Assignments and problems:

Introduction and real roots

(1) Let $p, g \in \mathbb{R}[x]$ be univariate polynomials. Generalize Sturm's theorem in order to count the number of distinct real zeroes of p in $[a, b]$ for which g is positive minus the number of distinct real zeroes of p in $[a, b]$ for which g is negative. Consider the initial sequence $p_0 := p$, $p_1 := p'g$.

(2) If two endormorphisms f and g on a finite-dimensional vector space V are diagonalizable and $f \circ g = g \circ f$, then there exists a joint diagonalization (i.e. a diagonalization with respect to a joint basis).

(3) Let I be a zero-dimensional, radical ideal in $R = \mathbb{C}[x_1, \ldots, x_n]$, and let $\mathcal{B}$ be a fixed monomial basis of R/I. Further let $f \in R$ such that the values $f(p)$ are distinct for all zeroes p.
  (a) For a given $p \in \mathrm{V}(I)$, show that there exists $g \in R$ with $g(p) = 1$ and $g(p') = 0$ for all zeroes $p' \neq p$.
  (b) Show that the coset $[g]$ from part (a) is an eigenvector of m_f, and that the geometric multiplicity of the corresponding eigenvalue is 1.

(4) Use Hermite's method to determine the number of common real zeros to

$$x^2 + y^2 = 1 \, , \quad \frac{1}{2}x^2 + 2y^2 = 1 \, .$$

Next, replace the factor 2 by a and investigate what happens for $a = 1$, $a > 1$.

(5) Write a Maple (or Singular) program to determine the trace form. Apply the program to the previous example.

(6) Given a monic quartic polynomial $p = x^4 + \sum_{i=0}^{3} a_i x^i$, deduce an exact characterization (in the coefficients a_i) for the property that p has exactly two roots.