Applicable Algebraic Geometry: Real Solutions, Applications, and Combinatorics

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Summary

While algebraic geometry is concerned with basic questions about solutions to equations, its value to other disciplines is through concrete objects and computational tools, as applications require knowledge of specific geometric objects and explicit, often real-number, solutions. Modern tools from computational algebraic geometry have great potential in applications, but their use requires a concerted effort to transfer this technology into the hands of applied scientists.

Sottile will work on projects ranging from basic research in real algebraic geometry and Schubert calculus to interdisciplinary work on applications of algebraic geometry, all with a substantial combinatorial component and an essential computational/experimental core. The intellectual merits of this activity include basic research to improve our understanding of real solutions to systems of polynomial equations through experimentation and conjecture and obtaining tighter fewnomial upper bounds while also developing a theory of lower bounds for the number of real solutions to structured polynomial systems. Other intellectual merits include disseminating ideas and techniques from algebraic geometry to other fields while following new research directions in mathematics inspired by problems from these fields.

This project will aid the development of human resources. Sottile has assembled a research team at Texas A&M of students and postdocs. In addition to their individual and collaborative research efforts, the team is working collectively on a series of vertically-integrated large-scale mathematical experiments. This team approach to research and training is consciously modeled on work patterns in the other sciences. It is intended to train undergraduates and new graduate students in research and to exploit the different strengths and knowledge of its members. Sottile's work will also broadly aid the professional development of his other young collaborators (of about 20 current collaborators, 5 are students and 7 are postdocs).

This project will have other impacts broader than training and deepening ties between mathematics and the applied sciences. The experimental projects are intended to demonstrate a style of collaboration not common in mathematics and to show that experimentation in mathematics can be more like experimentation in the other sciences. Sottile will continue to help organize large-scale international scientific meetings, such as the 2008 half-year program in real and tropical algebraic geometry at the Centre Interfacultaire Bernoulli at EPFL in Lausanne, Switzerland, and interdisciplinary conferences such as the workshop on "Non-linear computational geometry" at the IMA in June 2007. With Theobald, he will complete a textbook on "Applicable Algebraic Geometry".

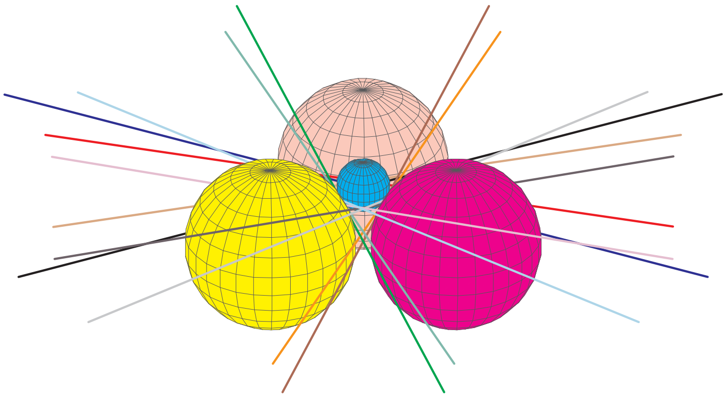
The main goals of this project are basic research in real algebraic geometry and Schubert calculus, training the next generation of mathematical scientists, and deepening ties between mathematics and the applied sciences.

Project Description

1. Results from previous NSF support

My scientific activity since August 1, 2002 has been supported by the grant "CAREER: Computation, Combinatorics, and Reality in Algebraic Geometry, with Applications", DMS-0134860 and DMS-0538734. (The grant was transferred.) This ends 31 July 2007, and is for \$344,577. I first describe my research activities and then their broader impacts. During the 50 months of this support, I have completed (and submitted) 26 research and expository papers. The numbers in parentheses refer to work supported by this CAREER grant which is listed at the end of this section, and those in brackets to the bibliography at the end of the proposal.

Nonlinear computational geometry. I work in nonlinear computational geometry, applying ideas from real algebraic geometry and computational algebraic geometry to solve geometric problems, typically in $\mathbb{R}^3$. This has involved line tangents to objects such as spheres, triangles, or line segments, or classifying degenerate configurations of these objects (2), (5), (7), (18), (24). I recently found four disjoint spheres with 12 real common tangents, settling an open problem. Here is a picture.



One reason for my interest in this area is that tools from real algebraic geometry and computational algebraic geometry are not yet widely used in nonlinear computational geometry. This is a pity, for the problems are intrinsically real and they involve varieties of low dimension and degree, so the inherent bad complexity of Gröbner bases is simply not an issue. I have other work in this area (1), (11), (26).

Hilbert's Theorem. In 1888, Hilbert [30] proved that a homogeneous quartic $q(x,y,z)$ in three variables which is non-negative ($q(x,y,z) \geq 0 \ \forall x,y,z \in \mathbb{R}$) has a witness for this fact in that it is the sum of three squares of real quadrics. Except for this case, non-negative forms of degree at least four and with at least three variables are not necessarily sums of squares. Coble [19] showed that when the quartic $q(x,y,z) = 0$ is smooth, representations as sums of complex squares correspond to non-trivial 2-torsion points on the Jacobian of the quartic. The Jacobian of a positive quartic has 15 such real 2-torsion points, corresponding to sums or differences of squares. Powers, Reznick, Scheiderer, and I studied this case in detail and used the Hochschild-Serre spectral sequence for étale cohomology with coefficients $\mathbb{G}_m$ to refine Hilbert's result and prove that such a quartic has exactly 8 inequivalent representations as a sum of squares of real quadrics (3).

Pieri formula for K-theory. Lenart and I gave a Pieri-type formula for the Grothendieck ring of a (type-A) flag variety (16). Such formulae are fundamental in the combinatorial study of the Schubert calculus. Lenart, Robinson and I then applied this result, together with the Billey-Braden theory of patterns [13], to obtain new formulas for Grothendieck polynomials (15). Later, Buch, Yong and I used some of these formals as well as formulas from [9] and [37] to show that the quiver coefficients are naturally Schubert structure constants and to give a new formula for quiver coefficients (6).

Combinatorial Hopf Algebras. With Bergeron and others, I used representation theory to explain the ubiquity of quasi-symmetric functions as generating functions in combinatorics [7]. Aguiar formulated an alternative theory based on categories [1] for infinitesimal Hopf algebras. Later, we three proved the equivalence of these approaches and much more (9). Due to this paper and a 2004 meeting at the Banff Institute that I helped organize, Hopf algebras with strong combinatorial structures are now called combinatorial Hopf algebras. Aguiar and I wrote three other papers [2], (8) and (10) elucidating basic structures of two fundamental combinatorial Hopf algebra, the Malvenuto-Reutenauer Hopf algebra of permutations and the Loday-Ronco Hopf algebra of trees.

Lower bounds. Work of Welschinger [67], Mikhalkin [51], and of Itenberg, et. al [33] gave lower bounds on the number of real rational curves of degree d interpolating $3d-1$ points in $\mathbb{RP}^2$. Concurrently, Eremenko and Gabrielov [23] gave a lower bound for the number of real solutions to certain problems in the Schubert calculus. These global results were featured in the MSRI 2004 winter semester: "Topological aspects of real algebraic geometry", of which I was the lead organizer. I wrote about this story in (4).

Inspired by this work, Soprunova and I (12) sought and found lower bounds for real solutions to certain sparse polynomial systems. This work involved a pleasing mixture of toric geometry, polyhedral combinatorics, and Gröbner degenerations.

New fewnomial upper bounds. In 1980, Khovanskii [35] proved that a system of n polynomials in n variables involving $n+k+1$ different monomials has at most

$$2^{\binom{n+k}{2}}(n+1)^{n+k}$$

nondegenerate *positive* solutions. This astronomical bound was not believed to be sharp.

I brought Bihan to MSRI in 2004 to work on Khovanskii's bound. With Bertrand, we proved the bound $2n+1$ for all real solutions, when the exponents of the monomials form a primitive vector configuration and $k=1$ (13). Bihan then gave the sharp bound of $n+1$ for nondegenerate positive solutions when $k=1$ [11]. Generalizing this work, Bihan and I gave the first improvement to Khovanskii's bound, lowering it to

$$\tfrac{e^2+3}{4}2^{\binom{k}{2}}n^k \ .$$

For this, we reduced the system to one consisting of k functions in k variables depending upon a vector configuration Gale dual to the exponents in the original system. Only solutions in a particular polyhedron are relevant, and we use polyhedral combinatorics, toric geometry, and the Khovanskii-Rolle theorem to obtain this bound.

Irrational decomposition. In 1988, Michel Brion [15] expressed the generating function (a multivariate polynomial) for the integer points in a rational polytope as a sum of rational generating functions of cones at each vertex. He used the Lefschetz-Riemann-Roch theorem in equivariant K-theory [4]. There are now vastly simpler proofs along the following lines. The result is easy for simplices, so you first triangulate the polytope, compute the generating functions, and add them. As the various cones and simplices overlap along faces, some device is needed to handle the subsequent overcounting.

Beck and I discovered the method of *irrational decomposition* to avoid such overcounting altogether. We inflate the polytope and translate it slightly so that there are no lattice points on any faces of the decomposition. This idea leads to simple proofs of fundamental theorems about generating functions for rational cones (17). With Haase, we used irrational decomposition to give a new proof of Brion's Theorem (21).

Quot scheme. Goresky, Kottwicz, and MacPherson [29] introduced a method to compute the torus-equivariant cohomology of a space which has finitely many torus-fixed points and finitely many torus-invariant curves. Brion [16] generalized this GKM-theory to torus-invariant Chow groups, and Evain [26] explained how to modify it when there are non-isolated, and hence infinitely many, torus-invariant curves.

The quot scheme $Q_{d,r,n}$ of degree d and rank r quotient sheaves of $\mathcal{O}^n_{\mathbb{P}^1}$ is a smooth projective variety of dimension $dn + r(n-r)$ [63]. When $r > 0$, it compactifies the space of rational curves in the Grassmannian of degree d, and it has a natural action of a torus having finitely many fixed points. Braden, Chen, and I (20) analyzed the structure of $Q_{,r,n}$, identifying all torus-invariant curves. The non-isolated curves come in families which are products of projective spaces. We strengthened Evain's tools and used our description of $Q_{d,r,n}$ to give a GKM-style presentation for the equivariant Chow rings $Q_{d,r,n}$. This is one of the few cases where GKM-theory has been used to compute the equivariant cohomology (or Chow) ring of a variety which was not known before.

Horn recursions in the Schubert calculus. In 1962 Horn conjectured that the set of eigenvalues of $n \times n$ hermitian matrices which sum to 0 are given recursively by smaller hermitian matrices which sum to zero and have integer eigenvalues. This conjecture was proven through work of Klyachko [36] linking such eigenvalues to, among other things, the Schubert calculus, and work of Knutson and Tao [38]. This implied that all translates of Schubert subvarieties in a Grassmannian must meet when their partition indices satisfy Horn inequalities indexed by intersections on smaller Grassmannians.

Belkale [5] used geometry to prove that the Horn inequalities were necessary and sufficient. In 1998, I discovered necessary Horn-type inequalities for any flag manifold, and showed they were sufficient for the Grassmannian of Lagrangian subspaces in 16-dimensional space. Purbhoo simplified Belkale's result [54], which motivated us to find recursive necessary and sufficient conditions for an intersection of Schubert varieties on a *cominuscule* flag variety to be non-empty (22).

These recursive conditions come from Schubert varieties on smaller cominuscule flag varieties. One result is a new Horn-type recursion for the Grassmannian which is completely different than Horn's recursion. We also give two completely different Horn-type recursions for the Lagrangian Grassmannian. These necessary and sufficient inequalities seem unrelated to the sufficient ones that I found in 1998.

Contribution to the Development of Human Resources.

Much of my scientific effort has been expended on teaching and training. I directly supported graduate students Pereira (Ph.D. expected December 2006), Ruffo (Ph.D. expected June 2007), and Irving (a new student), postdocs Soprunova (Summer 2003) and Bihan (Winter 2004), and provided students and postdocs with computers necessary for their work. I have helped many people to attend scientific meetings, this included some coauthors when they presented our joint work or came to work with me.

I have collaborated scientifically with over 36 people, including 1 undergraduate student, 4 graduate students, and 14 postdocs; this has helped them to develop as scientists.

Broader impacts.

My activity has also been directed beyond research and training. I wrote five survey articles/lecture nortes (1), (4), (21), (24), and (25). The first (1) was for an interdisciplinary collection on mathematics and geometric modelling, (4) was for the widely read MSRI newsletter the *Emissary*, and (25) is an 80 page manuscript from a course I taught at the Institut Henri Poincaré in November 2005. I gave 73 talks, including 11 invited talks at meetings and 18 colloquia. I also helped to organize and gave/will give three short courses to students, including the upcoming 2007 IMA PI Summer Graduate Program.

I helped/am helping to organize or serve on the programme committee of 21 scientific meetings. This includes being the lead organizer of the MSRI 2004 winter semester on real algebraic geometry, leading the organization of the IMA workshop on Nonlinear Computational Geometry in June 2007, and helping to organize workshops at the Banff centre on "Combinatorial Hopf Algebras", "Positive Polynomials and Optimization", and "Schubert Calculus". Of these conferences, 14 were international and 4 were interdisciplinary, fostering interactions with scientists outside of mathematics.

My work with Beck on irrational decompositions has led to improvements in practical algorithms. Using Brion's Theorem, Barvinok developed a polynomial-time algorithm to enumerate integer points in a poltyope [3]. This is based on *signed decomposition* of cones and the *polar trick*: using dual cones to avoid overcounting (for otherwise the algorithm is exponential). In practical implementations such as Latte [20], the large integer coefficients of dual cones create bottlenecks. Recently, Köppe [41] showed how irrational decomposition can be used to avoid this problem and give dramatic speed-ups in the performance of Barvinok's algorithm. This is now implemented in Latte.

Scientific projects supported by this CAREER grant.

(1) *Toric ideals, real toric varieties, and the moment map*, in "Topics in Algebraic Geometry and Geometric Modeling", Contemp. Math. 334, 2003. 225–240.
(2) *Common transversals and tangents to two lines and two quadrics in* $\mathbb{P}^3$, with G. Megyesi and T. Theobald. Discr. Comput. Geom., **30**, (2003), 543–571.
(3) *A new approach to Hilbert's theorem on ternary quartics*, with V. Powers, B. Reznick, and C. Scheiderer, CR Math. (Paris), **339**, (2004), 617–620.
(4) *Tropical interpolation*, Emissary, the newsletter of MSRI. Autumn 2004, 3–4.
(5) *The envelope of lines meeting a fixed line that are tangent to two spheres*, with G. Megyesi. Discr. Comput. Geom, **33**, (2005) 617–644.
(6) *Quiver coefficients are Schubert structure constants*, with A. Buch and A. Yong. Math. Res. Lett., **12**, (2005) 567–574.
(7) *Transversals to line segments in* $\mathbb{R}^3$, with H. Brönnimann, H. Everett, S. Lazard, and S. Whitesides. Discr. Comput. Geometry, **34**, (2005), 381–390.
(8) *Commutative Hopf algebras of permutations and trees*, with M. Aguiar. J. Alg. Combin., **22**, (2005), 451 – 470.
(9) *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*, with M. Aguiar and N. Bergeron. Comp. Math., **142**, (2006), 1–30.

(10) *Structure of the Loday-Ronco Hopf algebra of trees*, with M. Aguiar. J. Algebra, **295** (2006), 473–511.
(11) *Cremona convexity, frame convexity, and a theorem of Santaló*, with J. Goodman, A. Holmsen, R. Pollack, and K. Ranestad. Adv. Geom., **6**, (2006), 301–322.
(12) *Lower bounds for real solutions to sparse polynomial systems*, with E. Soprunova. Adv. Math., **204**, (2006), 116–151.
(13) *Polynomial systems with few real zeroes*, with B. Bertrand and F. Bihan. Math. Zeit., **253**, (2006), 361–385.
(14) *Experimentation and conjectures in the real Schubert calculus for flag manifolds*, with J. Ruffo, Y. Sivan, and E. Soprunova. Exper. Math., **15**, (2006), 199–221.
(15) *Grothendieck polynomials via permutation patterns and chains in the Bruhat order*, with C. Lenart and S. Robinson. Amer. J. Math., **128**, (2006), 805–848.
(16) *A Pieri-type formula in the K-theory of a flag manifold*, with C. Lenart. Trans. AMS, to appear.
(17) *Irrational proofs of three theorems of Stanley*, with M. Beck. Europ. J. Combinatorics, to appear.
(18) *Line tangents to four triangles in three-dimensional space*, with H. Brönnimann, O. Devillers, and S. Lazard. Discr. Comput. Geometry, to appear.
(19) *Real hessian curves*, with A. Ortiz-Rodríguez. Boll. Soc. Math. Mexico, to appear.
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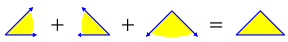
(21)  (Theorems of Brion, Lawrence, and Varchenko on rational generating functions for cones), with M. Beck and Ch. Haase, 2006. Submitted.
(22) *The recursive nature of cominuscule Schubert calculus*, with K. Purbhoo, 2006. Submitted.
(23) *New fewnomial upper bounds from Gale dual polynomial systems*, with Frédéric Bihan, 2006. Submitted.
(24) *Line problems in non-linear computational geometry*, with T. Theobald, 2006. Submitted.
(25) *Real solutions to equations from geometry*, Notes from a course at the IHP in November 2005. `math.AG/0609829`.
(26) *Linear precision for parametric patches*, with Luis Garcia, 2006.

2. Proposed research

I will discuss several lines of research and some specific projects. This level of activity is not excessive, but rather in line with what I have been doing recently. In the next three years I do not expect to be forced to look for and change jobs, run a semester at an Institute, or move four times, all of which I did in the last 50 months. These projects will be accomplished in collaboration with my students, postdocs at A&M and elsewhere, A&M faculty, and with colleagues in the US and abroad. For example, I am currently working on research with at least 20 different collaborators.

2.1. **The importance of collaboration.** For me, scientific activity is a collaborative activity. The two most important reasons are training the next generation of scientists and disseminating/sharing ideas between disciplines. This relevant, for more than half of this grant will be spent on students, visitors, their equipment, and travel—the most important components of any grant, and this proportion is likely underestimated; last year and this year I am forgoing 3 months of summer salary and a budgeted course buyout to support students, visitors, and travel that was not originally forseen.

While I will collaborate with anyone, I often work with students (not just my own) and postdocs. This teaches them the craft of research—generating ideas, studying good examples, computer experimentation, formulating results, writing reader-friendly papers, and giving clear talks. Also, working on a research project is a powerful incentive for students and postdocs to learn the body of knowledge on which we build our results.

There is a role for undergraduates in my research program. The seminal example in (14) (see Figure 2(b) or the cover illustration of that issue of Experimental Math.) was conceived by Sivan, then an undergraduate. I propose (Section 2.2.1) further experimentation which is particularly well-suited for undergraduates and new graduate students.

Building on my experiences in Prof. Gritzmann's Lehrstuhl in Munich last year and my time in experimental physics, I am creating a team approach to research in my group of students and postdocs at Texas A&M University. This involves fostering collaborations among them and with me, as well as supporting their research efforts.

Disseminating ideas between disciplines is another reason for collaboration. Some of the best science occurs when people with different backgrounds come together to solve a common problem, thereby enriching both subjects. This is true even when well-understood ideas from mathematics are used to solve applied problems, for the applications then shape future mathematical research. Interdisciplinary work requires significant preparation to identify collaborators and to learn enough of a different subject so that one's contributions are meaningful. Some of the requested resources for travel and visitors will be spent on this important preliminary work.

2.2. **Global information in real algebraic geometry.** The real numbers are fundamentally important in applications of mathematics, yet it is difficult to give useful global answers in real algebraic geometry. I propose research to address this difficult task. There are three themes, two of which are current lines of research, and one a new project.

2.2.1. *Experimentation and conjectures in the Schubert calculus.* In 1995, Boris and Michael Shapiro conjectured that a zero-dimensional intersection of Schubert varieties defined by flags which osculate the real rational normal curve would consist only of real points. This posited a rich class of polynomial systems which have only real solutions. I investigated their conjecture both experimentally and theoretically [59]. That generated significant interest [23, 24, 66] and inspired some of my own work [34, 57, 60, 58]. Last year, Mukhin, Tarasov, and Varchenko [52] proved the Shapiro conjecture using a surprising connection to representation theory—the necessarily real eigenvalues of the symmetric Shapovalov form give the points of intersection through the Bethe ansatz.

While most of this work, including [52], was for the Grassmannians, the original conjecture was for Schubert varieties in the classical (type-A) flag manifold. Unfortunately, the conjecture fails [59]. In fact, it fails for the first non-trivial problem on a flag variety. Consider the rational normal curve $\gamma\colon \mathbb{P}^1 \to \mathbb{P}^3$ and look for flags $\ell \subset H$, where the line

ℓ meets lines $\ell(s)$, $\ell(t)$, $\ell(u)$ tangent to γ at points s, t, and u, and the plane H meets γ in two specified points, $\gamma(v)$ and $\gamma(w)$, when all points are real. There will be two solution flags $\ell \subset H$, and the Shapiro conjecture posits that both solutions are real. Since $\gamma(v), \gamma(w) \in H$, the plane H contains the secant line $\lambda(v, w)$ that they span, and so ℓ meets this secant line. This reduces the problem to finding the lines ℓ meeting the three tangent lines and the secant line.

Lines meeting the three tangent lines $\ell(s)$, $\ell(t)$, and $\ell(u)$ form one ruling of a quadric surface Q in $\mathbb{P}^3$. Figure 1 illustrates this configuration. The lines that also meet the

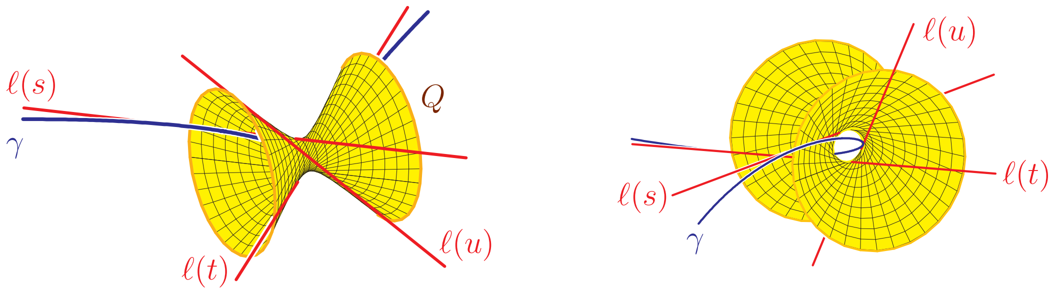


Figure 1. Two views of the quadric Q and tangent lines $\ell(s)$, $\ell(t)$, $\ell(u)$.

secant $\lambda(v, w)$ correspond to the points where it meets the quadric Q. In an expanded view down the throat of the quadric, Figure 2(a) shows a secant line $\lambda(v, w)$ which meets the hyperboloid in two points, and Figure 2(b) one which meets the hyperboloid in no real points. In case (a), there are two real lines ℓ meeting the tangents and the secant,

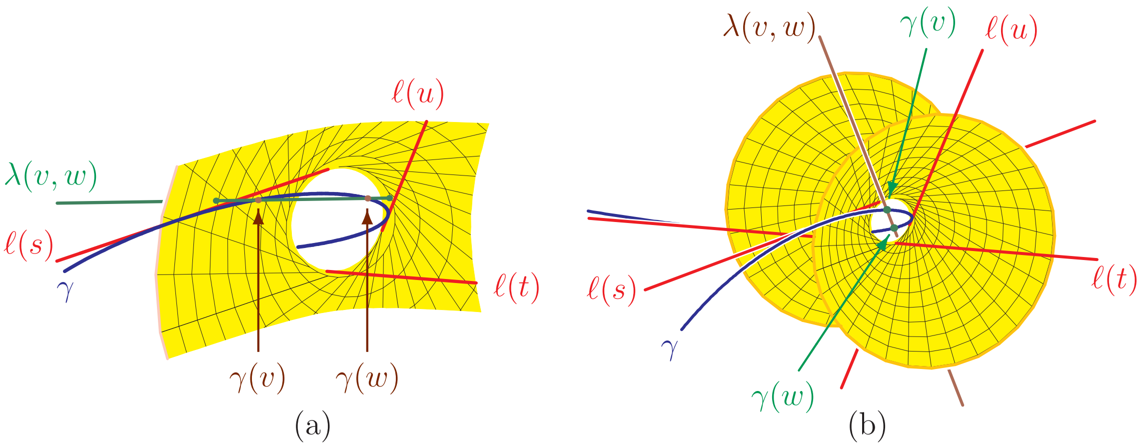


Figure 2. Two secant lines.

while in case (b), the two lines are complex. Thus Shapiro's conjecture fails.

This failure is quite interesting. If we label the points s, t, u with 1 (conditions on the line) and v, w by 2 (conditions on the plane), then along γ they occur in order

$$11122 \text{ in case (a)} \quad \text{and} \quad 11212 \text{ in case (b)}.$$

The sequence in case (a) is *monotone* and also in case (a) both solutions are always real.

In May 2003, I began a project with an undergraduate, a graduate student, and a postdoc to study this failure and possible refinements of the Shapiro conjecture for the flag manifold (14). We used over 15 gigaHertz-years of computing, solving over 500 million polynomial systems representing intersections of Schubert problems in over 1100 different enumerative problems on 27 different flag manifolds. These data are available on the web at `www.math.tamu.edu/~sottile/pages/Flags/index.html`.

Our experimentation uncovered and supported the Monotone Conjecture: For certain Grassmannian-type conditions, if the fixed flags appear along the rational curve γ in monotone order, then all solutions are real. We believe this to be the correct version of the Shapiro conjecture for the (type-A) flag manifold. Because this monotone condition is semi-algebraic, it is unlikely that the algebraic methods of [52] can settle this conjecture.

Inspired by (14), Eremenko, et. al [25] formulated another conjecture. The conditions in the Shapiro conjecture are Schubert varieties defined by flags that osculate the rational normal curve γ. A *secant flag $F_\bullet$* is one where every subspace F_i of $F_\bullet$ is spanned by its points of intersection with γ. Secant flags $F^1_\bullet, \ldots, F^s_\bullet$ are *disjoint* if there exist disjoint intervals $I_1, \ldots, I_s$ of γ such that the subspaces in flag $F^i_\bullet$ meet γ at points of I_i. They conjectured that, in a Grassmannian, an intersection of Schubert varieties defined by disjoint secant flags has only real points. In the limit as the interval I_i shrinks to a point the secant flag $F^i_\bullet$ becomes an osculating flag, and so this extends the Shapiro conjecture. They proved this secant conjecture for lines in $\mathbb{P}^n$, which implies the monotone conjecture for flags consisting of a point lying on a line in $\mathbb{P}^n$.

I will investigate this secant flag conjecture and an extension incorporating the monotone conjecture. This will be another large-scale experimental project similar to (14). It will involve my research team at Texas A&M (three students and two postdocs). We expect to write a paper describing our data and results about secant flags, and will archive our data on the web. This project is intended to train members of the team, not only in the Schubert calculus, symbolic computation, and real solutions to systems of equations, but also to work in collaboration in a way that is not common in mathematics.

There is more to be done here. Osculating flags are translates of a fixed flag by the flow generated by a principal nilpotent element of the Lie algebra. Thus there is a version of the Shapiro conjecture for any homogeneous space. I plan to investigate this conjecture on every small homogeneous space. That is, I will conduct preliminary investigations to see how the Shapiro conjecture may hold or fail. This will inform large-scale experimentation that I will undertake whose goal is to make and find evidence for reasoned conjectures, as well as search for new phenomena. Preliminary computations suggest that the Shapiro conjecture holds for the (type-D) orthogonal Grassmannian and it fails for the Lagrangian Grassmannian, again in a very interesting way (no solutions are real). My next experimental project will investigate the Shapiro conjecture on the Lagrangian Grassmannian and other homogeneous spaces for the symplectic group.

This is a long-term program, involving hundreds of gigaHertz-years of computing over many calendar years and several distinct projects/papers. I will archive the data intelligibly on the web. These projects will help train my students and postdocs, and in some summers I will hire an REU student, giving them real research experience with my team. While most experimentation will use workstations in TAMU faculty offices, I will purchase several Intel workstations dedicated to our more resource-intensive computations.

These experimental projects, with their large scale open-ended nature (we do not know what to expect), reliance on team work, and thoroughness (we will investigate *every* Schubert problem on *all* small homogeneous spaces) are quite different in nature from the norm in mathematics. Often experimentation in mathematics means to use a computer as very powerful pencil and paper, or to aid in computationally inspired speculation. These are useful roles, but I intend for these experimental projects to demonstrate that it is possible for experimentation to play another role in mathematics, one that is closer in spirit to the role of experimentation in the other sciences.

2.2.2. *Certificates of positivity for discriminants.* An approach to the Shapiro conjecture was to compute a discriminant of the corresponding polynomial system and show that it is non-negative, either on all of $\mathbb{R}^n$ or on a certain full-dimensional cone (for the monotone conjecture). It is particularly elegant to demonstrate positivity on $\mathbb{R}^n$ by writing a polynomial as a sum of squares, or by showing that it lies in a quadratic module [55] (for positivity on a cone). I used this in [59] to prove the Shapiro conjecture for 2-planes in $\mathbb{R}^4$. I computed a discriminant having degree 20 in 4 variables with a total of 788 different monomial terms (out of a possible 8855), and then used ad hoc methods to show that it was a sum of squares. Since not all non-negative polynomials of degree 20 in 4 variables are a sum of squares, this example suggests that all discriminants in the Shapiro conjecture might be sums of squares. In each of about 10 cases in which I have analyzed a discriminant, I have found such a certificate of positivity.

I propose to compute hundreds (or thousands) of such discriminants, and then look for such certificates for their positivity. As the polynomial systems are not sparse in the technical sense and symbolic computation is too feeble, I will use interpolation to compute the discriminants. Expressing some discriminants as sums of squares, or, for those from the monotone conjecture, as lying in a quadratic module, could pose a significant challenge, even to the powerful new tools from semi-definite programming that treat sums of squares numerically and algorithmically [53]. I expect that this will lead to a better understanding of these discriminants and produce extremely challenging test cases for software.

2.2.3. *Lower bounds for real solutions to polynomial equations.* Lower bounds on the numbers of real solutions to a system of equations are existence proofs for real solutions and may have enormous importance for the applications of mathematics. My work with Soprunova (12) was a 'proof-of-concept' that it is possible to have provable lower bounds. My goal is to produce a robust theory to understand, detect in examples, compute, and apply lower bounds for real solutions to polynomial systems.

That work with Soprunova had several ingredients. Consider a system of polynomials

$$(*) \qquad f_1(x_1, x_2, \ldots, x_n) \;=\; f_2(x_1, x_2, \ldots, x_n) \;=\; \cdots \;=\; f_n(x_1, x_2, \ldots, x_n) \;=\; 0$$

which is *unmixed* in that the f_i have the same Newton polytope Δ. Solutions to $(*)$ correspond to points in the inverse image of a map $\pi\colon Y_\Delta \to \mathbb{RP}^n$, where Y_Δ is the real toric variety associated to Δ. When a certain double cover $\widetilde{Y}_\Delta$ of Y_Δ is orientible, the degree of the lifted map $\widetilde{\pi}\colon \widetilde{Y}_\Delta \to S^n$ is a lower bound for the number of real solutions to $(*)$. We gave (i) conditions on Δ which implied that $\widetilde{Y}_\Delta$ is orientable, (ii) a method based on Gröbner degenerations and balanced triangulations to compute the degree of $\widetilde{\pi}$, and (iii) many examples based on order polytopes to which this theory applies.

I hope to generalize this basic theory to mixed systems (when the f_i have different Newton polytopes). This is non-trivial. While we may formulate the solutions to a mixed system as the fibres of a map, that map has a large base locus, and I will have to find a way around this obstruction. This further work will also (i) seek more general criteria which implies the orientibility of $\widetilde{Y}_\Delta$—we have examples having lower bounds to which our criteria do not hold, (*ii*) find more ways to compute the degree of $\widetilde{\pi}$, (*iii*) give more examples of systems possessing such a lower bound, and (*iv*) apply this theory to concrete problems.

The experimentation in (14) uncovered many Schubert problems which appear to have lower bounds on their numbers of real solutions. While it would seem that the approach of Eremenko and Gabrielov [23] might apply, these problems are again not naturally expressed as the fibres of a map. I feel that this obstruction will be overcome, for our data on these lower bounds are very convincing. (How can 10 million random instances lie?) This project may shed light on the more important problem just described about lower bounds to mixed polynomial systems.

Bihan and I recently gave a dramatic improvement to Khovanskii's celebrated fewnomial bound (23). For this, we reduced the original system to a Gale system, and then estimated the number of real solutions to the Gale system using the Khovanskii-Rolle Theorem. Both steps in that proof have applications.

The reduction to a Gale system should also work for complex solutions. The general picture should be that the set of solutions to the original system is fibred over the set of solutions to the Gale system, and the fibres are orbits of the same finite group. Some primitivity should imply the group is trivial and hence that the two systems are equivalent. In general, this will reduce solving the original system to solving the simpler Gale system (with few solutions) and solving the trivial fibre system. This might be used to bound the complexity of solving fewnomial systems over any field. I will have to work this out to see if the complexity bounds are an improvement over known bounds.

The arguments involving the Khovanskii-Rolle Theorem count points on smooth real curves. Following points along these curves should lead to a new numerical homotopy [56] algorithm for finding all real zeroes of a fewnomial system. What sets this algorithm apart is that the continuation is entirely real and no paths diverge, so it is efficient. It is also based on transcendental equations of the form $\sum a_i \log(p_i(x)) = 0$, where $p_i(x)$ is a linear function, and hence may be applied to a wider class of problems.

The current best fewnomial bound for systems of n polynomials in n variables with $n+k+1$ monomials is $\frac{e^2+3}{4}2^{\binom{k}{2}}n^k$. The polynomial dependence, n^k, is necessary, at least asymptotically in n with k fixed, but I believe that the term $2^{\binom{k}{2}}$ can be improved, perhaps to $2^{O(k)}$ or k^k. More attention needs to be spent on generating examples with many real solutions. Khovanskii's original proof was based on an induction, and it may be possible to alter the induction to improve the bound. Bihan, Rojas, and I plan a concerted effort to lower the fenomial upper bound and to construct more and better examples.

2.2.4. *Bredon cohomology of real toric varieties.* Abstractly, a real algebraic variety is a complex variety X equipped with an anti-holomorphic involution τ, and the real points $X_\mathbb{R}$ of X are the fixed points of τ. This viewpoint leads to the Smith-Thom [64] inequalities between the total Betti numbers of X and of $X_\mathbb{R}$. Finer invariants come when we consider the equivariant cohomology of X for the group $\mathbb{Z}_2 = \{1, \tau\}$. Still finer invariants are found in Bredon cohomology, a richer equivariant theory. This ring is bigraded by the real representation ring of $\mathbb{Z}_2$ and admits a cycle map from the motivic cohomology of X.

Recently, my colleague Lima-Fihlo with Dos Santos [22] computed the Bredon cohomology of real quadrics with coefficients in the constant MacKey functor $\underline{\mathbb{Z}}$. Their result shows that while the Bredon cohomology is highly non-trivial, it is possible to compute it for well-behaved varieties. That is what Lima-Filho and I propose to do. As quadrics are homogeneous spaces, one possibility is to study the Bredon cohomology of different real forms of homogeneous spaces.

Another possibility is to study exotic structures on real toric varieties. Delaunay [21] classified exotic real structures on toric varieties X_Δ associated to a polytope Δ. We may understand them as follows. Such a toric variety is equipped with a moment map $\mu\colon X_\Delta \to \Delta$. If (X_Δ, τ) is a toric real structure on X_Δ, then τ commutes with μ. The standard real structure fixes Δ while exotic real structures induce non-trivial involutions on Δ. Lima-Filho and I plan to compute the Bredon cohomology rings of these toric varieties equipped with exotic real structures. The explicit description of toric varieties, and the ineteraction of μ and τ should facilitate this effort. This should be an interesting challenge and lead to a better understanding of Bredon cohomology.

2.3. **Applications of Algebraic Geometry.** I believe that ideas and techniques from algebraic geometry can be fruitfully applied to solve problems and gain a deeper understanding in subjects outside of mathematics. I act on this belief by writing surveys, attending and organizing interdisciplinary meetings, and finding good problems to work on which are outside of my discipline. Let me describe some directions this effort will take me in the next few years.

2.3.1. *Linear precision.* In geometric modelling, pieces of curves and surfaces (and higher dimensional objects) are represented by parametrized patches, which are images of functions $\varphi\colon \Delta \to \mathbb{R}^n$, where Δ is some domain in $\mathbb{R}^d$. A *patch* consists of *basis functions* $\beta := \{\beta_a\colon \Delta \to \mathbb{R}_{\geq 0} \mid a \in \mathcal{A}\}$ which form a partition of unity, $\sum_a \beta_a(x) = 1$ for $x \in \Delta$. Here, the index set $\mathcal{A}$ is a finite subset of $\mathbb{R}^d$ whose convex hull is Δ. Then φ is given by

$$\varphi(x) \;=\; \sum_{a\in\mathcal{A}} \beta_a(x)\, b_a \;,$$

where $\{b_a \in \mathbb{R}^n \mid a \in \mathcal{A}\}$ are *control points*. The patch β has *linear precision* when it can replicate linear functions in a particular way, which is equivalent to requiring that

$$x \;=\; \sum_{a\in\mathcal{A}} \beta_a(x)\, a \;, \qquad \text{for} \quad x \in \Delta\,.$$

That is, if we take the control points to be the points of $\mathcal{A}$, we get the identity map. Such basis functions are a form of barycentric coordinates for Δ.

This formulation (which is more restrictive than is typical [27]) was developed in collaboration with Garcia (26). It is flexible enough to cover all examples in geometric modelling, yet sufficiently narrow so that it leads to precise results—any patch β has a unique reparametrization with linear precision, and precise criteria—this unique reparametrization is a rational function if and only if a certain algebraic variety has an exceptionally singular position with respect to a cannonical linear subspace.

This is part of a program began in (1) to better understand linear precision in geometric modelling. One goal is to determine which of Krasuaskas' multisided toric patches [42] have linear precision, and another goal is to introduce more and deeper mathematics into this subject. For example, for parametrized 3-folds the subtle difference between rationality and unirationality [18] matters. With Ranestad, we are using the structure of rational maps on surfaces to show that the only toric patches having linear presicion are the classical Bézier triangular patches and tensor product surfaces. Here, the points of $\mathcal{A}$ are taken to be the integer points in a lattice polygon Δ.

In our formulation of linear precision, the points $\mathcal{A}$ are fixed and their geometry plays a key role. More generally (and naturally) one may ask if it is possible to move the non-extreme points of $\mathcal{A}$ within Δ to achieve linear precision. The answer is yes for curves [27, p. 216]. This may be subtle in higher dimensions as often there are insufficient degrees of freedom. The precise geometric criteria we have developed should give us the tools to attack this problem. It is also interesting, both theoretically and practically, to understand patches of dimension 3 and higher, for geometric modelling can be used to represent higher-dimensional objects in a computer. I will apply our theory to study linear precision for higher dimensional toric patches.

2.3.2. *Mathematical kinematics.* This is a new direction for me which is related to non-linear computational geometry. A *mechanism* is an arrangement of rigid bodies coupled together at joints having specified motions. Fixing the combinatorial type (e.g. the types and numbers of joints between the different bodies) gives a class of mechanisms which depends upon continuous parameters (rod-length or placement and orientation of joints). For example, the Hexapod parallel robot or planar 4-bar mechanism are two common classes. We describe a given mechanism in this class as an incidence variety linking its positions in space to the motions of its joints. There are many interesting problems here. For example, classify all mechanisms in a given class having exceptional motions, analyze the motions of a given mechanism, or count all possible positions of a given mechanism after some or all joint motions are fixed. This last question is a problem in enumerative real algebraic geometry [61]. Kinematics should provide a rich class of geometric problems exhibiting interesting real-number phenomena.

This field is not new—mechanical engineers have been studying it for decades. However, after reading some of the literature and talking with people in the field, I see that interesting and practical advances can be made by applying ideas and techniques from algebraic geometry. The field is not yet informed by modern developments in computational, real, or enumerative algebraic geometry. For example, Husty and his collaborators [32] have made a beautifully ingenious study of some mechanisms, largely through manipulating polynomial systems by hand. I will attend a workshop at Notre Dame in March 2007 on the Geometry of Mechanism Science devoted to mathematical aspects of kinematics. I believe that this meeting, as well my contacts with people in the field, such as Husty, Selig (an editor of Robotica), and Wampler of General Motors (who lives not far from my mother), will help me to identify the contributions that I will make to this field.

2.3.3. *Convex geometry of orbits.* I have been working with Longinetti and Sgheri of Florence, Italy to gain a complete understanding of certain convex bodies which are the convex hull of an orbit of the compact group $SO(3)$ in a reducible representation. Our motivation comes from their work on numerical algorithms to understand aspects of the structure of metallo-proteins using paramagnetic data [28, 48].

Let $V = \mathbb{R}^3$ be the standard representation of $SO(3)$ and $W = (S^2V)^0$ be the space of trace-zero symmetric 3×3 matrices, a 5-dimensional irreducible representation of $SO(3)$.

We are studying the convex hulls of orbits $SO(3).w$, where $w \in W^k$. When $k = 1$, this is not too hard, as W is irreducible. We have an incomplete understanding when $k = 2$ and need to study all $k \leq 5$ (if $k > 6$, it reduces to $k \leq 5$). Their application needs the *Caratheodory number*, which is the minimum number n such that for every point x in the convex hull of $SO(3).w$, there exist $g_1, \ldots, g_n \in SO(3)$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $x = \lambda_1 g_1.w + \cdots + \lambda_n g_n.w$ and $1 = \lambda_1 + \cdots + \lambda_n$. We expect to determine the Caratheodory number by identifying the facial structure of this convex body.

What surprised me when I began to work on this problem was that such a simple and concrete problem has apparently not been studied. I propose to study the geometry of convex hulls of orbits for the orthogonal groups; at least when the ambient space has only few irreducible factors for then the geometry of subgroups of $SO(n)$ appears to dominate the analysis, rendering it feasible.

2.3.4. *Applicable algebraic geometry.* In my CAREER proposal, I had proposed to work on a text book intended to develop algebraic geometry with a perspective for applications, particularly in linear systems theory. I wrote a 110 page draft of a section developing algebraic geometry, but nothing further happened as my coauthors moved on scientifically. We have since abandoned our collaboration.

I remained committed to the idea, and now Theobald and I are working on a broader version of this abandoned project. Below is a current outline. Chapters 1, 2, 3, 7, 8, 9, 10, 11, and 12 exist in draft form, as we have each developed some material in LaTeXed notes for recent courses. (See, for example (25).)

Part I: Motivation and basics
1. Introduction and motivational applications.
2. Affine and projective geometry. Polynomials, ideals, and varieties.
3. Computational methods. Gröbner bases, resultants, and numerical homotopy.

Part II: The real world
4. Foundations of real algebraic geometry.
5. Computational real algebraic geometry.

Part III: Solving equations through deformations
6. Polyhedral geometry and toric varieties
7. Sparse equations. Kouchnirenko and Bernstein Theorems. Polyhedral homotopy.

Part IV: Algebraic optimization
8. Optimization. Linear and semidefinite programming, interior-point methods.
9. Algebraic certificates. Real Nullstellensatz, Putinar and Schmüdgen Theorems.
10. Sums of squares relaxations, 0-1 problems, and applications in engineering.

Part V: Spaces of linear spaces
11. Grassmannians, Schubert varieties and Schubert calculus.
12. Applications. Rational functions with real critical points. Pole placement.

Part VI: Non-linear computational geometry
13. Algebraic geometry in computational geometry, mechanisms.
14. Algebraic geometry in geometric modelling.

We are both lecturing at the 2007 IMA PI Summer Graduate Program on "Applicable Algebraic Geometry" which will be held at Texas A&M University in summer 2007. We will use our course to develop some material for this book. Our lectures will include topics from all parts except Part V.

During 2007-8 we plan to complete a draft of this book, and will likely visit each other that year to work on this project.

2.4. **New directions in the Schubert calculus.** Combinatorics underlies much, if not all, of the research I have proposed, and I maintain strong links to the subject—I am currently an associate editor of the SIAM Journal on Discrete Mathematics.

The structure of the cohomology rings of flag varieties is intertwined with the combinatorics of the Weyl groups of the associated algebraic groups. Much activity in the modern Schubert calculus is devoted to interactions between combinatorics and the geometry of flag varieties, with the particular goal of establishing positivity results using combinatorics. My current interests in the subject are more toward algebra and combinatorics than geometry. They range from continuations of recent work to new directions such as developing a theory of filtered combinatorial Hopf algebras and extending Schubert calculus beyond geometry to the complex reflection groups.

In work with Yong, we hope to use the Chevalley formulas in Grothendieck rings of flag manifolds [47] to extend formulas of Knutson and Yong [39] which they obtain by truncation. This will give new multiplication formulas in the Grothendieck and cohomology rings of flag manifolds of types B, C, and D. This is the furthest possible extension of formulas of Kogan [40] and Lascoux and Schützenberger's method to prove the Littlewood-Richardson rule [46]. We will work on this in summer 2007 when I visit Minnesota for the IMA program on applications of algebraic geometry.

Billey and Braden's geometric theory of permutations patterns [13] has led to new formulas for Schubert classes in cohomology [9, 10] and in Grothendieck rings (15). In principle this should work for any flag manifold and for generalized cohomology theories (K-theory and beyond). I plan to pursue this with Elizondo of UNAM in Mexico; he wants to learn some combinatorial aspects of algebraic geometry, and he knows more algebraic topology than me. We will start this when he visits me and my student Sanchez (who was working with Elizondo in Mexico) in 2007.

Lam [43] defined an analog of Stanley symmetric functions for the type-A affine Weyl group. His original definition may be recast into the theory of combinatorial Hopf algebras (9), and likely there is a common generalization of it and of the Billey-Haiman theory of Stanley symmetric functions [12] giving analogs of Stanley symmetric functions for all affine Weyl groups of classical type. Here, the peak quasi-symmetric functions and the analysis of [7, 8] will play a role. This is the first step toward generalizing this recent spectacular work of Lam and his coauthors applying Lam's affine Stanley symmetric functions to the cohomology of the affine Grassmannian [44, 45].

One source for the theory of combinatorial Hopf algebras comes from the Schubert calculus, where we associate symmetric or quasi-symmetric functions to posets with particular properties [7, 62]. This is a graded theory; the functions are homogeneous of degree equal to the rank of the poset. In the algebraic-combinatorial study of the Grothendieck ring of flag manifolds, the Schubert classes are inhomogeneous, and so the machinery of combinatorial Hopf algebras simply does not apply. I want to remedy this by developing a theory of filtered combinatorial Hopf algebras which would play the same role for the Grothendieck ring as the ordinary ones do for cohomology. There are already hints of such a theory in Buch's work on Grothendieck ring of the Grassmannian [17]: The target space was not the Hopf algebra of symmetric functions, but a sub bialgebra, and it was necessary to invert an element to define an antipode so that the target became Hopf. Another hint comes from a discussion that I had with Lou Billera; he and Brenti have defined an inhomogeneous quasi-symmetric function associated to intervals in the Bruhat order which is related to Kazhdan-Lusztig polynomials.

Hiller [31] developed a Schubert calculus for all Coxeter groups, demonstrating the essential combinatorial nature of the subject. Some (if not all) of this makes sense for complex reflection groups, in particularly those arising as wreath products of the symmetric groups with a finite cyclic group, which are also called coloured permutations. These groups are not the Weyl group of any algebraic group and therefore this 'Schubert calculus' has no direct relation to geometry. This program began with a very interesting paper of Totaro [65] where he computes the analog of a particular intersection number of a Grassmannian, but in the non-geometric case of the ring of polynomials invariant under the action of some complex reflection group. The formulas are beautiful and suggest there is much interesting mathematics to be uncovered in this direction.

Some of the beginnings of this are found in work of Malle [14, 50]. Any such theory should have some component relating it to combinatorial Hopf algebras, and parts of this already exist with the study of the coloured peak algebra by Bergeron and Hohlweg [6]. Totaro identified Schubert bases of these rings as Hall-Littlewood functions at roots of unity [49]. This suggests to me that much of the theory we have, from Schubert polynomials through to quasi-symmetric functions, should be generalized to functions of a parameter q so that when q is a root of unity we obtain the different coloured versions of our objects. This is a very fruitful area with many different possible directions.

3. Biographical Sketch

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Professional Preparation

Michigan State University, Honors B.S. in Physics, 1985.
University of Cambridge, CPGS, Maths Tripos Part III, with distinction, 1986.
University of Chicago, S.M. Mathematics, 1989, and Ph.D. in Mathematics, 1994.

Professional Experience

Professor, Texas A&M University, Since September 2006.
Professeur Invité, Université Paris 6, November 2005.
Associate Professor, Texas A&M University, August 2004 – August 2006.
Clay Mathematical Institute Senior Researcher, January-June 2004.
Assistant Professor, University of Massachusetts at Amherst, 2000-2004.
Van Vleck Assistant Professor, University of Wisconsin at Madison, 1999-2000.
MSRI Postdoctoral Fellow, Autumn 1998.
MSRI Postdoctoral Fellow, 1996-1997.
Term-Limited Assistant Professor, University of Toronto, 1994-1998.
Summer student at Lawrence Berkeley Laboratory 88" cyclotron.
Supervisor Joseph Cerny. Summer 1984.
Student worker at National Superconducting Cyclotron Laboratory,
Michigan State University. Supervisor Ed Kashy. 1982–1985.

Publications most related to project

(1) *Lower bounds for real solutions to sparse polynomial systems*, with E. Soprunova. Advances in Mathematics, **204**, (2006), 116–151.
(2) *Experimentation and conjectures in the real Schubert calculus for flag manifolds*, with J. Ruffo, Y. Sivan, and E. Soprunova. Exper. Math., **15**, (2006), 199–221.
(3) *A Pieri-type formula in the K-theory of a flag manifold*, with C. Lenart. Trans. AMS, to appear.
(4) *Real solutions to equations from geometry*, Notes from a course at the IHP in November 2005. `math.AG/0609829`.
(5) *New fewnomial upper bounds from Gale dual polynomial systems*, with F. Bihan. `math.AG/0609544`, 2006. Submitted.

Other Significant Publications

(1) *Intersection theory on spherical varieties*, with W. Fulton, R. MacPherson, and B. Sturmfels, J. Alg. Geom., **4** (1995), 181–193.
(2) *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds*, with N. Bergeron, Duke Math. J., **94** (1998), 273–423.
(3) *Real rational curves in Grassmannians*, J. Amer. Math. Soc., **13** (2000), 333–341.
(4) *Structure of the Malvenuto-Reutenauer Hopf algebra of permutations*, with M. Aguiar. Advances in Mathematics, 191 (2005), 225–275.

(5) *The recursive nature of cominuscule Schubert calculus*, with K. Purbhoo. Submitted.

Synergistic Activities

(1) Editorial Board, SIAM Journal on Discrete Mathematics, since January 2004.
(2) Chair of the organizing committee, MSRI semester on "Topological Aspects of Real Algebraic Geometry", January-May 2004.
(3) Co-organizer, with R. Krasauskas, of a workshop on Real Algebraic Geometry and Geometric Modeling, April 3-4, 2004, MSRI, Berkeley.

(4) Chair of organizing committee, IMA workshop on "Non-linear computational geometry", May 28 - June 2, 2006.
(5) Lead organizer and lecturer, IMA PI Summer Graduate School on Applicable Algebraic Geometry, July 23–10 August 2007.

Collaborators & Other Affiliations

Collaborators in past four years:

M. Aguiar (Texas A&M)
M. Beck, (San Fransico)
N. Bergeron, (York University, Toronto)
B. Bertrand, (Genève, Suisse)
F. Bihan, (Chambéry, France)
T. Braden, (Massachusetts)
H. Brönniman, (Brooklyn Polytechnic)
A. Buch, (Rutgers)
L. Chen, (Ohio State)
O. Devillers (INRIA, Sophia, France)
H. Everett, (INRIA, Nancy, France)
L. Garcia, (Texas A&M)
J. E. Goodman, (CUNY)
J. Griggs, (South Carolina)
C. Haase, (Freie Universität Berlin)
C. Hillar, (Texas A&M)
A. Holmsen, (Bergen, Norway)
S. Lazard, (INRIA, Nancy, France)
C. Lenart, (SUNY Albany)
M. Longinetti, (Firenze, Italy)
G. Megyesi, (Manchester, England)
A. Ortiz-Rodríguez, (UNAM, Mexico)
R. Pollack, (Courant)
V. Powers, (Emory University)
K. Purbhoo, (U. of British Columbia)
K. Ranestad, (Oslo, Norway)
B. Reznick, (Illinois)
J. Ruffo, (Texas A&M)
S. Robinson, (No affiliation)
C. Scheiderer, (Konsanz, Germany)
L. Sgheri, (CNR, Firenze, Italy)
Y. Sivan, (Massachusetts)
E. Soprunova, (Cleveland State)
T. Theobald, (Berlin)
S. Whitesides, (McGill University)
A. Yong, (Minnesota)

Thesis Advisor: William Fulton

Postdoctoral Advisors: Askold Khovanskii, Nantel Bergeron, and Bernd Sturmfels

Graduate Students: (All are current) Jim Ruffo, Corey Irving, and Abraham Martin Del Campo-Sanchez

Postdoctoral Advisees: Greg Warrington (2000-3), Evgenia Soprunova (2002-2004), Frédéric Bihan (2004), Seongchun Kwon (2004), Luis Garcia (current), Christopher Hillar (current), Maria Belk (current).

4. Budget justifications

Graduate student support: I currently supervise 3 advanced graduate students. I expect to replace the one graduating in August 2007 and will likely pick up more students during the life of this project, as the two others are just starting and my vigorous research program has plenty of good problems to interest students. Not only will my students work on their thesis research, but I am collaborating with them on other research (See Section 2.2.1). I plan to support them in the summers and support one for a term each year. This includes salary, tuition, fees, and fringe benefits. Texas A&M University pays their tuition when they are teaching, but not their fees, and so I am also asking for money to pay their fees when they are teaching.

Summer REU student: One summer, I plan to hire an REU student to assist with my experimental projects, as described in Section 2.2.1. I budget this for the first summer, as I do not yet know which year this will actually take place.

Visitors: Some of my most successful collaborations began with extended scientific visits of my collaborators. This was particularly true of my work with Theobald, with Aguiar, and with Bihan. I want to continue this, having one long-term (1 or more weeks) and several short-term visitors each year. In particular, I expect to have Theobald visit for a week or more in the first year of the grant to work on our textbook.

Travel by members of my team: I am asking for funds to support members of my team to travel to conferences, which is critical for their development as scientists. The requested amount will cover 2 1-week conferences and 3 weekend trips each year. While I will only advise 2 postdocs next year, that number will likely remain constant or rise as my department hires 6-8 postdocs each year. These 5 trips will be shared between 5 or more people each year.

Computer equipment: The type of work that we do requires that I and each of my students and postdocs have a laptop computer. I will also want us to have several workstations dedicated to our experimental projects.

PI Travel: In a typical year, I travel 3-4 times overseas to meetings, and make another 4-5 week-long trips within the US and Canada, as well as many weekend conferences. These are not only occasions to present work that I am doing, but more importantly to work on and start collaborations and plan other scientific meetings. I am only requesting enough funds to partially supprt this activity.

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