TROPICAL GEOMETRY EXERCISES
CIMPA SCHOOL - UNIVERSITY OF IBADAN, JUNE 2017

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1. Tropical arithmetic and polynomials

Exercise 1.1. Show that

$$\text{“}(x+0)^k\text{”} = \text{“}x^k + 0\text{”}$$

as functions. Remember that “a^k” = “$a \cdot \dots \cdot a$” $= ka$ for $a \in \mathbb{T}$.

Exercise 1.2. Draw the graphs of the tropical polynomials $P(x) =$ “$x^3 + 2x^2 + 3x + (-1)$” and $Q(x) =$ “$x^3 + (-2)x^2 + 2x + (-1)$”, and determine their tropical roots.

Exercise 1.3. Let $a \in \mathbb{R}$ and $b, c \in \mathbb{T}$. Determine the roots of the polynomial “ax^2+bx+c”. What is the tropical discriminant?

Exercise 1.4. Prove that x_0 is a tropical root of order at least k of $P(x)$ if and only if there exists a tropical polynomial $Q(x)$ such that $P(x) =$ “$(x + x_0)^k Q(x)$”.

Exercise 1.5. Prove that the tropical semi-field is algebraically closed. In other words that a tropical polynomial function of degree d has exactly d roots counted with multiplicity.

Exercise 1.6. Prove that there is no way to enlarge the tropical numbers $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ to include additive inverses. Hint “$a + a$” $= a$ for all $a \in \mathbb{T}$.

Exercise 1.7 (Maslov dequantisation)**.** Let $\mathbb{R}_{\geq 0}$ denote the non-negative real numbers and $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. Notice that $\log_t(\mathbb{R}_{\geq 0}) = \mathbb{T}$. Consider the semi-field $(\mathbb{T}, +_t, \times_t)$ where

$$x +_t y := \log_t(t^x + t^y) \qquad \text{and} \qquad x \times_t y := \log_t(t^x t^y).$$

(1) Show that for all $t > 1$ the semi-field $(\mathbb{T}, +_t, \times_t)$ is isomorphic to the semi-field $(\mathbb{R}_{\geq 0}, +, \times)$ equipped with the usual sum and multiplication
(2) Show that as t tends to ∞ the semi-field $(\mathbb{T}, +_t, \times_t)$ converges to a semi-field isomorphic the tropical semi-field $(\mathbb{T}, \max, +)$.

2. Tropical curves in the plane

Exercise 2.1. Draw the tropical curves defined by the tropical polynomials

$$\text{“}y + x^3 + 2x^2 + 3x + (-1)\text{”}$$

and

$$\text{“}7 + 4x + y + 4xy + 3y^2 + (-3)x^2\text{”},$$

as well as their dual subdivisions. Compare the 1st curve with the graph of $P(x)$ from Exercise 1.2.

Exercise 2.2. Prove the balancing property at a vertex of a tropical curve using the dual subdivision.

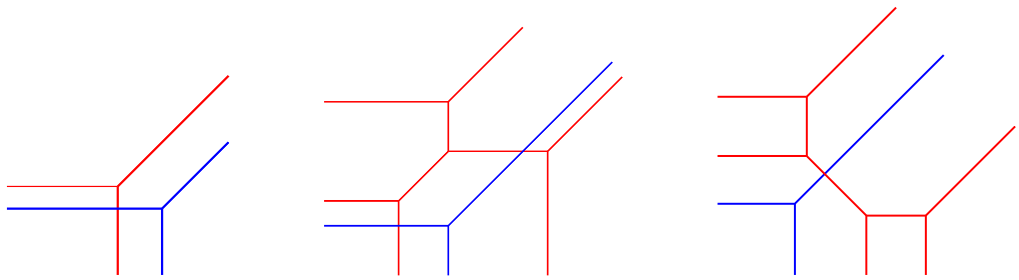
Exercise 2.3. A tropical curve C is called **nodal** if its dual subdivision consists of triangles and parallelograms. The **genus** of a nodal tropical curve of degree with Newton polygon Δ is

$$g(C) = \frac{\sigma - |\partial\Delta \cap \mathbb{Z}^2| + 2}{2},$$

where σ is equal to the number of triangles in the subdivision of Δ dual to C. A tropical curve is of **degree** d if its Newton polygon has vertices $(0,0), (d,0), (0,d)$.

(1) Show that a tropical curve of degree d has at most d^2 vertices.
(2) What is the genus of a non-singular tropical curve of degree d?

Exercise 2.4. Determine the stable intersection points of the following 3 pairs of tropical curves.



a) b) c)

Exercise 2.5. What is the self-intersection number of a tropical curve in $\mathbb{R}^2$ with Newton polygon Δ?

Exercise 2.6 (Tropical Bézout's Theorem)**.** Prove that if C_1 and C_2 are tropical curves in $\mathbb{R}^2$ of degrees d_1 and d_2, respectively, then the intersection number of C_1 and C_2 is $d_1 d_2$.

Exercise 2.7. Pick's formula for a lattice polygon Δ in $\mathbb{R}^2$ states that:

$$2\text{Area}(\Delta) = 2|\text{Int}(\Delta) \cap \mathbb{Z}^2| + |\partial\Delta \cap \mathbb{Z}| - 2$$

(1) Prove Pick's formula assuming that you have a primitive triangulation of Δ.
(2) Rewrite Pick's formula in terms of the genus, the self-intersection number, and the number of unbounded edges of non-singular tropical curve with Newton polygon Δ.

Exercise 2.8. Prove that every rational weighted 1-dimensional polyhedral complex in $\mathbb{R}^2$ which satisfies the balancing condition is the tropical curve of a tropical polynomial in 2 variables. Hint: Start by writing down the tropical polynomial when the polyhedral complex has a single vertex.