Abraham Martín del Campo and Frank Sottile, *Experimentation in the Schubert calculus*, *Advanced Studies in Pure Mathematics, Mathematical Society of Japan Summer Institute 2012 on Schubert Calculus*, to appear, 2013.
73. Karola Mészáros, Greta Panova, and Alexander Postnikov, *Schur times Schubert via the Fomin-Kirillov algebra*, Electronic Journal of Combinatorics **21** (2014), P1.39.
74. D. Monk, *The geometry of flag manifolds*, Proc. London Math. Soc. (3) **9** (1959), 253–286.
75. Lisa Nilsson and Mikael Passare, *Discriminant coamoebas in dimension two*, J. Commut. Algebra **2** (2010), no. 4, 447–471.
76. Mounir Nisse, *Maximally sparse polynomials have solid amoebas*, arXiv:0704.2216, 2007.
77. ———, *Geometric and combinatorial structure of hypersurface coamoebas*, arXiv:0906.2729, 2009.
78. ———, *Amoeba finite basis does not exist in general*, 2014, arXiv.org/1404.3912.
79. ———, *Amoeba finite basis of solutions to a system of n polynomials in n variables*, 2014, arXiv.org/1404.3593.
80. Mounir Nisse and Frank Sottile, *Non-archimedean coamoebae*, Tropical and Non-Archimedean Geometry, Contemporary Mathematics, vol. 605, American Mathematical Society, 2013, pp. 73–91.
81. ———, *The phase limit set of a variety*, Algebra Number Theory **7** (2013), no. 2, 339–352.
82. ———, *Higher convexity of coamoeba complements*, arXiv.org/1405.4900, 2014.
83. Mikael Passare, Timur Sadykov, and August Tsikh, *Singularities of hypergeometric functions in several variables*, Compos. Math. **141** (2005), no. 3, 787–810.
84. Mikael Passare and Frank Sottile, *Discriminant coamoebas through homology*, J. Commut. Algebra **5** (2013), no. 3, 413–440.
85. Elisa Postinghel, Frank Sottile, and Nelly Villamizar, *Degenerations of irrational toric varieties*, arXiv.org/1312.3834, 2013.
86. Alexander Postnikov, *On a quantum version of Pieri's formula*, Advances in geometry, Progr. Math., vol. 172, Birkhäuser Boston, Boston, MA, 1999, pp. 371–383.
87. Kevin Purbhoo, *A Nullstellensatz for amoebas*, Duke Math. J. **141** (2008), no. 3, 407–445.

88. Korben Rusek, Jeanette Shakalli, and Frank Sottile, *Dense fewnomials*, Randomization, relaxation, and complexity in polynomial equation solving, Contemp. Math., vol. 556, Amer. Math. Soc., Providence, RI, 2011, pp. 167–186.
89. Raman Sanyal, Frank Sottile, and Bernd Sturmfels, *Orbitopes*, Mathematika **57** (2011), no. 2, 275–314.
90. Franziska Schroeter and Timo de Wolff, *The boundary of amoebas*, 2013, `arXiv.org/1310.7363`.
91. S. Smale, *Newton's method estimates from data at one point*, The merging of disciplines: new directions in pure, applied, and computational mathematics (Laramie, Wyo., 1985), Springer, New York, 1986, pp. 185–196.
92. E. Soprunova and F. Sottile, *Lower bounds for real solutions to sparse polynomial systems*, Adv. Math. **204** (2006), no. 1, 116–151.
93. ______, *Lower bounds in real algebraic geometry and orientability of real toric varieties*, Discrete Comput. Geom. **50** (2013), no. 2, 509–519.
94. Frank Sottile, *Pieri's formula for flag manifolds and Schubert polynomials*, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 1, 89–110.
95. ______, *Real solutions to equations from geometry*, University Lecture Series, vol. 57, American Mathematical Society, Providence, RI, 2011.
96. ______, *Algebraic geometry*, *Princeton Companion to Applied Mathematics*, to appear, 2014.
97. Frank Sottile, Ravi Vakil, and Jan Verschelde, *Solving Schubert problems with Littlewood-Richardson homotopies*, ISSAC 2010—Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2010, pp. 179–186.
98. Frank Sottile and Jacob White, *Double transitivity of Galois groups in Schubert calculus of Grassmannians*, `arXiv.org/1312.5987`, 2013.
99. Frank Sottile and Chungang Zhu, *Injectivity of 2D toric Bézier patches*, Proceedings of 12th International Conference on Computer-Aided Design and Computer Graphics (Ralph Martin, Hiromasa Suzuki, and Changhe Tu, eds.), IEEE CPS, 15–17 September 2011, pp. 235–238.
100. David Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Adv. Geom. **4** (2004), no. 3, 389–411.
101. Müller Stefan, Elisenda Feliu, Georg Regensburger, Carsten Conradi, Anne Shiu, and Alicia Dickenstein, *Sign conditions for injectivity of generalized polynomial maps with applications to chemical reaction networks and real algebraic geometry*, 2013, `arXiv:1311.5493`.
102. William Strunk and E.B. White, *Elements of style, fourth edition*, Allyn and Bacon, 2000.
103. B. Sturmfels, *Gröbner bases and convex polytopes*, American Mathematical Society, Providence, RI, 1996.
104. Thorsten Theobald, *Computing amoebas*, Experiment. Math. **11** (2002), no. 4, 513–526.
105. R. Vakil, *A geometric Littlewood-Richardson rule*, Ann. of Math. (2) **164** (2006), no. 2, 371–422, Appendix A written with A. Knutson.
106. ______, *Schubert induction*, Ann. of Math. (2) **164** (2006), no. 2, 489–512.
107. Cynthia Vinzant, *Edges of the Barvinok-Novik orbitope*, Discrete Comput. Geom. **46** (2011), no. 3, 479–487.
108. V. E. Voskresenskiĭ, *Projective invariant Demazure models*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 2, 195–210, 431.
109. William Zinsser, *On writing well*, Harper-Colins, 2005.