Contemporary Schubert Calculus and Schubert Geometry

Frank Sottile
(Texas A& M)
Jim Carrell
(U.B.C.)

March 17-21, 2007

1 A Brief Overview of Schubert Calculus and Related Theories

Schubert calculus refers to the calculus of enumerative geometry, which is the art of counting geometric figures determined by given incidence conditions. For example, how many lines in projective 3-space meet four given lines? This was developed in the 19th century and presented in the classic treatise "Kälkul der abzählanden Geometrie" by Herman Cäser Hannibal Schubert in 1879. Schubert, Pieri, and Giambelli subsequently developed algorithms to solve enumerative geometric problems concerning linear subspaces of vector spaces, which we now understand to be computations in the cohomology ring of a Grassmannian. Their vision and technical skill exceeded the rigorous foundations of this subject, and Hilbert, in his 15th problem, asked for a rigorous foundation of the subject. This was largely completed by the middle of the 20th century, based on the cohomology of rings of Grassmannians.

The basic tools of Schubert calculus are the Grassmannians $G_{k,n}$ consisting of the k-dimensional linear subspaces of $\mathbb{P}^n$ and Schubert varieties, which are certain subvarieties of $G_{k,n}$ made up of the k-dimensional subspaces which satisfy certain incidence relations (called Schubert conditions) with respect to a fixed flag: A flag in $\mathbb{P}^n$ is an increasing sequence $W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset \mathbb{P}^n$ where W_i is a linear subspace of dimension i. Given a sequence of integers $0 \leq a_0 < a_1 < \cdots < a_k \leq n$, the Schubert variety $\Omega(a_0, a_1, \ldots, a_k)$ is the set of all $V \in G_{k,n}$ for which $\dim(V \cap W_{a_j}) \geq j$. Another description of $\Omega(a_0, a_1, \ldots, a_k)$ is obtained by noting that the group $GL_{n+1}(\mathbb{C})$ of nonsingular $n \times n$ matrices acts transitively on $G_{k,n}$, and let B denote the subgroup of all upper triangular matrices in $GL_{n+1}(\mathbb{C})$. Then $\Omega(a_0, a_1, \ldots, a_k)$ is the Zariski closure of the orbit BV_0, where V_0 is the (unique) k-plane in $\Omega(a_0, a_1, \ldots, a_k)$ spanned by standard coordinate points $e_{a_0}, \ldots, e_{a_k}$. The key fact, known as the "Basissatz", is that the Schubert varieties form a basis of the integral homology groups of $G_{k,n}$, and dually determine a basis of its cohomology algebra. The basic formulas of Schubert calculus are given by the famous identities of Giambelli and Pieri and the determinantal formula.

By the 1940's it had been proved that multiplication in cohomology ring of $G_{k,n}$ with the natural basis formed by the classes dual to the Schubert varieties is the same as multiplication of Schur functions in the algebra of symmetric functions. That is, if $\sigma(a_0, a_1, \ldots, a_k)$ denotes the class dual to $\Omega(a_0, a_1, \ldots, a_k)$ in the cohomology algebra $H^*(G_{k,n}, \mathbb{Z})$, then when one expresses the product $\sigma(a_0, a_1, \ldots, a_k)\sigma(b_0, b_1, \ldots, b_k)$ as a sum of classes $\sigma(c_0, c_1, \ldots, c_k)$, the coefficient of $\sigma(c_0, c_1, \ldots, c_k)$ is a particular Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$. To be precise, if μ is the partition $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_k$ where $\mu_i = n - i - a_i + i$ and ν and λ are the partitions defined analogously for $(b_0, b_1, \ldots, b_k)$ and $(c_0, c_1, \ldots, c_k)$, then

$$\{\mu\}\{\nu\} = \sum_{\lambda} c^{\lambda}_{\mu\nu}\{\lambda\},$$

where $\{\mu\}$, $\{\nu\}$ and $\{\lambda\}$ are the Schur functions corresponding to the three partitions. This is extremely fundamental since the Littlewood-Richardson rule gives a combinatorial formula for the $c_{\mu\nu}^{\lambda}$ in terms of Young tableaux. Furthermore, it establishes a close relationship between Schubert calculus and the representation ring of the general linear group with its basis of irreducible Weyl modules.

In a famous unpublished 1951 paper, Chevalley proposed a far reaching generalization of the notion of a Schubert variety which illuminated the close connection between Schubert geometry and the theory of algebraic groups and initiated the systematic investigation of the natural generalizations of Schubert varieties for an arbitrary algebraic homogeneous space: that is, a projective variety of the form G/P, where P is a parabolic subgroup of a semi-simple algebraic group over an algebraically closed field (see [6]). His major contribution was to define Schubert varieties in a G/P as the closures of B-orbits, where B is a Borel subgroup of G: that is, a maximal solvable connected subgroup of G. This paper motivated many, many papers about the structure of Schubert varieties, the nature of their singularities and the combinatorial properties of the Weyl group of G and its connection with the geometry of the flag variety G/B. The Littlewood-Richardson rule for the cohomology ring of G/B in particular remains an important open question and was one of the major problems that the workshop addressed. In fact, the Littlewood-Richardson problem now refers to the analogous question for the general cohomology theories relevant to algebraic geometry, such as equivariant cohomology, K-theory and quantum cohomology, as well as equivariant versions of these theories.

2 Recent Developments and Open Some Problems

In the last 20 years, Schubert calculus has come to refer to the study of the geometric, combinatoric and algebraic aspects of the Schubert basis in the various cohomological setting mentioned above along with their relation to the rest of mathematics. There has been an explosion of progress in the subject, especially since the major impetus provided by the Oberwolfach workshop in 1997 at which people from combinatorics and geometry met each other for the first time at the very time Gromov-Witten theory, quantum cohomology and moduli spaces were being studied at a year-long meeting in Sweden. At about the same time, Knutson and Tao [15] proved the saturation conjecture, which, with work of Klyachko, established the Horn conjecture about the eigenvalues of hermitian matrices. While their methods were purely combinatorial, the relation to the Horn problem was through the geometry of Schubert calculus. We put recent results into three categories.

The Quantum and Equivariant Schubert Calculus,

Here, the highlights include proofs by Aaron Bertram of the Pieri, Giambelli and determinantal formulas for Grassmannians in the setting of quantum cohomology [1]; the work of Allen Knutson and Ezra Miller [14] giving a natural cohomological setting for Schubert polynomials; positivity results of Graham [11] for equivariant cohomology of G/B (see also [12] and [8]); and Brion's work on positivity in the K-theory of the flag variety [2].

Generalizations of the Littlewood-Richardson Rule and Combinatorics

The equivariant Littlewood-Richardson rule for general cohomologies remains an important open problem where, so far, only partial results, many very recent, have been attained. Knutson, Tao and Woodward [16], using versions of the Littlewood-Richardson rule stated in terms of honeycombs and puzzles, re-prove all the main theorems about eigenvalues in the context of honeycomb borders. This gives a beautiful connection between the Littlewood-Richardson rule and the eigenvalue problem just mentioned. Also, Anders Buch [3] has proved the Littlewood-Richardson rule for the K-theory of Grassmannian; Furthermore, Ravi Vakil, working over an arbitrary field, obtained a purely geometric proof of the Littlewood-Richardson rule [25]. Shortly before the workshop, Izzet Coşkun announced a solution to the Littlewood-Richardson problem for the small quantum cohomology of a Grassmannian (see [7]). A key notion on the combinatorial side of the Littlewood-Richardson problem is jeu de taqin, introduced by Schützenberger [22]. Recently, using the language of *jeu de taqin*, Chaput and Perrin [5] have computed the Littlewood-Richardson constants for the Λ-miniscule elements of the Weyl group introduced by Peterson (see [4]). This generalizes the recent result of Thomas and Yong proving the Littlewood-Richardson rule for cominuscule homogeneous spaces [24].

Real Schubert Calculus

Another research thrust has concerned real solutions to Schubert problems. A by-product of Vakil's geometric Littlewood-Richardson rule was a proof that every Schubert problem on the Grassmannian can have only real solutions [26]. Sottile and his collaborators [23, 21] have studied a subtle conjecture of Boris and Michael Shapiro and its generalizations. Eremenko and Gabrielov [9] proved it for Grassmannians of codimension-2 planes by showing that a rational function with only real critical points must be real. Just before the workshop, Mukhin, Tarasov, and Varchenko gave a general proof for all Grassmannians [17], and generalized this to related geometric problems [18].

3 Presentation Highlights

The following talks were presented at the workshop.

Anders Buch (Rutgers)
Title: *Equivariant Gromov-Witten invariants of Grassmannians*
Abstract: I will speak about joint work with L. Mihalcea, in which we prove that all equivariant (3-point, genus zero) Gromov-Witten invariants on Grassmannians are equal to equivariant triple-intersections on two-step flag varieties. This is a continuation of work with A. Kresch and H. Tamvakis, which established this result for the ordinary Gromov-Witten invariants. The non-equivariant case was obtained by showing that the curves counted by a Gromov-Witten invariant are in bijection with their kernel-span pairs, which consist exactly of the points in a triple-intersection of two-step Schubert varieties. Since the equivariant Gromov-Witten invariants have no enumerative interpretation (and also because they are defined relative to Schubert varieties that are not in general position), the proof in the equivariant case must be based on intersection theory. The main new construction is a blow-up of Kontsevichs moduli space that makes it possible to assign a kernel-span pair of the expected dimensions to every curve. By utilizing a construction of Chaput, Manivel, and Perrin, these results can be extended to all (co)minuscule homogeneous spaces.

Izzet Coşkun (MIT)
Title: *The geometry of flag degenerations*
Abstract: My goal is to explain the flat limits of certain subvarieties of partial flag varieties under one-parameter specializations. I will explain how this study leads to many new Littlewood-Richardson rules. I will try to get the audience to actively participate in the calculations, so we will have graph paper and colored pencils.

Takeshi Ikeda (Okayama University of Science)
Title: *Equivariant Schubert calculus for isotropic Grassmannians*
Abstract: We describe the torus-equivariant cohomology ring of isotropic Grassmannians by using localization maps to any torus-fixed point. We present two formulas for equivariant Schubert classes of these homogeneous spaces. The first formula is given as a weighted sum over combinatorial ob jects which we call excited Young diagrams. The second one is written in terms of factorial Schur Q- or P -functions. As an application, we give a Giambelli-type formula for the equivariant Schubert classes. This is a joint pro ject with Hiroshi Naruse.

Allen Knutson (UC San Diego)
Title: *Matroids, shifting, and Schubert calculus*
Abstract: In A geometric Littlewood-Richardson rule, Vakil starts with an intersection of two Schubert varieties in a Grassmannian, and alternately degenerates, then decomposes into geometric components; the end result is a list of Schubert varieties. The primary geometric miracle is that each degeneration stays generically reduced, with the consequence that the original cohomology class (the product of two Schubert classes) is written as a multiplicity-free sum of Schubert classes. There is an obvious generalization of this procedure to the flag manifold (where the geometric miracle is as yet unknown) but it doesnt give a combinatorial rule, just a geometric one. In this talk Ill explain how to combinatorialize Ravis procedure, replacing varieties with matroids, and degeneration with shifting (originally invented by Erdos-Ko-Rado for extremal combinatorics). One upshot will be that each of Ravis varieties is defined by linear equations, allowing us to extend his proof to K-theory. Another is an easily stated conjecture for flag manifold Schubert calculus.

Thomas Lam (Harvard)
Title: *Affine Schubert calculus*
Abstract: I will discuss some recent progress in understanding the (co)homology of the affine Grassmannian from the point of view of Schubert calculus. In particular, I will explain how to obtain polynomial representatives for Schubert classes and analogues of Pieri rules. If time permits, I hope also to explain some connections with calculations of the (co)homology ring of the affine Grassmannian due to Bott, to Ginzburg, to Bezrukavnikov, Finkelberg and Mirkovic and to Peterson. Part of this talk is based on joint work with Lapointe, Morse and Shimozono.

Cristian Lenart (SUNY Albany)
Title: *K-theory and quantum K-theory of flag varieties*
Abstract: I previously presented Chevalley-type multiplication formulas in the T -equivariant K-theory of generalized flag varieties G/P; these formulas were derived in joint work with A. Postnikov. In the first part of this talk, I will present a model for $KT = G/P$ in terms of a certain braided Hopf algebra called the Nichols-Woronowicz algebra. This model is based on the Chevalley-type formulas mentioned above, and has potential applications to deriving more general multiplication formulas. In the second part of the talk, I will show the way in which a presentation of the quantum K-theory of the classical flag variety leads to the construction of certain polynomials, called quantum Grothendieck polynomials, that are conjectured to represent Schubert classes. We present evidence for this conjecture; this includes the fact that the quantum Grothendieck polynomials satisfy a multiplication formula which is the natural generalization of the Chevalley-type formula mentioned above and of the corresponding formula in quantum cohomology. This talk is based on joint work with T. Maeno.

Elena Marchisotto (California State University, Northridge)
Title: *Evaluating the research of Mario Pieri (1860-1913) in algebraic geometry*
Abstract: Mario Pieri was an active member of the research groups surrounding Corrado Segre and Giuseppe Peano at the University of Turin around the turn of the nineteenth century. Pieri played a ma jor role in foundations of mathematics. Can the same be said of his role in algebraic geometry? Was he the first to introduce the Schubert calculus to Italy? Did he accomplish all he set out to do with the appropriate rigor? Do his papers contain there valuable ideas for contemporary work? Are his contribu- tions to enumerative geometry limited to the multiplication formula and theorem for correspondences on an n-dimensional pro jective space that are discussed in Fultons book on intersection theory? In my talk I will share my initial research in attempts to answer the above questions. I will provide an overview of Pieri the man and Pieri the algebraic geometer. I will briefly discuss Pieris papers in algebraic geometry, in particular his results in enumerative geometry and his use of the Schubert calculus. I will report on existent letters of Castelnuovo, de Paolis, Enriques, Fouret, Schubert, Severi, and Zeuthen.

Leonardo Mihalcea (Duke)
Title: *Chern-Schwartz-MacPherson classes for Schubert cells in the Grassmannian*
Abstract: A conjecture of Deligne and Grothendieck states that there is a functorial theory of Chern classes on possibly singular varieties, viewed as a natural transformation from the group of constructible functions to the homology (or Chow group) of a compact variety. This conjecture was solved in 1973 by R. MacPherson; the classes he defined (now known as Chern-Schwarz-MacPherson - or CSM) turned out to be the same as those defined by M.H. Schwartz, by different methods. In joint work with Paolo Aluffi, we give explicit formulae for the CSM classes of the Schubert varieties in the Grassmannian. The main tools used in the computation are a new formula for the CSM classes recently discovered by P. Aluffi and a Bott-Samelson resolution of a Schubert variety. Given that Schubert varieties are singular, an unexpected feature is a certain effectivity satisfied by these classes; we have proved it in few cases, and it is conjectured to hold in general.

Steve Mitchell (University of Washington)
Title: *Smooth and palindromic Schubert varieties in affine Grassmannians*
Abstract: The affine Grassmannian associated to a simple complex algebraic group G is an infinite- dimensional projective variety that is very much analogous to an ordinary Grassmannian. In particular, it comes equipped with a decomposition into Schubert cells whose closures are ordinary projective varieties. It turns

out that for each fixed G, only finitely many of these Schubert varieties are smooth, and except in type A only finitely many are even palindromic. In fact it is possible to determine the smooth and palindromic Schubert varieties explicitly. For example, in type C a Schubert variety is smooth if and only if it is a closed parabolic orbit, in which case it is a symplectic Grassmannian. It is palindromic if and only if it is either smooth or a subcomplex of the Schubert variety associated to the lowest nontrivial anti-dominant coroot lattice element; this variety is just the Thom space of a line bundle over projective space. The most eccentric type is type A, where in each rank there are two infinite families of singular palindromics. These latter varieties have interesting topological properties, which were studied in the 80s by myself and (independently) Graeme Segal.

Evgeny Mukhin (IUPUI)
Title: *On generalizations of the B. and M. Shapiro conjecture*
Abstract: A proof of the B. and M. Shapiro conjecture is based on the consideration of the periodic quantum Gaudin model. We show what results are obtained in the similar way from quasiperiodic quantum Gaudin and from quasiperiodic XXX models.

Nicolas Perrin (Paris)
Title: *Some combinatorial aspects of cominuscule geometry*
Abstract: In this talk we shall explain combinatorics tools (quivers) generalising Young diagrams that appear in the study of cominuscule homogeneous spaces. We shall see how to read on these quivers some geometric properties of the Schubert varieties like their singularities, the fact that they are Gorenstein, locally factorial or even that they admit or not small resolutions. These quivers are used by A. Yong and H. Thomas for classical Schubert calculus. We shall see how these quivers also help for computing some Gromov-Witten invariants.

James Ruffo (TAMU)
Title: *A straightening law for the Drinfeld Lagrangian Grassmannian*
Abstract: We present a structured set of defining equations for the Drinfeld compactification of the space of rational maps of a given degree in the Lagrangian Grassmannian. These equations give a straightening law on a certain ordered set, which allows the use of combinatorial arguments to establish useful geometric properties of this compactification. For instance, its coordinate ring is Cohen-Macaulay and Koszul.

Mark Shimozono (Virginia Tech)
Title: *Kac-Moody dual graded graphs and affine Schubert calculus*
Abstract: Motivated by the dual Hopf algebra structures on the homology and cohomology of the affine Grassmannians, particularly their Schubert structure constants, we exhibit families of dual graded graphs (in the sense of Fomin) associated to any Kac-Moody algebra, dominant weight and positive central element, yielding enumerative identities involving chains in the strong and weak Bruhat orders.

Hugh Thomas (University of New Brunswick)
Title: *A combinatorial rule for (co)minuscule Schubert calculus*
Abstract: I will discuss a root system uniform, concise combinatorial rule for Schubert calculus of minuscule and cominuscule flag manifolds G/P . (The latter are also known as compact Hermitian symmetric spaces.) We connect this geometry to work of Proctor in poset combinatorics, thereby generalizing Schützenbergers jeu de taquin formulation of the Littlewood-Richardson rule for computing intersection numbers of Grassmannian Schubert varieties. I will explain the rule and, time permitting, discuss ideas used in its proof, including cominuscule recursions, a general technique relating the Schubert constants for different Lie types. I will also discuss cominuscule dual equivalence, a generalization of a concept due to Haiman. We use this to provide an independent proof of Proctors jeu de taquin results in our context.

Julianna Tymoczko (Iowa)
Title: *Divided difference operators for Grassmannians* Abstract: We construct divided difference operators for Grassmannians. More precisely, we give an explicit combinatorial formula for divided difference operators on the equivariant cohomology of G/P, for any parabolic subgroup P and any complex reductive linear

algebraic group G. One application generalizes a flag-variety result of Sara Billeys to compute the localizations of equivariant Schubert classes for G/P. Another provides equivariant Pieri rules for Grassmannians. These divided difference operators are constructed using GKM (Goresky-Kottwitz-MacPherson) the- ory, which gives a combinatorial construction of equivariant cohomology for many suitably nice algebraic varieties. We will focus on how the theory works for Grassmannians of k-planes in complex n-dimensional space.

Alexander Varchenko (North Carolina)
Title: *The B. and M. Shapiro conjecture in real algebraic geometry and Bethe ansatz.*
Abstract: I shall discuss the proof of Shapiros conjecture by methods of math physics and representation theory. Shapiros conjecture says the following. If the Wronskian of a set of polynomials has real roots only, then the complex span of this set of polynomials has a basis consisting of polynomials with real coefficients.

Jan Verschelde (University of Illinois-Chicago)
Title: *Numerical homotopy algorithms for enumerative geometry*
Abstract: In a 1998 paper on Numerical Schubert Calculus, Birk Huber, Frank Sottile, and Bernd Sturmfels proposed numerical Pieri homotopy algorithms to solve problems in enumerative geometry. Im- plementations of these algorithms ran on specific examples of intersection problems whose solution set is entirely real. Jointly with Yusong Wang, the Pieri homotopies were applied to the output pole placement problem in linear systems control. Following Vakils geometric proof of the Littlewood-Richardson rule, Sottile, Vakil, and I designed deformation methods to solve general Schubert problems. Current work is directed to implement these new Littlewood-Richardson homotopies.

Alex Yong (University of Illinois)
Title: *Governing singularities of Schubert varieties*
Abstract: We present a combinatorial and computational commutative algebra methodology for studying singularities of Schubert varieties of flag manifolds. We define the combinatorial notion of interval pattern avoidance. For reasonable invariants P of singularities, we geometrically prove that this governs (1) the P -locus of a Schubert variety, and (2) which Schubert varieties are globally not P. The prototypical case is P=singular; classical pattern avoidance applies admirably for this choice [Lakshmibai-Sandhya 90], but is insufficient in general. Our approach is analyzed for some common in- variants, including Kazhdan-Lusztig polynomials, multiplicity, factoriality, and Gorensteinness, extending [Woo-Yong05]; the description of the singular locus (which was independently proved by [Billey-Warrington 03], [Cortez 03], [Kassel-Lascoux-Reutenauer03], [Manivel01]) is also thus reinterpreted. Our methods are amenable to computer experimentation, based on computing with Kazhdan-Lusztig ideals (a class of generalized determinantal ideals) using Macaulay 2. This feature is supplemented by a collection of open problems and conjectures.

4 Scientific Progress Made

An emerging theme at the workshop was the affine Schubert calculus. Previously, the primary focus in Schubert calculus was on the (generalized) cohomology rings of finite-dimensional flag varieties G/P. Work of Lam, Shimozono, and others pointed to the possibility to understand the homology and cohomology Hopf algebras of the infinite dimensional affine flag varieties. The workshop also provided an opportunity for some to understand Coşkun's new geometric proof of the that Littlewood-Richardson rule in the quantum cohomology of the Grassmannian.

5 Outcome of the Meeting

This meeting, on the 10th anniversary of the original meeting on Schubert calculus at Oberwolfach was an opportunity for the major participants in Schubert calculus to asses the current state of affairs of the subject. Some of the advances since that meeting had their genesis there. Discussions about the importance of transversality helped inspire Mukhin, Tarasov, and Varchenko's second proof of the Shapiro Conjecture [19]. Purbhoo became interested in transversality and their work, and this led to his geometric construction of the basic algorithms of Young tableaux [20]. Subsequent to the meeting, significant progress has been made in the equivariant K-theory Schubert calculus; again that had some beginnings at the meeting. Also, the affine Schubert calculus has matured since, living up to its early progress.

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