COMBINATORIAL ALGEBRAIC GEOMETRY AND DISCRETE PERIODIC OPERATORS

A Dissertation
by
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ABSTRACT

Discrete periodic operators belong to a classical class of operators arising from the tight-binding approximation from solid state physics. We study the dispersion relation of discrete periodic operators using methods from algebraic and discrete geometry. In particular, we will discuss the spectral edges of the dispersion relation, as well as methods for determining when the dispersion relation is (ir)reducible.

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NOMENCLATURE

Γ	A periodic graph $\mathbb{C}$
G	A group
X, Z	An affine or projective variety
Y	A parameter space.
T	A subset.
f, g, h, κ	general functions
$\circ$	composition of functions
d	Typical ambient dimension
$\mathbb{C}^d$	The d-dimensional vector space of complex numbers
$\mathbb{C}^*$	The algebraic torus, that is $\mathbb{C} \smallsetminus \{0\}$
$\mathbb{R}^d$	The d-dimensional vector space of real numbers
$x, y, z = z_1, \ldots, z_d$	indeterminates or vectors of indeterminates.
$\mathbf{0}$	The zero vector.
υ	A monomial or term of a polynomial.
$\mathbb{C}[z]$	The ring of polynomials with coefficients in $\mathbb{C}$ in d variables.
$\mathbb{C}[z^{\pm}]$	The ring of Laurent polynomials with coefficients in $\mathbb{C}$ in d variables.
$\mathbb{T}$	The compact torus or complex unit circle, that is $z \in \mathbb{C}$ such that $\lvert z \rvert = 1$
$\mathbb{Z}^d$	The d-dimensional lattice of integer vectors
$\mathbb{N}^d$	The d-dimensional vectors of natural numbers
$\mathbb{P}^d$	The d-dimensional complex projective space
$\sim$	A relation

R	A ring
a	an element of $\mathbb{Z}^d$
S	a set
P, K	Polytopes (typically integral)
s, p	a point especially of a set s or polytope P, respectively
i, j, k, l, m, n	an integer
t, r	an element or indeterminate of $\mathbb{R}$ or $\mathbb{Q}$
V	a potential and a function that send the vertices of some graph to a field of scalars
E	an edge labeling, a function that sends the edges of some graph to a field of scalars
$\mathcal{V}(\cdot)$	The vertices of the input graph
$\mathcal{E}(\cdot)$	The edges of the input graph
Δ, Δ_E	The discrete Laplacian and weighted discrete Laplacian
Q, A	Integer vector, or vector of natural numbers
Q/A	Coordinate-wise division
U	An open set or a finite subset
$\mathbb{U}$	A unitary operator
w	An vector over $\mathbb{R}$ or $\mathbb{Z}$, typically an inner normal vector
σ	A finite subset of $[d]$
τ	A permutation
F	A face of a polytope
$f\|_F$	A restriction to a facial form
$\mathcal{N}(\cdot)$	A Newton polytope associated to the input, typically a polynomial or periodic graph
I	the identity matrix or operator
$\mathcal{I}$	An ideal

$\bar{\cdot}$	Complex conjugation, or set complement
$\sqrt{}$	The square root function
$\binom{[d]}{k}$	The collection of all sets of k elements chosen from $[d]$ with no replacement
ϕ, ψ, χ	general functions or elements of a vector space or Hilbert space
c	coefficient or complex scalar
$\mathcal{A}$	the support of a function or fundamental domain; a finite subset of $\mathbb{Z}^d$ in either case
$\phi_{\mathcal{A}}$	A map embedding the algebraic torus
MV	Mixed volume
vol	Euclidean volume
conv	The convex hull
Aff	The affine span
u, v, ω	Vertices of a polytope or graph
ω	A vertex, especially of a fundamental domain
$\mathbf{V}(\cdot)$	Variety of an ideal or set of functions
$\mathbf{I}(\cdot)$	Ideal of a variety
rad	The radical of an ideal
$\mathbb{C}[Z]$	Coordinate ring of a variety
Spec$(\cdot)$	Variety of a coordinate ring
J	The Jacobian of a system of polynomials
det	The determinant of a matrix
$[n]$	The set of the first n positive integers
$\langle$list$\rangle$	ideal generated by a list of polynomials
$\langle \cdot, \cdot \rangle$	inner product
$\|\| \cdot \|\|$	norm on a vector space

$\lvert \cdot \rvert$	scalar norm
$\ell^2(\cdot)$	Hilbert space of square summable functions on a lattice or vertex set
$L^2(\mathbb{T})$	Hilbert space of square integrable functions
W	A fundamental domain of vertices
$\sigma(\cdot)$	The spectrum of an operator
$\rho(\cdot)$	The resolvent set of an operator
λ	The elements of the spectrum, either as a fixed value or indeterminate
H	An operator on a vector space
L	An operator on a Hilbert space, especially a discrete periodic operator
$\oplus$	direct sum
$Q\mathbb{Z}$	The free full rank subgroup $\oplus_{i=1}^{d} q_i\mathbb{Z}$ of $\mathbb{Z}^d$ where $Q = (q_1, \ldots, q_d)$
$\mathcal{F}$	Restriction of a Face, or the integer points of a face.
$\mathscr{F}$	Fourier or Floquet transform
$\hat{\cdot}$	A function or operator after a Fourier or Floquet transform
μ, γ	Elements of a group of unity
$\mathcal{U}_Q$	A direct product of groups of unity
D	The dispersion polynomial
$.^T$	Transpose of a matrix or vector
$.^*$	Adjoint of a matrix or vector
e_i	Basis vectors or functions
$\mathscr{V}, \mathscr{W}$	A vector space
$\mathscr{H}$	A Hilbert space
$\mathscr{L}$	A linear subspace

H	An operator
$\mathscr{U}$	An operator, especially a unitary one
$\cdot$	dot product
$\sigma \odot Q$	A characteristic type function on a vector
$\mid$	Divides
$\gcd$	Greatest common divisor
∇	Gradient
$\nabla_{\mathbb{T}}$	Toric Gradient
$\mathcal{L}$	A linear subspace
$\mathscr{L}, \mathscr{L}(t)$	A collection of linear subspaces, possibly continuously varying in t.
$X_{\mathcal{A}}$	An affine toric variety.
$\mathcal{X}_{\mathcal{A}}$	An projective toric variety.
N	A height-one pyramid.
M	A face of a height-one pyramid.
$\mathcal{N}(\Gamma)$	Generic Newton polytope of a periodic graph.
π	A projection map.
$\mathcal{G}$	Integer points on the base of a polytope.
Θ	A subvariety or set.
Υ	An incidence correspondence.

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1. INTRODUCTION

A discrete periodic operator L is a periodically weighted graph Laplacian with a periodic potential that acts on functions on the vertices of a $\mathbb{Z}^d$-periodic graph. Belonging to this class of operators is the well-known discrete periodic Schrödinger operator. In mathematical physics, the discrete Schrödinger operator with periodic potential (arising from the tight-binding approximation) plays an important role in solid state theory. Studying the spectrum of this operator is useful for many applications, such as nano-materials, topological insulators, and photonic crystal theory [31].

Through the Floquet transform, one can show that there exists a finite matrix $L(z)$ with Laurent polynomial entries such that collecting the eigenvalues of $L(z)$ restricted to z in the compact torus recovers the spectrum. Equivalently, the spectrum is given by the coordinate projection of the vanishing set, known as the Bloch variety, of the dispersion polynomial $D(z, \lambda) = \det(L(z) - I\lambda)$, restricted to z in the d-dimensional compact torus ($\mathbb{T}^d$). The Bloch variety lives in $\mathbb{T}^d \times \mathbb{R}$ and is a real algebraic variety.

Instead of restricting z to the compact torus, one can also consider the (complex) Bloch variety: the vanishing set of $D(z, \lambda)$ for z in the larger d-dimensional algebraic torus $(\mathbb{C}^\times)^d$. When z lies outsides of the d-dimensional compact torus, $L(z)$ is no longer self adjoint and thus may have complex eigenvalues; therefore, the (complex) Bloch variety lives in $(\mathbb{C}^\times)^d \times \mathbb{C}$. Unlike its real counterpart, the Bloch variety is an algebraic variety, which allows for the application of algebraic geometry to its study. A Fermi variety of an eigenvalue λ_0 is the collection of $z \in (\mathbb{C}^\times)^d$ such that $L(z)$ has λ_0 as an eigenvalue, which is given by the vanishing set of $D(z, \lambda_0)$ in $(\mathbb{C}^\times)^d$.

Using algebraic geometry in the study of the spectral theory of these operators dates back to at least the 1980s (e.g [27]); with much progress in the 1990s, when Giesker, Knörrer, and Trubowitz [23] employed algebraic geometry to study a wide range of algebraic properties of the Bloch varieties of discrete periodic operators associated with the square lattice. Their investigations covered the irreducibility of the Fermi varieties, Floquet isospectrality, the density of states, and more. Recently, there has been a surge of interest in using algebraic methods to study the spectral theory of these operators.

Although there have been many exciting recent developments in Floquet and Fermi isospectrality, the density of states, and many more as a result of the use of algebro-geometric methods in the study of the spectral theory of discrete periodic operators, we will limit our scope to the use of algebraic and discrete geometric methods in the study of the extrema of the Bloch variety and the (ir)reducbility of the Bloch and Fermi varieties. For a survey of the exciting developments that are outside of our scope, we refer the interested reader to [32, 40].

In Chapters 2 and 3, we will introduce the necessary background in discrete geometry, algebraic geometry, spectral theory, and discrete periodic operators for the sections to follow. In Chapter 4, we will discuss some history and recent developments in the study of the critical points of discrete periodic operators. The developments that we will discuss in Chapter 4 are based on the author's work with Frank Sottile in [14]. In Chapter 5, we will discuss some history and recent developments in the study of the reducibility of the Bloch and Fermi varieties. The developments we discuss in Chapter 5 is based on the author's work with Jordy Lopez in [12]. Finally, in Chapter 6, we provide some concluding remarks.

2.1 Discrete Geometry and Algebra

We begin with a brief discussion on polytopes and other associated convex sets. For more details we refer the reader to [11,25,53].

2.1.1 An Introduction to Geometric Combinatorics

A set $S \subseteq \mathbb{R}^d$ is *convex* if for any points $a, b \in S$, the line segment connecting a and b is contained in S. The *convex hull* of a set $S \subseteq \mathbb{R}^d$ is the smallest convex set containing S. For a finite set $S = \{s_1, \ldots, s_n\}$, the convex hull of S is explicitly

$$\text{conv}(S) = \Big\{ \sum_{i=1}^{n} t_i s_i \mid \sum_{i=1}^{n} t_i = 1 \text{ and } 0 \leq t_i \leq 1 \text{ for each } i \Big\};$$

in this case conv(S) is called a *polytope*. We call the elements $s \in \mathbb{Z}^d$ *lattice* vectors. If S is a finite subset of $\mathbb{Z}^d \subset \mathbb{R}^d$, then conv$(S)$ is an *integral (or lattice) polytope.* A compact convex set is called a *convex body*.

A set $S \subseteq \mathbb{R}^d$ is *affine* if for any points $a, b \in S$, the line connecting a and b is contained in S. The *affine span* of a set $S \subseteq \mathbb{R}^d$ is the smallest affine set containing S, in particular

$$\text{Aff}(S) = \Big\{ \sum_{i=1}^{n} t_i s_i \mid \sum_{i=1}^{n} t_i = 1 \text{ and } t_i \in \mathbb{R} \Big\}.$$

The *dimension* of a polytope $P \subset \mathbb{R}^d$ is the dimension of its affine span. We say a polytope P is *full-dimensional* its affine span is $\mathbb{R}^d$.

Let $\cdot$ denote the dot product. As a polytope P is a closed and bounded set on $\mathbb{R}^d$, given $w \in \mathbb{R}^d$, the magnitude of the linear function $s \mapsto w \cdot s$ is bounded on P. Given a polytope P and $w \in \mathbb{R}^d$, the *face exposed* by w is given by $P_w := \{p \in P \mid w \cdot p = \min_{a \in P} w \cdot a\}$. A face of a polytope is itself a polytope; moreover, the intersection of two faces of a polytope is also a face of that polytope. For a d-dimensional polytope P, the $(d-1)$-dimensional faces of P are its *facets*, the 0-dimensional faces of P are its *vertices*, and the 1-dimesional faces of P are its *edges*. In general, a k-dimensional face of P is called a k-face. Moreover, any face is given by the intersection of finitely many facets ([11, Theorem 1.11]). When P is an integral polytope, we only need integer vectors $w \in \mathbb{Z}^d$ to expose all faces of P.

The *Minkowski sum* of polytopes P and K is given by

$$P + K := \{p + k \mid p \in P, k \in K\}.$$

Note that the Minkowski sum $P + K$ is itself a polytope, and that the class of all polytopes in $\mathbb{R}^d$ forms a monoid under the Minkowski sum. Moreover, if F is a face of $P + K$, then there exist faces P_F and K_F of P and K respectively such that $F = P_F + K_F$ ([11, Theorem 1.5 Chapter IV]).

Suppose that $t \in \mathbb{R}_{\geq 0}$ and P is a polytope in $\mathbb{R}^d$, then $tP = \{tp \mid p \in P\}$ is a polytope in $\mathbb{R}^d$. We call the polytope tP a *multiple* of P. One can deduce that $2P = P + P$. Given polytopes $P_1, \ldots, P_n$ and positive real scalars $t_1, \ldots, t_n$, we say that $t_1P_1 + \cdots + t_nP_n$ is a linear combination of polytopes.

Let $\text{vol}(\cdot)$ denote the d-dimensional Euclidean volume. The volume of the linear combination of polytopes $t_1P_1 + \cdots + t_dP_d$ has the following formula ([11, Chapter IV Section 3]) as a polynomial of the t_i,

$$\text{vol}(t_1P_1 + \cdots + t_dP_d) = \sum_{a_i \in [d]} c_{P_{a_1}, \ldots, P_{a_d}} t_{a_1} \ldots t_{a_d}.$$

In particular, when treating the t_i as indeterminates, we have that $\text{vol}(t_1P_1 + \cdots + t_dP_d)$ is a homogeneous polynomial of degree d in $\mathbb{R}[t_1, \ldots, t_d]$ with coefficients which are dependent on the P_i. The coefficient of $t_1t_2 \cdots t_d$ in $\text{vol}(\lambda_1P_1 + \cdots + \lambda_dP_d)$ is called the the *mixed volume* of $P_1, P_2, \ldots, P_d$, and is denoted by $MV(P_1, \ldots, P_d)$.

Mixed volume has the following properties:

1. $MV(P, \ldots, P) = d!\text{vol}(P)$.

2. MV is symmetric in its arguments.

3. MV is monotone, that is $P_i \subseteq K_i$ then $MV(P_1, \ldots, P_d) \leq MV(K_1, \ldots, K_d)$.

4. MV is multi-linear.

Mixed volume can be computed explicitly as follows ([11, Theorem 3.7 Chapter IV]):

$$MV(P_1, \ldots, P_d) = \sum_{p \subset [d]} (-1)^{d-|p|} \text{vol}\Big(\sum_{i \in p} P_i\Big).$$

An integral polytope P is *indecomposable* if whenever P is a Minkowski sum $P = K_1 + K_2$, then one of K_1 or K_2 is a single point. Otherwise, P is *decomposable*. An integral polytope K is *homothetic* to P if there exists a rational number $r \geq 0$ and a point $a \in \mathbb{Z}^d$ such that $K = rP + a$. An integral polytope P is *only homothetically decomposable* if whenever $P = K_1 + K_2$, then one of these summands is homothetic to P. In this case, both summands are homothetic to P.

Example 2.1.1. Let $S \subset \mathbb{R}^d$ be a finite set of lattice points with an affine span of dimension less than d, and let t be a lattice point in $\mathbb{R}^d \setminus \text{Aff}(S)$. The convex hull of $S \cup \{t\}$ is a *pyramid* (see Figures 2.1 and 5.3). Pyramids are only homothetically decomposable [21]. ◇

A *strong chain of faces* of a polytope P is a sequence of faces $F_1, \ldots, F_k$ of P such that for each $i \in \{1, \ldots, k-1\}$, $\dim F_i \cap F_{i+1} \geq 1$. A strong chain of faces is called *only homothetically decomposable* if every face F_i in the chain is only homothetically decomposable. A strong chain is said to *join* two vertices u and v if u is a vertex of F_1 and v is a vertex of F_k.

In [46], it was shown that if P is a polytope such that any two vertices of P can be joined by an only homothetically decomposable strong chain of faces, then P itself must be only homothetically decomposable.

We say a face F *touches* a strong chain $F_1, \ldots, F_k$ if $F \cap (\cup_{i=1}^k F_i)$ is non-empty. In [41], it was shown that if a polytope P has an only homothetically decomposable strong chain of faces that touches each facet of P, then P is only homothetically decomposable.

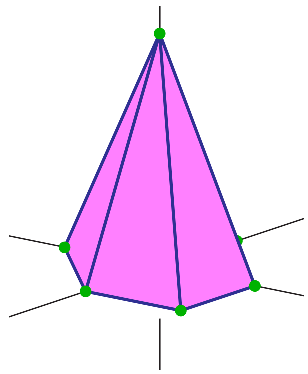


Figure 2.1: A Pyramid.

2.1.2 Polynomials to Polytopes

A *Laurent* polynomial $f \in \mathbb{C}[z^{\pm}]$ is a finite sum of monomials with complex coefficients,

$$f = \sum_{a \in \mathbb{Z}^d} c_a z^a \quad c_a \in \mathbb{C}.$$

The *support* of f, denoted by $\mathcal{A}(f)$, is the set of $a \in \mathbb{Z}^d$ such that $c_a \neq 0$, and the convex hull of $\mathcal{A}(f)$ is called the *Newton polytope* of f, denoted by $\mathcal{N}(f)$. Since the vertices of $\mathcal{N}(f)$ lie in $\mathcal{A}(f) \subseteq \mathbb{Z}^d$, it is an integral polytope. The *monomials* of f are the z^a such that $a \in \mathcal{A}(f)$. We will often refer to the product $c_a z^a$, when $c_a \neq 0$, as a *term* of f. Let $[z^a]f := c_a$ be the *coefficient* of z^a in f.

For a polynomial $f = \sum_{a \in \mathcal{A}(f)} c_a z^a$ and a face $F = \mathcal{N}(f)_w$, the *facial polynomial or facial form* of F is $f|_F := \sum_{a \in F \cap \mathcal{A}(f)} c_a z^a =: f|_w$. In other words, the facial polynomial of f identified by F is given by restricting to the terms $c_a z^a$ of f such that $a \in F$. A monomial z^a such that a is a vertex of $\mathcal{N}(f)$ is an *extreme monomial* of f. Given polynomials f and g, the Newton polytope of their product fg is given by $\mathcal{N}(fg) = \mathcal{N}(f) + \mathcal{N}(g)$.

A polynomial f is *quasi-homogeneous* with *quasi-homogeneity* $w \in \mathbb{Z}^d$ if there is a number $0 \neq w_f$ such that

$$a \in \mathcal{A}(f) \implies w \cdot a = w_f\,.$$

The quasi-homogeneities of f are those $w \in \mathbb{Z}^d$ whose dot product is constant on $\mathcal{A}(f)$. For $t \in \mathbb{C}^\times$ and $w \in \mathbb{Z}^d$, let $t^w := (t^{w_1}, \ldots, t^{w_d}) \in (\mathbb{C}^\times)^d$.

Lemma 2.1.2. Suppose that f has a quasi-homogeneity $w \in \mathbb{Z}^d$. Then

1. For $t \in \mathbb{C}^\times$ and $z \in (\mathbb{C}^\times)^d$, we have $f(t^w \cdot z) = t^{w_f} f(z)$.
2. We have
$$w_f\, f \;=\; \sum_{i=1}^{d} w_i\, z_i \frac{\partial f}{\partial z_i}\,.$$

Proof. Note that for $a \in \mathbb{Z}^d$, $(t^w \cdot z)^a = t^{w \cdot a} z^a$. The first statement follows. For the second, note that $a_i z^a = z_i \frac{\partial}{\partial z_i} z^a$. $\square$

Notice that the invertible elements in $\mathbb{C}[z^{\pm}]$ make up the collection of all nonzero terms, which forms the group of units, $(\mathbb{C}[z^{\pm}])^{\times}$. Indeed, any nonzero term cz^a has a multiplicative inverse $c^{-1}z^{-a}$ in $\mathbb{C}[z^{\pm}]$. A Laurent polynomial f is *irreducible* if it is not a monomial, and when there exist Laurent polynomials g and h such that $f = gh$, then either h or g is a monomial.

A Laurent polynomial f is *only homothetically reducible* if it is not a monomial, and if $f = gh$ implies that either $\mathcal{N}(g)$ or $\mathcal{N}(h)$ is homothetic to $\mathcal{N}(f)$. An irreducible Laurent polynomial is only homothetically reducible.

In this way, we view irreducibility and only homothetic reducibility as the polynomial analogs of indecomposablity and only homothetic decomposability for integral polytopes that were discussed in the previous section.

Example 2.1.3. A polynomial with an only homothetically decomposable Newton polytope is only homothetically reducible. The converse is false. Consider the reducible polynomial $f(x, y) = (xy + x + y + 2)^2$. Here $\mathcal{N}(f)$ is a square and thus can be decomposed into the two segments $\mathcal{N}(1 + y + y^2)$ and $\mathcal{N}(1 + x + x^2)$, neither of which is homothetic to $\mathcal{N}(f)$. However, the polynomial is only homothetically reducible as each factor $xy + x + y + 2$ is irreducible and $\mathcal{N}(f) = 2\mathcal{N}(xy + x + y + 2)$. $\diamond$

We end this section with a closing remark regarding our choice of definitions and notation.

Remark 2.1.4. Upon looking into previous works, particularly on only homothetic decomposability in the theory of polytopes such as [41, 46], one will see what we call "only homothetically decomposable" referred to as indecomposable. Our choice of language is due to working with Newton polytopes (and therefore restricting to integral polytopes), rather than general polytopes, which are never indecomposable in the way we define it. This decision provides us the following two implications given a Laurent polynomial $f \in \mathbb{C}[z^{\pm}]$:

$$\begin{aligned} \mathcal{N}(f) \text{ is indecomposable } &\implies f \text{ is irreducible,} \\ \mathcal{N}(f) \text{ is only homothetically decomposable } &\implies f \text{ is only homothetically reducible.} \end{aligned}$$

Finally, we should add that in [21], a Laurent polynomial f such that $\mathcal{N}(f)$ is indecomposable is referred to as an "absolutely irreducible" polynomial. $\diamond$

2.1.3 From Only Homothetic Decomposability to Only Homothetic Reducibility

This section is only necessary for Section 5.3, and it is based on [12, Section 3]. In it, we will extend the decomposability results of [46] to a class of Laurent polynomials.

As discussed in Section 2.1.1, only homothetic decomposability was considered in [46]. They showed that if enough faces of a polytope are only homothetically decomposable, then the polytope itself must be only homothetically decomposable.

We will abuse notation slightly throughout the following proofs. In particular, if f, g, and h are Laurent polynomials such that $f = gh$ and F is a face of $\mathcal{N}(f)$, we will write $f|_F = g|_F h|_F$ as the factorization of $f|_F$. Recall that there exists an inner normal $w \in \mathbb{R}^d$ that exposes F (and is such that $f|_w = f|_F$), and so really we mean that $g|_F = g|_w$ and $h|_F = h|_w$. We also assume that for any polytope P, $0P = \{\mathbf{0}\}$.

Remark 2.1.5. If f is only homothetically reducible then there exists Laurent polynomials g, h, and $r, t \in \mathbb{Q}$ such that $f = gh$, $r\mathcal{N}(f) = \mathcal{N}(g)$, and $t\mathcal{N}(f) = \mathcal{N}(h)$. Indeed, by the original definition of only homothetic irreducibility there exists a_g and a_h in $\mathbb{Z}^d$ so that $r\mathcal{N}(f) + a_g = \mathcal{N}(g)$, $t\mathcal{N}(f) + a_h = \mathcal{N}(h)$, and so $r\mathcal{N}(f) + a_g + t\mathcal{N}(f) + a_h = \mathcal{N}(f)$. It follows that $a_g + a_h = \mathbf{0}$, and thus there exists $g' = z^{a_h} g$ and $h' = z^{a_g} h$ such that $r\mathcal{N}(f) = \mathcal{N}(g')$ and $t\mathcal{N}(f) = \mathcal{N}(h')$.

Lemma 2.1.6. Let f, g, and h be Laurent polynomials and suppose that $f = gh$. Let F_1 and F_2 be faces of $\mathcal{N}(f)$ with $\dim F_1 \cap F_2 \geq 1$ whose corresponding facial polynomials, $f|_{F_1}$ and $f|_{F_2}$, are only homothetically reducible. If $\mathcal{N}(g|_{F_1}) = r\mathcal{N}(f|_{F_1})$ and $\mathcal{N}(h|_{F_1}) = t\mathcal{N}(f|_{F_1})$ for some pair $r, t \in \mathbb{Q}$, then $\mathcal{N}(g|_{F_2}) = r\mathcal{N}(f|_{F_2})$ and $\mathcal{N}(h|_{F_2}) = t\mathcal{N}(f|_{F_2})$.

Proof. As $f = gh$, we have that $f|_{F_1} = g|_{F_1} h|_{F_1}$. As $f|_{F_1}$ is only homothetic reducibility we have that $r\mathcal{N}(f|_{F_1}) = \mathcal{N}(g|_{F_1})$ and $t\mathcal{N}(f|_{F_1}) = \mathcal{N}(h|_{F_1})$ for some $r, t \in \mathbb{Q}$. Let $F' = F_1 \cap F_2$. As $F' \subset F_1$, it follows that $\mathcal{N}(g|_{F'}) = r\mathcal{N}(f|_{F'})$ and $\mathcal{N}(h|_{F'}) = t\mathcal{N}(f|_{F'})$. The polynomial $f|_{F_2}$ is only homothetically reducible and must agree with its restriction to F'; it follows that $r\mathcal{N}(f|_{F_2}) = \mathcal{N}(g|_{F_2})$ and $t\mathcal{N}(f|_{F_2}) = \mathcal{N}(h|_{F_2})$. □

Theorem 2.1.7. Let f be a Laurent polynomial, and suppose $f = gh$. If for each pair (a, b) of distinct vertices of $\mathcal{N}(f)$ there is a strong chain of faces $F_1, \ldots, F_n$ such that $a \in F_1$, $b \in F_n$, and, for each F_i, the corresponding facial polynomial $f|_{F_i}$ is homothetically reducible, then f is only homothetically reducible.

Proof. By Lemma 2.1.6, there exist a pair of rational numbers $r, t \in \mathbb{Q}$ such that $r\mathcal{N}(f|_{F_i}) = \mathcal{N}(g|_{F_i})$ and $t\mathcal{N}(f|_{F_i}) = \mathcal{N}(h|_{F_i})$ for all $i = 1, \ldots, n$. As $a \in F_1$ and $b \in F_n$, we have that $r\mathcal{N}(f|_a) = \mathcal{N}(g|_a)$, $t\mathcal{N}(f|_a) = \mathcal{N}(h|_a)$, $r\mathcal{N}(f|_b) = \mathcal{N}(g|_b)$, and $t\mathcal{N}(f|_b) = \mathcal{N}(h|_b)$. This is the case for all vertex pairs $(a, b) \in \mathcal{N}(f)$. In particular, we may fix a and let b vary over the other vertices. As any vertex of $\mathcal{N}(f)$ must come from the Minkowski sum of a pair of vertices u, v where $u \in \mathcal{N}(g)$ and $v \in \mathcal{N}(h)$, and any vertex u of $\mathcal{N}(g)$ or v of $\mathcal{N}(h)$ must be a Minkowski summand for some vertex of $\mathcal{N}(f)$; it follows that $r\mathcal{N}(f) = \mathcal{N}(g)$ and $t\mathcal{N}(f) = \mathcal{N}(h)$. □

Corollary 2.1.8. Suppose that f is only homothetically reducible. If there is a face F of $\mathcal{N}(f)$ such that $f|_F$ is irreducible, then f is irreducible.

Proof. Suppose that f is only homothetically reducible and let F be a face of $\mathcal{N}(f)$ such that $f|_F$ is irreducible. Suppose that g, h are Laurent polynomials such that $f = gh$. As f is only homothetically reducible, there exists $r, s \in \mathbb{Q}$ such that $r\mathcal{N}(f) = \mathcal{N}(g)$ and $t\mathcal{N}(f) = \mathcal{N}(h)$, and so for any face F' of $\mathcal{N}(f)$ we have that $r\mathcal{N}(f|_{F'}) = \mathcal{N}(g|_{F'})$ and $t\mathcal{N}(f|_{F'}) = \mathcal{N}(h|_{F'})$. Notice that $f|_F$ is irreducible and therefore, one of $g|_F$ or $h|_F$ is a monomial. As one of $h|_F$ or $g|_F$ must be a monomial (which by Remark 2.1.5 we can assume to be the constant monomial), either t or r is zero. □

2.2 Some Algebraic Geometry

In this section, we will provide a brief introduction to some aspects of algebraic geometry. For more on algebraic geometry, see [5, 7, 45].

An (affine) *variety* is the set of common zeroes of some collection of polynomials $f_1, \ldots, f_n \in \mathbb{C}[z_1, \ldots, z_d] = \mathbb{C}[z]$,

$$\mathbf{V}(f_1, \ldots, f_n) := \{x \in \mathbb{C}^d \mid f_1(x) = \cdots = f_n(x) = 0\}.$$

We will also refer to this as the set of *solutions* to the system of equations

$$f_1 = 0, f_2 = 0, \ldots, f_n = 0,$$

or as the *vanishing set(or locus)* of $f_1, \ldots, f_n$. If $n = 1$, then $\mathbf{V}(f_1)$ is a *hypersurface*, which is an affine variety of dimension $d - 1$. If Z_1 and Z_2 are varieties such that $Z_2 \subseteq Z_1$, we say that Z_2 is a *subvariety* of Z_1.

Given an affine variety Z, the *ideal* $\mathbf{I}(Z)$ is the collection of all polynomials that vanish on Z. In particular,

$$\mathbf{I}(Z) = \{f \in \mathbb{C}[z_1, \ldots, z_d] \mid f(x) = 0 \text{ for all } x \in Z\}.$$

This set is an ideal of the polynomial ring $\mathbb{C}[z]$. We write $\langle f_1, \ldots, f_n \rangle$ for the ideal generated by the polynomials $f_1, \ldots, f_n$.

Similarly, we may obtain an affine variety given an ideal $\mathcal{I} \subset \mathbb{C}[z]$:

$$\mathbf{V}(\mathcal{I}) = \{x \in \mathbb{C}^d \mid f(x) = 0 \text{ for all } f \in \mathbb{C}[z]\}.$$

By the Hilbert Basis Theorem, every ideal $\mathcal{I}$ of $\mathbb{C}[z]$ is finitely generated. Therefore, $\mathbf{V}(\mathcal{I})$ is an affine variety.

Notice that, for any $f \in \mathbb{C}[z]$ we have that $\mathbf{V}(\langle f^2 \rangle) = \mathbf{V}(\langle f \rangle)$, and so we see that the varieties of two distinct ideals can agree. Moreover, $\mathbf{IV}(\langle f^2 \rangle) = \langle f \rangle$. The reason for this observation is summarized by the following property of $\mathbf{I} \circ \mathbf{V}$:

$$\mathbf{I}(\mathbf{V}(\mathcal{I})) = \operatorname{rad}(\mathcal{I}) = \{f | f^k \in I \text{ for some } k \in \mathbb{Z}_{\geq 1}\}.$$

Here, $\operatorname{rad}(\mathcal{I})$ is referred to as the *radical* of the ideal $\mathcal{I}$. There is a bijection between affine varieties of $\mathbb{C}^d$ and radical ideals of $\mathbb{C}[z]$.

The functions $\mathbf{I}$ and $\mathbf{V}$ also both satisfy the following reverse inclusion relationships:

1. $\mathcal{I}_1 \subseteq \mathcal{I}_2 \implies \mathbf{V}(\mathcal{I}_2) \subseteq \mathbf{V}(\mathcal{I}_1)$

2. $Z_1 \subseteq Z_2 \implies \mathbf{I}(Z_2) \subseteq \mathbf{I}(Z_1)$

A variety is said to be *irreducible* if it is not the union of two proper subvarieties. That is, Z is irreducible if whenever there exist $Z_1, Z_2 \subseteq Z$ such that $Z = Z_1 \cup Z_2$, then we have either $Z = Z_1$ or $Z = Z_2$. If $\operatorname{rad}(\mathcal{I})$ is prime, then $\mathbf{V}(\mathcal{I})$ is irreducible; in particular, if f is an irreducible polynomial, then $\mathbf{V}(\langle f \rangle)$ is an irreducible variety. The *dimension* of a variety $Z \subseteq \mathbb{C}^d$ is the length of the longest chain of strictly decreasing irreducible subvarieties of Z, that is

$$(Z \supseteq) Z_0 \supsetneq Z_1 \cdots \supsetneq Z_k \neq \emptyset, \text{ where each } Z_i \text{ is irreducible.}$$

If this is indeed the longest chain, we say that Z has dimension k. The codimension of Z is $d - k$. It follows that if Z is irreducible, then any proper subvariety has a smaller dimension. Consider the set $\mathscr{L}$ of linear subspaces of dimension $d - k$ such that $Z \cap \mathcal{L}$ has only finitely many points for each $\mathcal{L} \in \mathscr{L}$. The *degree* of Z is given by $\max_{\mathcal{L} \in \mathscr{L}} |Z \cap \mathcal{L}|$.

The collection of varieties on $\mathbb{C}^d$ induces a topology whose closed sets are the affine varieties. Therefore, the open sets are exactly the complements of these varieties in $\mathbb{C}^d$. Moreover, the open sets given by complements of hypersurfaces of $\mathbb{C}^d$ give us a basis for our topology. This induced topology is known as the *Zariski topology*. Notably, if $Z(\neq \mathbb{C}^d)$ is a proper affine variety, then the Zariski open set $\mathbb{C}^d \setminus Z$ is a dense open set. If a property holds in a dense Zariski open set, we say that the property is *generic*, or holds for a generic element of the space being considered.

The *coordinate ring* of a variety Z is given by $\mathbb{C}[Z] = \mathbb{C}[z]/\mathbf{I}(Z)$. The elements of $\mathbb{C}[Z]$ are called the *regular functions* on Z. Just as varieties of $\mathbb{C}^d$ are in bijection with radical ideals of $\mathbb{C}[z]$, there is a bijection between the radical ideals of $\mathbb{C}[Z]$ and the affine subvarieties of Z. Additionally, if Z is irreducible, then $\mathbf{I}(Z)$ is prime and so $\mathbb{C}[Z]$ is an integral domain.

We can also define maps between the varieties $Z_1 \subseteq \mathbb{C}^{d_1}$ and $Z_2 \subseteq \mathbb{C}^{d_2}$. A function $\phi : Z_1 \to Z_2$ is said to be a *regular map* if there exists polynomials $f_1, \ldots, f_{d_2} \in \mathbb{C}[Z]$ such that $\phi(z_1, \ldots, z_{d_1}) = (f_1(z_1, \ldots, z_{d_1}), \ldots, f_{d_2}(z_1, \ldots, z_{d_1}))$ and $\phi(Z_1) \subseteq Z_2$. A regular map induces a map ϕ^* between the coordinate rings, $\mathbb{C}[Z_2] \to \mathbb{C}[Z_1]$, given by $g \mapsto g \circ \phi$.

One can similarly consider subvarieties of the *algebraic torus* $(\mathbb{C}^\times)^d = \mathbb{C}^d \setminus \{\mathbf{0}\}$ and of the projective space $\mathbb{P}^d = \mathbb{P}(\mathbb{C}^{d+1})$. A Laurent monomial is a map $(\mathbb{C}^\times)^d \to \mathbb{C}^\times$. In general, the affine varieties of $(\mathbb{C}^\times)^d$ are given by the vanishing of a collection of finitely many Laurent polynomials. Indeed, the affine varieties of the algebraic torus are exactly given by the vanishing sets of polynomials in

$$\mathbb{C}[z_1^\pm, \ldots, z_d^\pm] = \mathbb{C}[z^\pm] \simeq \mathbb{C}[z_1, \ldots, z_d, t]/\langle t z_1 \cdots z_d - 1\rangle.$$

To define $\mathbb{P}^d$, let us first define the relation $\sim$:

$$(x_0, \ldots, x_d) \sim (y_0, \ldots, y_d) \text{ if there exists } c \in \mathbb{C}^\times \text{ such that } (y_0, \ldots, y_d) = (cx_0, \ldots, cx_d).$$

With this equivalence relation, we define projective space as

$$\mathbb{P}^d = (\mathbb{C}^{d+1} \setminus \{0\})/\sim.$$

The points of $\mathbb{P}^d$ are denoted by $[x_0 : \cdots : x_d]$ where $x_i \in \mathbb{C}$. These points each correspond to a line through the origin in $\mathbb{C}^{d+1}$. The *degree* of a monomial is given by the sum of the coordinates of its exponent vector. A *homogeneous* polynomial $f \in \mathbb{C}[z_0, z_1, \ldots, z_d]$ is a polynomial where each of its terms have the same degree. Notice that if a homogeneous polynomial f of degree n vanishes on $(x_0, \ldots, x_d)$, then it vanishes on $(cx_0, \ldots, cx_d)$ whenever $c \neq 0 \in \mathbb{C}$ as $f(cx_0, \ldots, cx_d) = c^n f(x_0, \ldots, x_d)$; so the vanishing sets of homogeneous polynomials are well-defined on $\mathbb{P}^d$. Thus a collection of homogeneous polynomials $f_1, \ldots, f_d \in \mathbb{C}[z]$ defines a *projective variety*.

Consider the affine variety $\mathbf{V}(\langle f_1, \ldots, f_k\rangle)$ generated by $f_i \in \mathbb{C}[z_1, \ldots, z_d]$. Let n_i be the max degree amoung all terms of f_i. Consider the the new system of homogeneous polynomials

$$z_0^{t_1} f_1(z_1/z_0, \ldots, z_d/z_0) = \cdots = z_0^{t_k} f_k(z_1/z_0, \ldots, z_d/z_0) = 0.$$

This defines a projective variety in $\mathbb{P}^d$. We call the vanishing set of this new system the *projective closure* of $\mathbf{V}(\langle f_1, \ldots, f_k\rangle)$.

Suppose that $Z = \mathbf{V}(f_1, \ldots, f_n) \subset \mathbb{C}^d$, where $f_1, \ldots, f_n$ generate $\mathbf{I}(Z)$. The *Jacobian* of $f_1, \ldots, f_n \in \mathbb{C}[z]$ is the matrix $J(f_1, \ldots, f_n)$ given by $J(f_1, \ldots, f_n)_{i,j} = \frac{\partial f_i}{\partial z_j}$, where $\frac{\partial f_i}{\partial z_j}$ denotes the first derivative of f_i with respect to the variable z_j. The *smooth (nonsingular)* locus of Z is the open subset of points of Z where $J(f_1, \ldots, f_n)$ has maximal rank.

Let f be a square-free polynomial (that is, $\langle f\rangle = \mathrm{rad}(\langle f\rangle)$). A point x is a *smooth (regular) point* on the hypersurface $\mathbf{V}(f)$ if the gradient $\nabla f(x) = (\frac{\partial f}{\partial z_1}(x), \ldots, \frac{\partial f}{\partial z_d}(x))$ is nonzero. The point $x \in \mathbf{V}(f)$ is *singular* if all partial derivatives of f vanish at x. The *multiplicity* (or local degree) at a point $x = (x_1, \dots, x_d) \in \mathbf{V}(f)$ is m if for a generic choice of $c \in \mathbb{C}^d$ we have that the polynomial $f(t) = f(x_1 + c_1 t, \dots, x_d + c_d t)$ is such that $t^m | f(t)$ but t^{m+1} does not divide $f(t)$ (see [5, Definition 1 Chapter 3 Section 4]).

Let $f\colon Z_1 \to Z_2$ be a regular map of varieties over a field of characteristic 0, with $f(Z_1)$ dense in Z_2; then there is an open subset U of Z_2 such that if $y \in Z_2$, then $\dim f^{-1}(y) + \dim Z_2 = \dim Z_1$. Given such a map f, *Bertini's Theorem* states that if Z_1 is smooth, then U may be chosen so that for every $y \in U$, the fiber $f^{-1}(y)$ is smooth [45, Theorem 2.27].

A *toric variety* is an irreducible variety Z with $(\mathbb{C}^\times)^d$ as a dense open subset, together with an action of $(\mathbb{C}^\times)^d$ on Z which extends the action of $(\mathbb{C}^\times)^d$ on itself. Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a finite subset of $\mathbb{Z}^d$ such that its affine span contains $\mathbb{Z}^d$. We may embed the algebraic torus $(\mathbb{C}^\times)^d$ into $\mathbb{P}^{n-1}$ through the map $\phi_\mathcal{A}$:

$$\begin{aligned} \phi_\mathcal{A} : (\mathbb{C}^\times)^d &\to \mathbb{P}^{n-1} \\ z = (z_1, \dots, z_d) &\mapsto [z_1^a : \dots : z_n^a]. \end{aligned}$$

Taking the closure of $\phi_\mathcal{A}((\mathbb{C}^\times)^d)$ in $\mathbb{P}^{n-1}$ yields the toric variety $X_\mathcal{A}$.

Given a Laurent polynomial with support a subset of $\mathcal{A}$, $f = \sum_{i=1}^n c_{a_i} z^{a_i}$, the space $X_\mathcal{A}$ effectively provides a space where f can be viewed as a homogeneous linear function. That is, if $f(x_1, \dots, x_d) = 0$ for some $x \in (\mathbb{C}^\times)^d$, then the linear form $\Lambda_f = \sum_{i=1}^n c_{a_i} z_{i-1}$ vanishes at $\phi_\mathcal{A}(x_1, \dots, x_d)$ in $\mathbb{P}^{n-1}$. For more on toric varieties see [6, 7, 19].

We finish with a brief discussion on bounding the number of isolated solutions to a system of Laurent polynomials. Suppose that we are given a system of d generic Laurent polynomials

$$f_1 = f_2 = \dots = f_d = 0, \text{ where } f_i \in \mathbb{C}[z^\pm],$$

such that $\mathcal{N}(f_i) = \mathcal{N}(f_j)$ for all $i, j \in [d]$. By *Kuchnirenko's Theorem*, there are exactly $d!\mathrm{vol}(\mathcal{N}(f_1))$ isolated solutions in $(\mathbb{C}^\times)^d$. When the system is not generic, the number of isolated solutions counted with multiplicity in $(\mathbb{C}^\times)^d$ is less than $d!\mathrm{vol}(\mathcal{N}(f_1))$ [22, Chapter 6 Section 2].

In Chapter 4, we will present a mild extension of Kushnirenko's Theorem to polynomials in the ring $\mathbb{C}[z^\pm, \lambda]$, which contains the polynomials we are interested; that is characteristic polynomials of matrices with Laurent polynomial entries, which the next chapter will introduce.

3.1 Periodic Graphs

We begin by recalling some basic notions from group theory. An *abelian group* $G = (G, +)$ is a set equipped with a binary operation $+ : G \times G \to G$, such that: (identity) there exists element $0 \in G$ such that $a = 0 + a = a + 0$ for any $a \in G$; (commutes) for any $a_1, a_2 \in G$ we have $a_1 + a_2 = a_2 + a_1$; and (inverse) for any $a \in G$ there exists a unique element $-a$ such that $a + (-a) = 0$. We say a group G is finitely generated if there exists some $a_1, \ldots, a_n \in G$ such that every element of G can be written as a linear combination of the elements $a_1, \ldots, a_n$.

A *homomorphism* of groups $f : G_1 \to G_2$ is a linear map from G_1 to G_2, that is $f(a_1 + a_2) = f(a_1) + f(a_2)$. A homomorphism is an *isomorphism* if $f(G_1) = G_2$ (surjective) and $f(a) = 0$ only if $a = 0$ (injective). If there exists an isomorphism between two groups G_1 and G_2, then we say they are *isomorphic*. A finitely generated abelian group G is *free* if it is isomorphic to $\mathbb{Z}^d$ for some $d \in \mathbb{N}$, in which case we say that G has *rank* d.

A *(right) group action* of an abelian group G on a set S is a function $\gamma : S \times G \to S$ such that for $x \in S$ we have $\gamma(x, 0) = x$ and $\gamma(x, a + b) = \gamma(\gamma(a, x), b)$, where 0 denotes the identity element of G. From here on we will write $\gamma(x, a) = x + a$. The *orbit* of an element $x \in S$ under the action G is the set $x + G = \{x + a \mid a \in G\}$. We denote the collection of distinct orbits of S under the action G by S/G. We say the G action is *free* on S if for all $x \in S$: if $x + a = x$ for some $a \in G$, then $a = 0$.

Definition 3.1.1. Let G be a finitely generated free abelian group. A *G-periodic graph* $\Gamma = (\mathcal{V}(\Gamma), \mathcal{E}(\Gamma))$ is an infinite simple graph (undirected with no multiple edges or loops) with vertices $\mathcal{V}(\Gamma)$ and edges $\mathcal{E}(\Gamma) \subset \mathcal{V}(\Gamma) \times \mathcal{V}(\Gamma)$ that satisfy the following:

1. G acts freely on both the vertices $\mathcal{V}(\Gamma)$ and the edges $\mathcal{E}(\Gamma)$.
2. Both $\mathcal{V}(\Gamma)/G$ and $\mathcal{E}(\Gamma)/G$ are finite sets.

The action of G on $\mathcal{E}(\Gamma)$ is denoted by $(u, v) + a = (u + a, v + a) \in \mathcal{E}(\Gamma)$ for $a \in G$ and $(u, v) \in \mathcal{E}(\Gamma)$. In this context, G is often called the *(abstract) period lattice*. ⋄

Notice that we do not require a $\mathbb{Z}^d$-periodic graph to be connected, as is often the case in the literature.

A *fundamental domain* is a finite set $W \subset \mathcal{V}(\Gamma)$ of representatives of the orbits $\mathcal{V}(\Gamma)/G$. If U is a collection of vertices and a an element of G, we write $U + a := \bigcup_{u \in U} u + a$. In this way, if W is a fundamental domain we have that $\mathcal{V}(\Gamma)$ is a disjoint union of the translates of W. Figure 3.1 illustrates two $\mathbb{Z}^2$-periodic graphs realized in $\mathbb{R}^2$.

Let W be a fundamental domain of the $\mathbb{Z}^d$-periodic graph Γ. The *support* of a collection of vertices $U \subseteq W$, denoted $\mathcal{A}(U)$, is the collection of $a \in \mathbb{Z}^d$ such that there exists an edge in $\mathcal{E}(\Gamma)$ between some vertex of U and some vertex of $U + a$. In particular

$$\mathcal{A}(U) := \{a \in \mathbb{Z}^d | \text{ there exists } (u, v + a) \in \mathcal{E}(\Gamma) \text{ for some } u, v \in U\}.$$

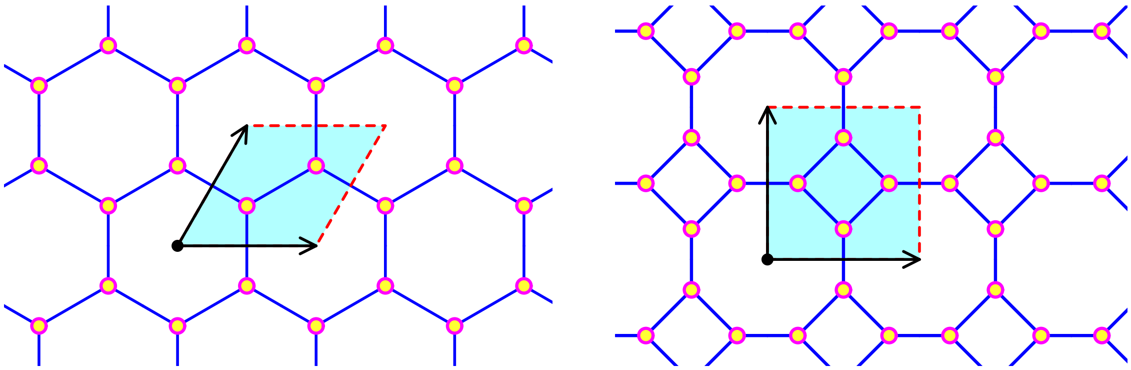


Figure 3.1: The left figure is the hexagonal lattice. The right figure is an abelian cover of K_4.

Let Γ be a $\mathbb{Z}^d$-periodic graph with fundamental domain W. For $Q := (q_1, \dots, q_d) \in (\mathbb{N})^d$, let $Q\mathbb{Z}$ denote the finite-index subgroup $\bigoplus_{i=1}^d q_i\mathbb{Z}$ of $\mathbb{Z}^d$. As Γ is also $Q\mathbb{Z}$-periodic, the fundamental domain W induces a natural fundamental domain for Γ as a $Q\mathbb{Z}$-periodic graph, $W_Q = \bigcup_{a\in\mathbb{Z}^d|0\leq a_i<q_i} W + a$, where $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$. We will sometimes refer to W_Q as an *expansion* of W to emphasize relationship between W_Q and W. Figure 3.2 highlights the fundamental domain $W_{(3,2)}$ of the hexagonal lattice shown in Figure 3.1. Let Γ be a G-periodic graph

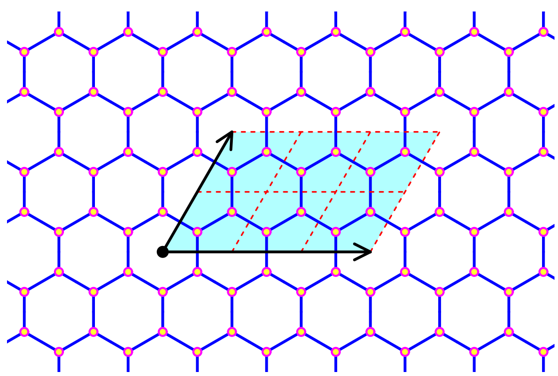


Figure 3.2: The $(3, 2)\mathbb{Z}$-periodic hexagonal lattice.

with fundamental domain W, and for each $(u, v) \in W$ let $k_{(u,v)}$ denote the largest integer such that there exists $A_1, \dots, A_{k_{(u,v)}} \in G$ such that $(u, v+A_i) \in \mathcal{E}(\Gamma)$ and $(u, v+A_i)+G = (u, v+A_j)+G$ only if $i = j$. Let Γ/G be the finite multi-graph with vertices $\mathcal{V}(\Gamma/G) := W$ and edges

$$\mathcal{E}(\Gamma/G) := \{(u, v)_i | u, v \in W, i \in [k_{(u,v)]}\}.$$

We call Γ/G the *quotient graph* of Γ. In other words, Γ/G is exactly the finite graph given by the natural realization of a graph with vertices given by the vertex orbits $\mathcal{V}(\Gamma)/G$ and edges given by the edge orbits $\mathbf{E} := \mathcal{E}(\Gamma)/G$; that is, it is the image of the natural projection $\Gamma \to \Gamma/\mathbb{Z}^d$ (see

Figure 3.3). It is natural to view Γ as an *abelian covering* of the finite graph Γ/G. This point of view is explored in [51].

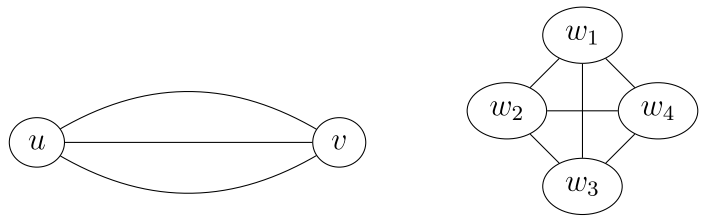


Figure 3.3: Respective quotient graphs of the periodic graphs from Figure 3.1.

3.2 Basic Spectral Theory

Given a $\mathbb{Z}^d$-periodic graph $\Gamma = (\mathcal{V}(\Gamma), \mathcal{E}(\Gamma))$, we wish the study the spectrum of a class of bounded linear operators (often selfadjoint) acting on $\ell^2(\mathcal{V}(\Gamma))$, the Hilbert space of square summable complex-valued functions on the vertices of Γ. In this section, we will introduce the basics of spectral theory on bounded linear operators. Much of our discussion follows [8, Chapter 1]. Another great source is [52].

A *norm* on a vector space $\mathscr{V}$ over the complex numbers is a function $\|\cdot\| : \mathscr{V} \to \mathbb{R}$ that satisfies the following properties for all $\phi, \psi \in \mathscr{V}$ and all $c \in \mathbb{C}$:

1. $\|\phi\| \geq 0$.
2. $\|\phi\| = 0$ if and only if $\phi = 0$.
3. $\|c\phi\| = |c|\|\phi\|$.
4. $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$.

Then $(\mathscr{V}, \|\cdot\|)$ is a *normed space*. A norm $\|\cdot\|$ on $\mathscr{V}$ induces a metric on $\mathscr{V}$ via

$$d(\phi, \psi) = \|\phi - \psi\|.$$

Let $\mathscr{V}$ be a vector space over the complex numbers. An *inner product* is a function $\langle \cdot, \cdot \rangle : \mathscr{V} \times \mathscr{V} \to \mathbb{C}$ such that for all $\phi, \psi, \chi \in \mathscr{V}$ and all $c_1, c_2 \in \mathbb{C}$ we have:

1. $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$, where the overline denotes complex conjugation.
2. $\langle \phi, \phi \rangle \geq 0$ (By Condition (1), $\langle \phi, \phi \rangle$ is real as $\langle \phi, \phi \rangle = \overline{\langle \phi, \phi \rangle}$).
3. $\langle \phi, \phi \rangle = 0$ if and only if $\phi = 0$.
4. $\langle c_1\phi + c_2\psi, \chi \rangle = c_1\langle \phi, \chi \rangle + c_2\langle \psi, \chi \rangle$.

Every inner product on $\mathscr{V}$ gives rise to a norm on $\mathscr{V}$, defined by

$$\|\phi\| := \sqrt{\langle \phi, \phi \rangle}.$$

We call such a normed spaced $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ an *inner product space*. When an inner product space is also complete, it is called a *Hilbert space*.

Example 3.2.1. Let $\ell^2(\mathbb{Z})$ be the collection of square summable sequences on $\mathbb{Z}$. Explicitly,

$$\ell^2(\mathbb{Z}) = \Big\{ \phi : \mathbb{Z} \to \mathbb{C} \mid \sum_{n \in \mathbb{Z}} |\phi_n|^2 < \infty \Big\}.$$

For $\phi, \psi \in \ell^2(\Gamma)$, we may define the inner product:

$$\langle \phi, \psi \rangle = \sum_{n \in \mathbb{Z}} \phi(n) \overline{\psi(n)}.$$

Therefore, the induced norm of an element $\phi \in \ell^2(\Gamma)$ is given by

$$\|\phi\| = \sqrt{\sum_{n \in \mathbb{Z}} \phi(n) \overline{\phi(n)}} = \sqrt{\sum_{n \in \mathbb{Z}} |\phi(n)|^2}.$$

One can show that this norm induces a complete metric, and therefore $\ell^2(\mathbb{Z})$ ($= (\ell^2(\mathbb{Z}), \langle \cdot, \cdot \rangle)$ is a Hilbert space (see [8, Theorem 1.2.7] for details). ◇

Let $\mathscr{V}$ and $\mathscr{W}$ be normed spaces. A bounded linear operator $L : \mathscr{V} \to \mathscr{W}$ is a map from $\mathscr{V}$ to $\mathscr{W}$ with the following properties:

1. (linear) for $\phi, \psi \in \mathscr{V}$ and $b, c \in \mathbb{C}$ we have $L(b\phi + c\psi) = bL(\phi) + cL(\psi)$.
2. (bounded) The *operator norm* is bounded, that is

$$\|L\| := \sup\{\|L(\phi)\| \mid \phi \in \mathscr{V}, \|\phi\| = 1\} < \infty.$$

Suppose that L is a bounded linear operator from $L : \mathscr{V} \to \mathscr{W}$. Let $I_{\mathscr{V}}$ denote the identity operator from $\mathscr{V}$ to $\mathscr{V}$ given by the map $I_{\mathscr{V}}(\phi) = \phi$ for all $\phi \in \mathscr{V}$. An operator L such that there exists $H : \mathscr{W} \to \mathscr{V}$ such that $LH = I_W$ and $HL = I_{\mathscr{V}}$ is called *invertible*. In this case we write $L^{-1} = H$ and call H the inverse of L. When the ambient space is clear we will simply write I for $I_{\mathscr{V}}$.

Suppose that $L : \mathscr{V} \to \mathscr{V}$ is a bounded linear operator. The collection of $\lambda \in \mathbb{C}$ such that the operator $L - \lambda I_{\mathscr{V}}$ (= $L - \lambda I$) is invertible is called the *resolvent set* of L, denoted by $\rho(L)$. The *spectrum* of L, denoted by $\sigma(L)$, is given by the complement of the resolvent set in $\mathbb{C}$. That is,

$$\sigma(L) = \mathbb{C} \setminus \rho(L) = \{\lambda \in \mathbb{C} \mid L - \lambda I \text{ is not invertible}\}.$$

If $\mathscr{V}$ is a Banach space, then for any bounded linear operator $L : \mathscr{V} \to \mathscr{V}$, $\sigma(L)$ is a nonempty compact subset of $\mathbb{C}$ [8, Proposition 1.3.9].

A $\lambda \in \sigma(L)$ is called an *eigenvalue* if there exists $\phi \in \mathscr{V}$ such that $L\phi = \lambda\phi$; that is, $L - \lambda I$ has a nontrivial kernel. The collection of eigenvalues, however, does not necessarily make up the entire spectrum and may even be empty (as we will see later in this section). The collection of $\lambda \in \sigma(L)$ such that λ is an eigenvalue is known as the *discrete spectrum*, and its complement in $\sigma(L)$ is called the *essential spectrum*.

Oftentimes the operators that we consider will be selfadjoint. Let $\mathscr{V}$ be a Hilbert space and let $L : \mathscr{V} \to \mathscr{V}$ be an operator. There exists an operator L^* such that for each $\phi, \psi \in \mathscr{V}$ we have

$$\langle L\phi, \psi\rangle = \langle \phi, L^*\psi\rangle.$$

Here L^* is called the *adjoint* of L. We say that L is *selfadjoint* if $L = L^*$. For a bounded linear selfadjoint operator L, the spectrum is a compact subset of $\mathbb{R}$ [8, Proposition 1.4.7].

Let L be a bounded linear operator on a Hilbert space $\mathscr{V}$ and let $\lambda \in \mathbb{C}$. We call a sequence of functions $\phi_n \in \mathscr{V}$ a *Weyl sequence* for L at λ if $\|\phi_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|(L - \lambda)\phi_n\| = 0.$$

If there is a Weyl sequence for L at λ, then $\lambda \in \sigma(L)$; moreover, if L is selfadjoint, then the converse holds ([8, Theorem 1.4.20] and [52, Lemma 2.17]).

An operator $\mathbb{U} : \mathscr{V} \to \mathscr{V}$ is unitary if its adjoint is also its inverse, that is $\mathbb{U}\mathbb{U}^* = \mathbb{U}^*\mathbb{U} = I$. Let $L : \mathscr{V} \to \mathscr{V}$ be a bounded linear operator and let $U : \mathscr{V} \to \mathscr{V}$ be a unitary operator. Notice that $L - \lambda I$ is invertible if and only if $\mathbb{U}^*(L - \lambda I)\mathbb{U} = \mathbb{U}^*L\mathbb{U} - \lambda I$ is, and so it follows that $\sigma(L) = \sigma(\mathbb{U}^*L\mathbb{U})$.

If $\mathbb{U} : \mathscr{H}_1 \to \mathscr{H}_2$ is a linear map between two Hilbert spaces, we say that it is *unitary* if it is invertible and if

$$\langle \mathbb{U}\phi, \mathbb{U}\psi\rangle_{\mathscr{H}_2} = \langle \phi, \psi\rangle_{\mathscr{H}_1} \text{ for all } \phi, \psi \in \mathscr{H}_1.$$

If such a $\mathbb{U}$ exists between two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, then we say that $\mathscr{H}_1$ and $\mathscr{H}_2$ are *isomorphic* as Hilbert spaces. For a unitary operator $\mathbb{U}$, we write $\mathbb{U}^*$ for its inverse. Similar to before, we have that $L - \lambda I : \mathscr{H}_1 \to \mathscr{H}_1$ is invertible if and only if $\mathbb{U}(L - \lambda I)\mathbb{U}^* = \mathbb{U}L\mathbb{U}^* - \lambda I$ is, and so it follows that $\sigma(L) = \sigma(\mathbb{U}L\mathbb{U}^*)$.

The case of the discrete Laplacian on $\mathbb{Z}$

We finish our discussion with an analysis of the spectrum of the discrete Laplacian on $\mathbb{Z}$. The discrete Laplacian $\Delta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is given by,

$$(\Delta\phi)(n) = \phi(n+1) + \phi(n-1), \phi \in \ell^2(\mathbb{Z}), n \in \mathbb{Z}.$$

We start by showing that Δ is a bounded selfadjoint linear operator on $\ell^2(\mathbb{Z})$.

1. Δ is obviously linear.

2. Δ is bounded:

$$\langle \Delta\phi, \Delta\phi\rangle = \sum_{n\in\mathbb{Z}} (\phi_{n+1} + \phi_{n-1})\overline{(\phi_{n+1} + \phi_{n-1})} \leq \sum_{n\in\mathbb{Z}} (|\phi_{n+1}| + |\phi_{n-1}|)^2.$$

Recall that $\|\Delta\phi\| = \sqrt{\langle \Delta\phi, \Delta\phi \rangle}$. By the Minkowski inequality,

$$\sqrt{\sum_{n\in\mathbb{Z}}(|\phi_{n+1}|+|\phi_{n-1}|)^2} \leq \sqrt{\sum_{n\in\mathbb{Z}}|\phi_{n+1}|^2} + \sqrt{\sum_{n\in\mathbb{Z}}|\phi_{n-1}|^2} = 2\|\phi\|.$$

Thus we see that $\|\Delta\phi\| \leq 2\|\phi\|$ for any $\phi \in \ell^2(\mathbb{Z})$, and so Δ is bounded.

3. Δ is selfadjoint: Let $\phi, \psi \in \ell^2(\mathbb{Z})$. We have

$$\langle \Delta\phi, \psi \rangle = \sum_{n\in\mathbb{Z}}(\phi_{n+1}+\phi_{n-1})\overline{\psi_n}.$$

Collecting the coefficients of ϕ_n we may rewrite this sum as

$$\sum_{n\in\mathbb{Z}}(\overline{\psi_{n+1}}+\overline{\psi_{n-1}})\phi_n = \sum_{n\in\mathbb{Z}}(\overline{\Delta\psi_n})\phi_n = \langle \phi, \Delta\psi \rangle.$$

Thus we conclude $\langle \Delta\phi, \psi \rangle = \langle \phi, \Delta\psi \rangle$; that is, Δ is selfadjoint.

Next we will see that this operator has no eigenvalues, but rather its spectrum consists entirely of essential spectrum. We will accomplish this by showing that Δ has no eigenfunctions in $\ell^2(\mathbb{Z})$.

Let us start by assuming that there exists $\phi \neq 0$ such that $L\phi = \lambda\phi$. Notice that then we have

$$\phi(n+1) + \phi(n-1) = \lambda\phi(n).$$

If there is a ϕ that satisfies this, then it satisfies the recurrence

$$\phi(n) = \lambda\phi(n-1) - \phi(n-2).$$

This recurrence has characteristic equation $r^2 - \lambda r + 1 = 0$; which has roots $\frac{\lambda\pm\sqrt{\lambda^2-4}}{2}$. Notice that if r_1, r_2 are the roots, then ϕ must be of the form $\phi = c_1 r_1^n + c_2 r_2^n$ for some $c_1, c_2 \in \mathbb{C}$. Let us consider the possible ϕ that satisfy this recurrence via the various cases of λ with respect to whether the discriminant is positive or not.

1. Case $-2 \leq \lambda \leq 2$: If we set $\lambda = 2\cos(\theta)$, than we see that the roots are exactly $e^{-i\theta}$ and $e^{i\theta}$, and thus we must have that $\phi(n) = c_1 e^{i\theta n} + c_2 e^{-i\theta n}$. All non-zero functions of this form are either constant or oscillating as n goes to $\pm\infty$, and thus cannot be in $\ell^2(\mathbb{Z})$.

2. Case $|\lambda| > 2$: The discriminant is positive, and so the roots are real. Moreover, one root will be a real number with a magnitude greater than 1 and one root will be a real number with a magnitude less than 1; denote these roots r_1 and r_2, respectively.

   As we have that $\phi(n) = c_1 r_1^n + c_2 r_2^n$; when $c_1 \neq 0$, we have $|r_1| > |r_2|$. Therefore, as $n \to \infty$ the $c_1 r_1^n$ term will dominate, and the magnitude of $\phi(n)$ will diverge to infinity.

   Similarly, when $c_2 \neq 0$, we have that $|\frac{1}{r_2}| > |\frac{1}{r_1}|$; thus as $n \to -\infty$ the $c_2 r_2^n$ term will dominate, and the magnitude of $\phi(n)$ will diverge to infinity.

   In either case, we see that ϕ cannot be in $\ell^2(\mathbb{Z})$.

It follows that any non-zero ϕ satisfying this recurrence cannot be in $\ell^2(\mathbb{Z})$, and so Δ has no eigenfunctions in this space. We conclude that $\Delta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ has no eigenvalues.

This leaves us to study the remaining essential spectrum of Δ, which we will show results in $\sigma(\Delta) = [-2, 2]$. For this, we turn to the Fourier transform. Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, and let $L^2(\mathbb{T})$ denote the space of square integrable functions on $\mathbb{T}$,

$$L^2(\mathbb{T}) = \Big\{ f : \mathbb{T} \to \mathbb{C} \mid \int_{\mathbb{T}} |f(z)|^2 dz < \infty \Big\},$$

where dz is the Haar measure on $\mathbb{T}$, that is $\displaystyle\int_{\mathbb{T}} dz = 1$.

In this way, $L^2(\mathbb{T})$ is a Hilbert space with a norm induced by the inner product

$$\langle f, g\rangle = \int_{\mathbb{T}} f(z)\overline{g(z)}\, dz.$$

The Fourier transform $\mathscr{F}$ sends functions in $\ell^2(\mathbb{Z})$ to functions in $L^2(\mathbb{T})$ in the following way:

$$\mathscr{F} g = \hat{g}(z) = \sum_{n\in\mathbb{Z}} g(n) z^{-n}.$$

This transform is invertible. Indeed, we may define $\mathscr{F}^*$ to be the following operator:

$$\mathscr{F}^* \hat{g} = g(n) = \int_{z\in\mathbb{T}} \hat{g}(z) z^n \, dz.$$

We claim that $\mathscr{F}$ is unitary. Indeed, if $f, g \in \ell^2(\Gamma)$, then we have

$$\langle \mathscr{F} f, \mathscr{F} g\rangle_{L^2(\mathbb{T})} = \int_{z\in\mathbb{T}} \sum_{n\in\mathbb{Z}} f(n) z^{-n} \sum_{m\in\mathbb{Z}} \overline{g(m)}\overline{z^{-m}} dz = \int_{z\in\mathbb{T}} \sum_{n\in\mathbb{Z}} f(n) z^{-n} \sum_{m\in\mathbb{Z}} \overline{g(m)} z^m \, dz.$$

When $m \neq n$, we have that $\int_{z\in\mathbb{T}} f(n) z^{-n} \overline{g(m)} z^m dz = 0$. It follows that

$$\langle \mathscr{F} f, \mathscr{F} g\rangle_{L^2(\mathbb{T})} = \int_{z\in\mathbb{T}} \sum_{n\in\mathbb{Z}} f(n)\overline{g(n)} dz = \sum_{n\in\mathbb{Z}} f(n)\overline{g(n)} = \langle f, g\rangle_{\ell^2(\mathbb{Z})}.$$

Thus we see that $\mathscr{F}$ is a unitary operator, and so the spectrum of Δ is equal to the spectrum of $\mathscr{F}\Delta\mathscr{F}^*$. In other words, $\Delta - \lambda I$ is bijective if and only if $\mathscr{F}\Delta\mathscr{F}^* - \lambda I$ is.

Let us consider how $\mathscr{F}\Delta\mathscr{F}^* - \lambda I$ acts on $\hat{g}(z)$ for a fixed $z_0 \in \mathbb{T}$. We have

$$\Big((\mathscr{F}\Delta\mathscr{F}^* - \lambda I)\hat{g}\Big)(z_0) = \left(\mathscr{F} \int_{z\in\mathbb{T}} \hat{g}(z) z^{n+1} + \hat{g}(z) z^{n-1} dz\right)(z_0) - \lambda \hat{g}(z_0) = (z_0 + z_0^{-1} - \lambda)\hat{g}(z_0).$$

Thus, we see that $(\mathscr{F}\Delta\mathscr{F}^* - \lambda I)$ acts on $L^2(\mathbb{T})$ as a multiplication operator. It is easy to see that this operator is a bijection if and only if $(z_0 + z_0^{-1} - \lambda) \neq 0$ for each $z_0 \in \mathbb{T}$. To show this, first notice that if λ is such that $\lambda \neq z_0 + z_0^{-1}$ for all $z_0 \in \mathbb{T}$, then the operator $H : L^2(\mathbb{T}) \to L^2(\mathbb{T})$

given by $\hat{g}(z) \mapsto \frac{\hat{g}(z)}{z-z^{-1}-\lambda}$ is well-defined and exactly the inverse to $(\mathscr{F}\Delta\mathscr{F}^* - \lambda I)$, and thus $\lambda \in \rho(\Delta)$.

On the other hand, if $\lambda = z_0 + z_0^{-1}$ for some $z_0 \in \mathbb{T}$, then $(\mathscr{F}\Delta\mathscr{F}^*)$ has a Weyl sequence at λ. Indeed, let $e^{2\pi i\theta} = z_0$ and consider the sequence of functions $f_n \in L^2(\mathbb{T})$ such that $f_n(z) = \sqrt{\frac{n}{2}}$ when $z = e^{2\pi i(\theta+t)}$ for some $t \in [-\frac{1}{n}, \frac{1}{n}]$ and is 0 otherwise. One can verify that $\|f_n(z)\| = 1$ for each $n \in \mathbb{N}$, but $\lim_{n\to\infty} \|(\mathscr{F}\Delta\mathscr{F}^* - \lambda I)\phi_n\| = 0$. Thus ϕ_n is a Weyl sequence of Δ at λ, and so $\lambda \in \sigma(\Delta)$.

It follows that $\lambda \in \sigma(\Delta)$ if and only if λ is a root of $(z + z^{-1} - \lambda)$ for some $z \in \mathbb{T}$. Notice that when $z \in \mathbb{T}$, we have $z+z^{-1} = 2Re(z)$ and $Re(z) \in [-1, 1]$, and so it follows that $\sigma(\Delta) = [-2, 2]$.

As the action of $\mathscr{F}\Delta\mathscr{F}^* : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ on a function $\hat{g}(z) \in L^2(\mathbb{T})$ for each fixed $z \in \mathbb{T}$ is just multiplication by $(z + z^{-1})$, we may express our operator as

$$\Big((\mathscr{F}\Delta\mathscr{F}^* - \lambda I)\hat{g}\Big)(z) = (z + z^{-1})\hat{g}(z).$$

As we saw before, Δ has no eigenfunctions in $\ell^2(\mathbb{Z})$ (similar to how $\mathscr{F}\Delta\mathscr{F}^*$ has no eigenfunctions in $L^2(\mathbb{T})$). However, we can ask whether there is a way to interpret the elements λ of the essential spectrum as eigenvalues of L on the larger space of functions $\phi : \mathbb{Z} \to \mathbb{C}$. The answer is yes, and we call such functions *generalized eigenfunctions*. We say a function $\phi : \mathbb{Z} \to \mathbb{T}$ is a generalized eigenfunction of Δ if it satisfies $\Delta\phi = \lambda\phi$ and $|\phi(n)| \leq c|1 + |n||^t$ for some constants $c, t > 0$ and every $n \in \mathbb{Z}$ [8, Definition 2.4.1]. The collection of λ such that there exists a generalized eigenfunction ϕ of Δ are called the *generalized eigenvalues* of Δ; the closure of the collection of generalized eigenvalues of Δ recovers $\sigma(\Delta)$ [8, Theorem 2.4.2].

Let us identify the generalized eigenfunctions of Δ. Suppose that $\lambda = z_0 + z_0^{-1}$ for some $z_0 \in \mathbb{T}$. Recall that then $\lambda \in [-2, 2]$, and so we already saw what the generalized eigenfunctions of λ are in this case; they are functions of the form $\phi(n) = c_1e^{i\theta n} + c_2e^{-i\theta n}$, when $\lambda = 2\cos(\theta)(= e^{i\theta} + e^{-i\theta})$. As $\lambda = z_0 + z_0^{-1}$, there is a $\theta \in [0, 2\pi)$ such that $e^{i\theta} = z_0 \in \mathbb{T}$ and $e^{-i\theta} = z_0^{-1} \in \mathbb{T}$, and so $\phi(n) = c_1z_0^n + c_2z_0^{-n}$. One can quickly check that any function of the form $\phi(n) = cz_0^{\pm n}$ satisfies $\Delta\phi(n) = (z_0 + z_0^{-1})\phi(n)$, and so all functions of the form $\phi(n) = c_1z_0^n + c_2z_0^{-n}$, such that c_1 or c_2 is non-zero, are generalized eigenfunctions of Δ corresponding to the generalized eigenvalue $\lambda = z_0 + z_0^{-1}$. Thus, we see that every $\lambda \in \sigma(\Delta)$ is a generalized eigenvalue.

3.3 Discrete Periodic Operators and Floquet Theory

The Schrödinger operator is a well known and deeply studied operator. This operator is composed of a Laplacian with a perturbing potential. The choice of potential in studying Schrödinger operators has deep implications; entire branches of spectral theory hinge on whether the potential function is random, almost-periodic, quasiperiodic, continuous and periodic, or, as in our case, discrete and periodic. The discrete periodic Schrödinger operator arises from the tight-binding model, and the study of this family of operators is one of the most classical directions in mathematical physics. We will examine the broader class of discrete periodic operators, which are operators composed of a periodically weighted discrete Laplacian and a periodic potential.

3.3.1 Discrete Periodic Operators

Let G be a finitely generated free abelian group acting on a set S. We say that a function $f : S \to \mathbb{C}$ is *G-periodic* if for all $s \in S$ and for all $a \in G$, we have $f(s + a) = f(s)$. If S/G is finite, it follows that the range of f in $\mathbb{C}$ is given by a finite set of complex numbers.

Definition 3.3.1. Given a G-periodic graph Γ and a G-periodic function $E : \mathcal{E}(\Gamma) \to \mathbb{C}$, a *weighted discrete Laplacian* (also called a Laplace-Beltrami operator), Δ_E, is an operator that acts on functions $f : \mathcal{V}(\Gamma) \to \mathbb{C}$ in the following way:

$$(\Delta_E f)(u) = \sum_{(u,v)\in\mathcal{E}(\Gamma)} E((u,v))(f(u) - f(v)).$$

We call the function E an *edge labeling*, and say that $E((u,v))$ is the *weight (or label)* of the edge $(u,v) \in \mathcal{E}(\Gamma)$. ◇

Definition 3.3.2. Given a G-periodic graph Γ and a free abelian group G that acts on Γ, a *G-periodic potential*, V, is a multiplication operator that acts on functions on the vertices of Γ with values in $\mathbb{C}$; that is, for each $f : \mathcal{V}(\Gamma) \to \mathbb{C}$ and $u \in \mathcal{V}(\Gamma)$,

$$(Vf)(u) = V(u)f(u).$$

Here, we abuse notation by also using V to denote the G-periodic function $V : \mathcal{V}(\Gamma) \to \mathbb{C}$. ◇

Definition 3.3.3. Given a G-periodic graph Γ, the sum of a weighted graph Laplacian Δ_E and a G-periodic potential V defines a *discrete periodic operator* $L := V + \Delta_E$. In particular, L acts on functions $f : \mathcal{V}(\Gamma) \to \mathbb{C}$ in the following way:

$$(Lf)(u) = V(u)f(u) + \sum_{(u,v)\in\mathcal{E}(\Gamma)} E((u,v))(f(u) - f(v)).$$

When all edge weights are 1, this is known as a *discrete periodic Schrödinger operator*. ◇

Together, we say that the pair of G-periodic functions (V, E) give a *labeling* of Γ. As L is dependent on the choice of labeling (V, E), we will often say that L is the discrete periodic operator given by (V, E); that is, L is the discrete periodic operator $V + \Delta_E$.

A discrete periodic operator L is a bounded linear operator on $\ell^2(\mathcal{V}(\Gamma))$. Moreover, when E and V are taken to be real-valued functions, L is selfadjoint and therefore has only real spectrum. The proof of these facts is similar to the special case of the discrete Laplacian presented in Section 3.2. For example, one can verify that if $(u_1, v_1), \ldots, (u_k, v_k)$ are representatives of the k edge orbits $\mathcal{E}(\Gamma)/G$, then

$$\|L\| \le \sum_{u\in W} |V(u)| + 2\sum_{i=1}^{k} |E((u_i, v_i))|.$$

3.3.2 Floquet Theory

Let Γ be a $\mathbb{Z}^d$-periodic graph with a fundamental domain W. We wish to study the spectrum of a discrete periodic operator L acting on $\ell^2(\mathcal{V}(\Gamma))$, the Hilbert space of square summable functions on $\mathcal{V}(\Gamma)$. Fix a fundamental domain W of Γ. The *Floquet transform* $\mathscr{F}$ of a function g on $\mathcal{V}(\Gamma)$ is given by:

$$g(u) \mapsto \hat{g}(z, u) = \sum_{a\in\mathbb{Z}^d} g(u + a)z^{-a}.$$

Notice that $(\mathscr{F}g(u+a))(z) = z^a(\mathscr{F}\hat{g}(u))(z)$. One can see that $\hat{g}(z,u)$ is just the Fourier transform of g restricted to vertices in the orbit $u+\mathbb{Z}^d$. If $g \in \ell^2(\mathcal{V}(\Gamma))$, then

$$\sum_{u \in W} \int_{\mathbb{T}^d} |\hat{g}(z,u)dz|^2 < \infty.$$

That is, $\hat{g}(z,u)$ is in $L^2(\mathbb{T}^d)$, the Hilbert space of square integrable functions on $\mathbb{T}^d$, for each $u \in W$. Suppose that $W = \{\omega_1, \ldots, \omega_m\}$. Given a function $g \in \ell^2(\mathcal{V}(\Gamma))$, we can view $\mathscr{F}g = \hat{g}(z,\cdot) = (\hat{g}(z,\omega_1), \hat{g}(z,\omega_2), \ldots, \hat{g}(z,\omega_m))^T$ as an element of the Hilbert space

$$L^2(\mathbb{T}^d)^W = \{\hat{g}(z,\cdot) \mid \hat{g}(z,\omega_i) \text{ is in } L^2(\mathbb{T}^d) \text{ for each } i \in [n]\}.$$

Indeed, $L^2(\mathbb{T}^d)^W$ is a Hilbert space with the inner product:

$$\langle \hat{f}(z,\cdot), g(z,\cdot) \rangle = \sum_{i=1}^{m} \int_{\mathbb{T}^d} f(z,w_i)\overline{g(z,w_i)}dz.$$

Like the Fourier transform, the Floquet transform $\mathscr{F} : \ell^2(\mathcal{V}(\Gamma)) \to L^2(\mathbb{T}^d)^W$ is a unitary operator; so L and $\mathscr{F}L\mathscr{F}^*$ are unitarily equivalent, where $\mathscr{F}^*$ denotes the inverse Floquet transform.

Define $L(z)$ to be the $W \times W$ matrix such that, for each $u, v \in W$,

$$L(z)_{u,v} = \delta_{u,v}\left(V(u) + \sum_{(u,\omega) \in \mathcal{E}(\Gamma)} E((u,\omega))\right) - \sum_{(u,v+a) \in \mathcal{E}(\Gamma)} E((u,v+a))z^a, \tag{3.1}$$

where δ is the Kronecker delta function (that is, $\delta_{u,v} = 1$ when $u = v$ and $\delta_{u,v} = 0$ otherwise).

We will now show that, for any $\hat{g}(z,\cdot) \in L^2(\mathbb{T}^d)^W$,

$$\mathscr{F}L\mathscr{F}^*\hat{g}(z,\cdot) = L(z)\hat{g}(z,\cdot). \tag{3.2}$$

This equality is because, for each $z \in \mathbb{T}^d$ and $u \in W$, we have

$$\mathscr{F}L\mathscr{F}^*(\hat{g})(z,u) = V(u)\hat{g}(z,u) + \sum_{(u,v+a) \in \mathcal{E}(\Gamma), v \in W} E((u,v+a))(\hat{g}(z,u) - z^a\hat{g}(z,v)).$$

Notice that $L(z)$ is defined such that for each $z \in \mathbb{T}^d$ and any $\hat{g}(z,\cdot) \in L^2(\mathbb{T}^d)^W$, we have

$$L(z)_{u,v} = [\hat{g}(z,v)](\mathscr{F}L\mathscr{F}^*(\hat{g})(z,u)),$$

where $[\hat{g}(z,v)]f$ denotes extracting the coefficient of $\hat{g}(z,v)$ in f. In this way,

$$\sum_{v \in W} L(z)_{u,v}\hat{g}(z,v) = \mathscr{F}L\mathscr{F}^*(\hat{g})(z,u).$$

It follows that for each $z \in \mathbb{T}^d$ and any $\hat{g}(z,\cdot) \in L^2(\mathbb{T}^d)^W$, we have $\mathscr{F}L\mathscr{F}^*\hat{g}(z,\cdot) = L(z)\hat{g}(z,\cdot)$. That is, $\mathscr{F}L\mathscr{F}^*$ acts as multiplication by a finite matrix on the vector $\hat{g}(z,\cdot) \in L^2(\mathbb{T}^d)^W$ for each fixed $z \in \mathbb{T}^d$. We obtain the following fact.

Fact 3.3.4. $\mathscr{F}L\mathscr{F}^* - \lambda I$ is a bijection if and only if each $L(z) - \lambda I$ is. It follows that, the spectrum of L is given by the union of the eigenvalues of $L(z)$ for each $z \in \mathbb{T}^d$.

Proof. As in the case of Section 3.2, for a fixed $z_0 \in \mathbb{T}^d$, we can consider $(\mathscr{F}L\mathscr{F}^* - \lambda I)$ acting on a vector $\hat{g}(z, \cdot)$, and we see that

$$(\mathscr{F}L\mathscr{F}^* - \lambda I)\hat{g}(z_0, \cdot) = L(z_0)\hat{g}(z_0, \cdot) - \lambda I\hat{g}(z_0, \cdot)) = (L(z_0) - \lambda I)\hat{g}(z_0, \cdot).$$

Unless λ is an eigenvalue of $L(z)$ for some $z \in \mathbb{T}$, the operator $(\mathscr{F}L\mathscr{F}^* - \lambda I) : L^2(\mathbb{T}^d)^W \to L^2(\mathbb{T}^d)^W$ has a well-defined inverse, as each $L(z) - \lambda I$ does, and is thus a bijection ($\lambda \in \rho(\mathscr{F}L\mathscr{F}^*)$).

Moreover, similar to the discrete Laplacian, one can show that if λ is an eigenvalue of $L(z_0)$ for some $z_0 \in \mathbb{T}^d$, the operator $(\mathscr{F}L\mathscr{F}^*)$ has a Weyl sequence at λ. To show this, let $\theta \in \mathbb{R}^d$ be such that $z_0 = (e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_d})$, let $\int = (\int_1, \ldots, \int_m) \in \mathbb{R}^m$ be an eigenvector of $L(z_0)$ such that $L(z_0)\int = \lambda\int$, and let $c = \frac{1}{\sum_{i=1}^m |\int_i|^2}$. Define $\phi_n(z) = ((\phi_n)_1(z), \ldots, (\phi_n)_m(z))$ where

$$(\phi_n)_i(z) = \left(\sqrt{c\frac{n^d}{2^d}}\right)\int_i \text{ when } z = (e^{2\pi i(\theta_1+t_1)}, \ldots, e^{2\pi i(\theta_d+t_d)}) \text{ for some } t \in \left[-\frac{1}{n}, \frac{1}{n}\right]^d,$$

and is zero otherwise. Notice that $\|\phi_n\| = 1$ for each $n \in \mathbb{N}$, but $\lim_{n\to\infty} \|(\mathscr{F}L\mathscr{F}^* - \lambda I)\phi_n\| = 0$. Thus, ϕ_n is a Weyl sequence of $\mathscr{F}L\mathscr{F}^*$ at λ, and so we have that $\lambda \in \sigma(\mathscr{F}L\mathscr{F}^*)$. □

When z is viewed as an indeterminate, it is easy see that $L(z)$ is just a $|W| \times |W|$ matrix with Laurent polynomial entries in $\mathbb{C}[z^{\pm}]$. Given a discrete periodic operator L, we will often refer to $L(z)$ as the *Floquet matrix* of L.

Remark 3.3.5. A *Floquet* function $f : \mathcal{V}(\Gamma) \to \mathbb{C}$ with *Floquet multipler* $z \in \mathbb{T}^d$ is a quasi-periodic with respect to the $\mathbb{Z}^d$ group action. That is, f is such that for each $u \in \mathcal{V}(\Gamma)$ and $a \in \mathbb{Z}^d$, we have $f(u + a) = z^a f(u)$. Such functions are also sometimes called Bloch functions, or functions satisfying "Floquet-boundary" condition.

We may also obtain the spectrum of L on $\ell^2(\mathcal{V}(\Gamma))$ by studying the spectrum of L acting on the spaces of Floquet functions $f : \mathcal{V}(\Gamma) \to \mathbb{C}$ with Floquet multipler z for each $z \in \mathbb{T}^d$. As in the example given in Section 3.2, one can show that such quasi-periodic functions on $\mathcal{V}(\Gamma)$ make up the generalized eigenfunctions of L.

Indeed, first notice that each quasi-periodic function f is determined by the values it takes on the fundamental domain, and thus it can be represented by the vector $(f(\omega_1), f(\omega_2), \ldots, f(\omega_m))$. Now consider L acting on the vector $(f(\omega_1), f(\omega_2), \ldots, f(\omega_m))^T$, where the fundamental domain is given by $\{\omega_1, \ldots, \omega_m\} = W$, and where f is such that $f(\omega_i + a) = z_0^a f(\omega_i)$ for each $i \in [m]$ and all $a \in \mathbb{Z}^d$. It is easy to verify that L acts on this vector as multiplication by the matrix $L(z_0)$. Notice that if $u \in \mathcal{V}(\Gamma)$, then there exists a such that $u = a + \omega_i$ for some $i \in [m]$. Therefore, if $Lf(\omega_i) = \lambda f(\omega_i)$ for each $i \in [m]$, then we have that $Lf(u) = \lambda f(u)$ for all $u \in \mathcal{V}(\Gamma)$. It follows that such functions f are indeed generalized eigenfunctions of L, and that the eigenvalues of $L(z_0)$ are generalized eigenvalues of L.

Example 3.3.6. Let Γ be the hexagonal lattice from Figure 3.1. Figure 3.4 shows a labeling in a neighborhood of its fundamental domain. Thus $W = \{u, v\}$ consists of two vertices, and there are

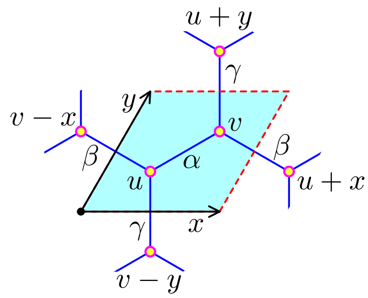


Figure 3.4: A labeling of the hexagonal lattice.

three (orbits of) edges, with labels α, β, γ. Let $(x, y) \in \mathbb{T}^2$. The operator $\mathscr{F} L \mathscr{F}^*$ is

$$\begin{aligned}
\mathscr{F} L \mathscr{F}^*(\hat{f})((x,y),u) &= (V(u) + \alpha + \beta + \gamma)\hat{f}((x,y),u) \ + \ (-\alpha - \beta x^{-1} - \gamma y^{-1})\hat{f}((x,y),v)\,, \\
\mathscr{F} L \mathscr{F}^*(\hat{f})((x,y),v) &= (V(v) + \alpha + \beta + \gamma)\hat{f}((x,y),v) \ + \ (-\alpha - \beta x - \gamma y)\hat{f}((x,y),u)\,.
\end{aligned}$$

Collecting coefficients of $\hat{f}(u)$ and $\hat{f}(v)$, we obtain the 2×2-matrix $L(x, y)$:

$$L(x,y) \ = \ \begin{pmatrix} V(u) + \alpha + \beta + \gamma & -\alpha - \beta x^{-1} - \gamma y^{-1} \\ -\alpha - \beta x - \gamma y & V(v) + \alpha + \beta + \gamma \end{pmatrix}\,, \tag{3.3}$$

whose entries are Laurent polynomials in x and y. Notice that the support $\mathcal{A}(W)$ equals the set of exponents of monomials which appear in L. Observe that for $(x, y) \in \mathbb{T}^2$, we have $L^T(x, y) = L(x^{-1}, y^{-1}) = L(\bar{x}, \bar{y})$, and so when V and E are both real, $L(x, y)$ is Hermitian. ⋄

This observation holds in general, that is, when both E and V take only real values, $L(z)$ is Hermitian for $z \in \mathbb{T}^d$. In this case, for each $z \in \mathbb{T}^d$, the spectrum is real and consists of its $|W|$ eigenvalues:

$$\lambda_1(z) \ \leq \ \lambda_2(z) \ \leq \ \cdots \ \leq \ \lambda_{|W|}(z)\,. \tag{3.4}$$

These eigenvalues vary continuously with $z \in \mathbb{T}^d$, and $\lambda_j(z)$ is called the jth *spectral band function*, $\lambda_j \colon \mathbb{T}^d \to \mathbb{R}$. Its image is an interval in $\mathbb{R}$, called the *jth spectral band*. The eigenvalues (3.4) are the roots of the characteristic polynomial restricted to $z \in \mathbb{T}^d$,

$$D(z, \lambda) \ := \ \det(L(z) \ - \ \lambda I)\,, \tag{3.5}$$

which we call the *dispersion polynomial*. Its vanishing set on this restriction defines a real algebraic hypersurface

$$BV_{\mathbb{R}}(L) = \mathbf{V}_{\mathbb{R}}(D(z,\lambda)) = \{(z,\lambda) \in \mathbb{T}^d \times \mathbb{R} \mid D(z,\lambda) = 0\}\,, \tag{3.6}$$

called the (real) *Bloch variety* (or dispersion relation) of the operator L. As we wish to apply techniques from algebraic geometry, we will often also consider the (complex) Bloch variety $BV(L) = \mathbf{V}(D(z, \lambda)) \subset (\mathbb{C}^*)^d \times \mathbb{C}$. The image of the (real) Bloch variety under the projection to $\mathbb{R}$ is the spectrum $\sigma(L)$ of the operator L. This projection is a function λ on the Bloch variety. That is, $\lambda : BV_{\mathbb{R}}(L) \to \mathbb{R}$ (or $\lambda : BV(L) \to \mathbb{C}$) is the coordinate projection $\lambda(z, \lambda) = \lambda$ with domain restricted to the Bloch variety. Identifying the jth branch/graph with $\mathbb{T}^d$, the restriction of λ to that branch gives the corresponding spectral band function λ_j.

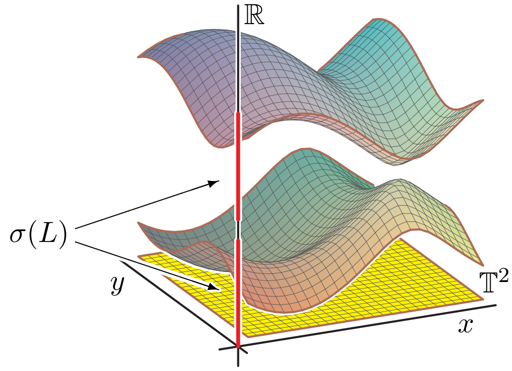


Figure 3.5: A Bloch variety and spectral bands for the hexagonal lattice.

Figure 3.5 shows this for the operator L on the hexagonal lattice with edge weights $6, 3, 2$ and potential $V(u) = V(v) = 0$—for this we unfurl $\mathbb{T}^2$, representing it by $[-\frac{\pi}{2}, \frac{3\pi}{2}]^2 \subset \mathbb{R}^2$, which is a fundamental domain in its universal cover.

It has two branches with each the graph of the corresponding spectral band function. An endpoint of a spectral band (*spectral edge*) is the image of an extremum of some band function $\lambda_j(z)$. For the hexagonal lattice at these parameters, each band function has two nondegenerate extrema, and together these give the four spectral edges. These edges are also local extrema of the function λ on the Bloch variety.

3.3.3 Changing the Period Lattice of the Potential

We present an additional way of constructing the dispersion polynomial as one varies the period of the potential which will be used in Chapter 5; where we will explore the effect that changing the period of the potential has on the reducibility of the dispersion polynomial.

Let Γ be a $\mathbb{Z}^d$-periodic graph with fundamental domain W, where $m := |W|$. Fix $Q \in \mathbb{N}^d$ and let $|Q| := \prod_i q_i$. We wish to study the dispersion polynomial given a labeling (V_Q, E) where $E : \mathcal{E}(\Gamma) \to \mathbb{C}$ is a $\mathbb{Z}^d$-periodic edge labeling and $V_Q : \mathcal{V}(\Gamma) \to \mathbb{C}$ is $Q\mathbb{Z}^d$-periodic potential, rather than a $\mathbb{Z}^d$-periodic potential. As $Q\mathbb{Z}$ is a free full rank subgroup, we have that Γ is also a $Q\mathbb{Z}$-periodic graph, and E is a $Q\mathbb{Z}$-periodic edge labeling. Thus, as discussed in 3.1, W_Q is a fundamental domain of Γ equipped with the free action $Q\mathbb{Z}$ (that is, as a $Q\mathbb{Z}$-periodic graph). As Q will vary throughout our discussion in Chapter 5, we will denote the Floquet matrix of L with respect to the $Q\mathbb{Z}$-periodic graph Γ with fundamental domain W_Q by $L_Q(z)$. Recall that W_Q contains exactly $|Q|m$ vertices, and so, by Equation (3.1), $L_Q(z)$ is a $(|Q|m \times |Q|m)$-matrix with Laurent polynomial entries.

We next discuss an alternative representative of $L_Q(z)$ that comes from a change of basis. Recall that for each $z \in \mathbb{T}^d$, we can view $L_Q(z)$ as acting on the space of Floquet functions on $\mathcal{V}(\Gamma)$ with Floquet multiplier z (see Remark 3.3.5). We begin by considering the surjective group homomorphism

$$\begin{array}{rccc} \phi : & (\mathbb{C}^\times)^d & \longrightarrow & (\mathbb{C}^\times)^d \\ & (z_1, \ldots, z_d) & \longmapsto & (z_1^{q_1}, \ldots, z_d^{q_d}), \end{array} \tag{3.7}$$

with *kernel group* $\mathcal{U}_Q := \prod_{i=1}^d \mathcal{U}_{q_i}$, where $\mathcal{U}_{q_i}$ is the multiplicative group of q_ith roots of unity.

Fix $z \in \mathbb{T}^d$. The set consisting of the functions that satisfy

$$e_\mu(v+a) := (\mu_1 z_1)^{a_1} \cdots (\mu_d z_d)^{a_d} e_\mu(v) = (\mu z)^a e_\mu(v) \tag{3.8}$$

for $\mu \in \mathcal{U}_Q$, $a \in \mathbb{Z}^d$, and $v \in W$, forms a basis for functions $\psi : \mathcal{V}(\Gamma) \to \mathbb{C}$ satisfying

$$\psi(Q_i + a + v) = z_i^{q_i} \psi(a+v) \text{ for each } v \in W \text{ and each } i = 1, \ldots, d.$$

Here, $Q_i \in \mathbb{Z}^d$ is q_i in the ith component and 0 elsewhere. That is, the e_μ form a basis for functions that are quasiperiodic Floquet functions with Floquet multiplier z^Q with respect to the $Q\mathbb{Z}$ group action. Notice that for each z, these are exactly the generalized eigenfunctions of L_Q after the cover map (3.7), and thus we may obtain a new matrix representation for $L_Q(z_1^{q_1}, \ldots, z_d^{q_d})$ in the basis given by the functions of (3.8). The weighted discrete Laplacian in the basis (3.8) is

$$(\Delta_E e_\mu)(u) = \sum_{(u,v+a) \in \mathcal{E}(\Gamma)} E_{(u,v+a)}(e_\mu(u) - (\mu z)^a e_\mu(v)),$$

where $u, v \in W$ and $\mu \in \mathcal{U}_Q$. The Floquet matrix of the Δ_E in the basis $\{e_\mu(v) \mid \mu \in \mathcal{U}_Q, v \in W\}$ is the $m|Q| \times m|Q|$ matrix $\hat{\Delta}_E(z)$. This matrix is given by $|Q| \times |Q|$ blocks, indexed by $\mathcal{U}_Q \times \mathcal{U}_Q$, of $m \times m$ matrices, indexed by $W \times W$. Explicitly, it is the block-diagonal matrix

$$\hat{\Delta}_E(z)_{\mu,\mu'} := \Delta_E(\mu z)\delta_{\mu,\mu'},$$

where $\Delta_E(z)$ is the Floquet matrix of the weighted discrete Laplacian on the $\mathbb{Z}^d$-periodic graph Γ.

In order to discuss the potential V in this new basis, we will take a discrete Fourier transform. For $\mu \in \mathcal{U}_Q$, $v \in W$, and $a \in \mathbb{Z}^d$ such that $v + a \in W_Q$, the discrete Fourier transform of the potential V is

$$(Ve_\mu)(v+a) = \sum_{\rho \in \mathcal{U}_Q} \hat{V}_{\rho,\mu}(v) e_\rho(v+a) = \sum_{\rho \in U_Q} \hat{V}_{\rho,\mu}(v) \rho^a e_\rho(v),$$

where $\hat{V}_{\rho,\mu}(v)$ are the Fourier coefficients (see also [16, Equation 4.5]). In order to obtain a matrix multiplication operator representing V in this new basis, we solve for these coefficients and find that

$$\hat{V}_{\rho,\mu}(v) e_\rho(v) = \frac{e_\mu(v)}{|Q|} \sum_{v+a \in W_Q} V(v+a)(\mu\rho^{-1})^a.$$

Let $\hat{V}$ be the matrix representation of V in the basis given by the functions of (3.8); that is, by the discrete Fourier transform $\hat{V}$ acts on the basis function $e_\rho(v)$ by

$$\hat{V}(e_\rho(v)) = \sum_{\mu \in \mathcal{U}_Q} \hat{V}_{\rho,\mu}(v) e_\rho(v) = \sum_{\mu \in \mathcal{U}_Q} \frac{e_\mu(v)}{|Q|} \sum_{v+a \in W_Q} V(v+a)(\mu\rho^{-1})^a \quad \big(= V(e_\rho)(v) \big).$$

This is a $Q \times Q$ block matrix with $m \times m$ entries, indexed the same as $\hat{\Delta}_E(z)$. Here, each $\hat{V}_{\mu,\mu'}$ is an $m \times m$ diagonal matrix such that $(\hat{V}_{\mu,\mu'})_{u,u} = \hat{V}_{\mu,\mu'}(u)$.

Remark 3.3.7. If V is also $\mathbb{Z}^d$-periodic, then

$$\begin{aligned}\hat{V}_{\rho,\mu}(v)e_\rho(v) &= \frac{1}{|Q|}\sum_{v+a\in W_Q} V(v+a)e_\mu(v)(\mu\rho^{-1})^a \\ &= \frac{1}{|Q|}\sum_{v+a\in W_Q} V(v)e_\mu(v)(\mu\rho^{-1})^a \\ &= \frac{V(v)e_\mu(v)}{|Q|}\sum_{v+a\in W_Q}(\mu\rho^{-1})^a.\end{aligned}$$

Thus, $\hat{V}_{\rho,\mu}(v) = V(v)$ when $\rho = \mu$ and is 0 otherwise. That is, $\hat{V}$ is a diagonal matrix. ◇

It follows that the $m|Q| \times m|Q|$ matrix $L_Q(z)$ with respect to the basis $\{e_\mu(v) \mid \mu \in \mathcal{U}_Q, v \in W\}$, is

$$\hat{L}_Q(z) = \hat{V} + \hat{\Delta}_E(z).$$

Let $D_Q(z,\lambda) = \det(L_Q(z) - \lambda I)$ and $\hat{D}_Q(z,\lambda) = \det(\hat{L}_Q(z) - \lambda I)$. We conclude that,

$$D_Q(z^Q,\lambda) = \det(L_Q(z^Q) - \lambda I) = \det(\hat{L}_Q(z) - \lambda I) = \hat{D}_Q(z,\lambda).$$

Example 3.3.8. Let us continue Example 3.3.6. When we view the hexagonal lattice as $\mathbb{Z}^2$-periodic, as in the case of Figure 3.1, with a $\mathbb{Z}^2$-periodic potential V; we get the Floquet matrix

$$L(x,y) = \begin{pmatrix} V(u)+\alpha+\beta+\gamma & -\alpha-\beta x^{-1}-\gamma y^{-1} \\ -\alpha-\beta x-\gamma y & V(v)+\alpha+\beta+\gamma \end{pmatrix}.$$

Let $Q = (2,1)$ and let V_Q be a $Q\mathbb{Z}$-periodic potential, then $L_Q(x,y)$ is given by the matrix

$$\begin{pmatrix} V_Q(u)+\alpha+\beta+\gamma & -\alpha-\gamma y^{-1} & 0 & -\beta x^{-1} \\ -\alpha-\gamma y & V_Q(v)+\alpha+\beta+\gamma & -\beta & 0 \\ 0 & -\beta & V_Q((1,0)+u)+\alpha+\beta+\gamma & -\alpha-\gamma y^{-1} \\ -\beta x & 0 & -\alpha-\gamma y & V_Q((1,0)+v)+\alpha+\beta+\gamma \end{pmatrix}.$$

If V satisfies $V(u) = \frac{V_Q(u)+V_Q((1,0)+u)}{2}$ and $V(v) = \frac{V_Q(v)+V_Q((1,0)+v)}{2}$, then $\hat{L}_Q(x,y)$ is a 2×2 block matrix with each entry a 2×2 matrix. Explicitly,

$$\hat{L}_Q(x,y) = \begin{pmatrix} L(x,y) & (\hat{V}_Q)_{1,-1} \\ (\hat{V}_Q)_{-1,1} & L(-x,y) \end{pmatrix}, \text{ where}$$

$$(\hat{V}_Q)_{1,-1} = (\hat{V}_Q)_{-1,1} = \begin{pmatrix} V_Q(u) - V_Q(u+(1,0)) & 0 \\ 0 & V_Q(v) - V_Q(v+(1,0)) \end{pmatrix}.$$ ◇

See [16, Section 4] for a treatment where the Floquet transform is directly used to obtain $\hat{L}_Q(x,y)$; rather than through the finite change of basis on the space of generalized eigenfunctions that we used here.

3.4 Prelude to Algebraic Geometry for Discrete Periodic Operators

Suppose that Γ is a $\mathbb{Z}^d$-periodic graph with fundamental domain W. Let $D(z,\lambda)$ denote the dispersion polynomial of a discrete periodic operator L given by the labeling (V,E) of Γ. That is, $D(z,\lambda)$ is the characteristic polynomial of the finite matrix with Laurent polynomial entries; in particular, it is the characteristic polynomial of the Floquet matrix $L(z)$. As $L(z)^T = L(z^{-1})$, the dispersion polynomial $D(z,\lambda)$ is a polynomial in $\mathbb{C}[z^{\pm},\lambda]$ with the property that $D(z,\lambda) = D(z^{-1},\lambda)$. The Bloch variety is the vanishing set of $D(z,\lambda)$ in $(\mathbb{C}^{\times})^d \times \mathbb{C}$. We define the *Fermi variety* at $\lambda \in \mathbb{C}$ to be $FV_\lambda = \{z \in (\mathbb{C}^{\times})^d \mid D(z,\lambda) = 0\ \}$. In this larger space, the Bloch and Fermi varieties are complex algebraic varieties (hypersurfaces); however, in passing to the algebraic torus, we may no longer distinguish the branches $\lambda_j(z)$ of λ on the Bloch variety. At a smooth point (z_0,λ_0) whose projection z to $(\mathbb{C}^{\times})^d$ is regular (in that $\frac{\partial D}{\partial \lambda}(z_0,\lambda_0) \neq 0$), there is a locally defined function f of z with $\lambda_0 = f(z_0)$ and $D(z,f(z)) = 0$ on its domain, but this is not a global function of z.

As the Bloch and Fermi varieties are affine subvarieties of $(\mathbb{C}^{\times})^d \times \mathbb{C}$ and $(\mathbb{C}^{\times})^d$ respectively, this perspective will enable us to use methods from algebraic geometry to study the spectrum in a meaningful manner. In the remaining chapters, we will specifically focus on using algebraic geometry to study the critical point equations of $D(z,\lambda)$ (Chapter 4) and the reducibility of $D(z,\lambda)$ (Chapter 5). As we will discuss in these chapters, these purely algebraic questions have important consequences for the spectral theorist, particularly regarding the spectral edges, the existence of embedded eigenvalues, and quantum ergodicity.

4.1 The Spectral Edges Conjecture

This chapter is adapted from and based on [14].

4.1.1 What is the spectral edges conjecture?

An old and widely believed conjecture in mathematical physics concerns the structure of the Bloch variety near the edges of the spectral bands. Namely, that for a sufficiently general discrete periodic operator L, the extrema of the band functions λ_j on the Bloch variety are nondegenerate in that their Hessians are nondegenerate quadratic forms. This *spectral edges nondegeneracy conjecture* is stated in [31, Conjecture 5.25] and in [32, Conjecture 8.5], and it also appears in [4, 30, 42, 43]. Important notions, such as effective mass in solid state physics, the Liouville property, Green's function asymptotics, Anderson localization, homogenization, and many other assumed properties in physics, depend upon this conjecture.

The spectral edges nondegeneracy conjecture is one component of the *spectral edges conjecture*. The spectral edges conjecture states that for generic parameters, each extreme value is attained by a single band, the extrema are isolated, and the extrema are nondegenerate. We discuss progress for discrete operators on periodic graphs. In 2000, Klopp and Ralston [28] proved that for a generic Schrödinger operator each extreme value is attained by a single band. In 2015, Filonov and Kachkovskiy [17] gave a class of two-dimensional Schrödinger operators for which the extrema are isolated. They also show [17, Section 6] that the spectral edges conjecture may fail for a generic Schrödinger operator. In [10], Do, Kuchment, and Sottile prove that the spectral edges conjecture holds for a generic discrete periodic operator associated to a particular dense periodic graph. In [39], Liu proved that the extrema are isolated for the Schrödinger operator acting on the square lattice. Of course, the spectral edges conjecture is not limited to discrete periodic operators; for more on the history of this conjecture we point the reader to [10, 31, 32].

4.1.2 The Spectral Edges Conjecture for Discrete Periodic Operators

In the case of discrete periodic operators, the *spectral edges conjecture* [31, Conjecture 5.25] for a $\mathbb{Z}^d$-periodic graph Γ asserts that for a generic labeling (V, E) of Γ, the spectral edges conjecture holds. Here, generic means that there is a nonconstant polynomial $p(V, E)$ on the space of all labelings, that is the space containing all possible pairs of a $\mathbb{Z}^d$-periodic potential and $\mathbb{Z}^d$-periodic edge labeling, such that when $p(V, E) \neq 0$, these desired properties hold.

We will see later that nondegeneracy of the spectral edges is implied by the stronger condition that all critical points of the function λ on the complex Bloch variety are nondegenerate. Our aim is to bound the number of (isolated) critical points of λ on the Bloch variety of a discrete periodic operator L, give criteria for when the bound is attained, prove that it is attained for generic discrete periodic operators on a class of graphs, and finally to use these results to prove the spectral edges conjecture for $2^{19} + 2$ graphs. We treat these in the remainder of this chapter.

A family of operators has the *critical points property* if for almost all operators in the family, all critical points of the function λ (not just the extrema) are nondegenerate. Algebraic geometry was used in [10] to prove the following dichotomy: For a given algebraic family of discrete periodic operators, either the critical points property holds for that family, or almost all operators in the family have Bloch varieties with degenerate critical points.

In [10], this dichotomy was used to establish the critical points property for the family of Laplace-Beltrami difference operators on the $\mathbb{Z}^2$-periodic diatomic graph of Figure 5.3. Bloch varieties for these operators were shown to have at most 32 critical points. A single example was computed to have 32 nondegenerate critical points. Standard arguments from algebraic geometry (see Section 2.2) implied that for this family the critical points property, and therefore also the spectral edges nondegeneracy conjecture, holds (see [10, Section 5.4]).

We begin by recasting the extrema of spectral band functions in terms of constrained optimization. Suppose that Γ is a $\mathbb{Z}^d$-periodic graph, and let L be a discrete periodic operator with a real-valued labeling. The complex Bloch variety is the hypersurface $\mathbf{V}(D(z,\lambda))$ defined by the vanishing of the dispersion polynomial $D(z,\lambda)$. Critical points of the function λ on the Bloch variety are points of the Bloch variety where the gradients in $(\mathbb{C}^\times)^d \times \mathbb{C}$ of λ and $D(z,\lambda)$ are linearly dependent. That is, a critical point is a point $(z,\lambda) \in (\mathbb{C}^\times)^d \times \mathbb{C}$ with $D(z,\lambda) = 0$ such that either the gradient $\nabla D(z,\lambda)$ vanishes or we have $\frac{\partial D}{\partial z_i}(z,\lambda) = 0$ for $i = 1,\dots,d$ and $\frac{\partial D}{\partial \lambda}(z,\lambda) \neq 0$ (as $\nabla\lambda = (0,\dots,0,1)$). In either case, we have

$$D(z,\lambda) \;=\; 0 \qquad \text{and} \qquad \frac{\partial D}{\partial z_i} \;=\; 0 \qquad \text{for } i = 1,\dots,d\,.$$

Since $z_i \neq 0$, we obtain the equivalent system

$$D(z,\lambda) \;=\; z_1\frac{\partial D}{\partial z_1} \;=\; \cdots \;=\; z_d\frac{\partial D}{\partial z_d} \;=\; 0\,, \tag{4.1}$$

which we call the *critical point equations*.

Proposition 4.1.1. A point $(z,\lambda) \in (\mathbb{C}^\times)^d \times \mathbb{C}$ is a critical point of the function λ on the Bloch variety $\mathbf{V}(D(z,\lambda))$ if and only if (4.1) holds.

Proof. We already showed that at a critical point of λ, the equations (4.1) hold. Suppose now that $(z,\lambda) \in (\mathbb{C}^\times)^d \times \mathbb{C}$ is a solution to (4.1). As $D(z,\lambda) = 0$, the point lies on the Bloch variety. As $z \in (\mathbb{C}^\times)^d$, no coordinate z_i vanishes, which implies that $\frac{\partial D}{\partial z_i}(z,\lambda) = 0$ for $i = 1,\dots,d$. Thus the gradients $\nabla\lambda$ and ∇D are linearly dependent at (z,λ), showing that it is a critical point. □

Remark 4.1.2. Recall the notion of a spectral band function from Section 3.3.2. A point $(z_0,\lambda_0) \in \mathbb{T}^d \times \mathbb{R}$ such that $\lambda_0 = \lambda_j(z_0)$ is an extreme value of the spectral band function λ_j is also a critical point of the Bloch variety. Indeed, either the gradient ∇D vanishes at (z_0,λ_0) or it does not vanish. If $\nabla D(z_0,\lambda_0) = \mathbf{0}$, then (z_0,λ_0) is a critical point. If $\nabla D(z_0,\lambda_0) \neq 0$, then the Bloch variety is smooth at (z_0,λ_0) and thus is a smooth point of the graph of λ_j. As the point is smooth, the tangent plane must be a hyperplane; moreover, as $\lambda_0 = \lambda_j(z_0)$ is an extreme value of λ_j, the tangent plane is horizontal at (z_0,λ_0). If the tangent plane was not horizontal, then, by standard arguments, there would exist a curve $c(t) : \mathbb{T}^d \times \mathbb{R}$ such that $c(0) = (z_0,\lambda_0)$, $c(t) \subset \lambda_j$ for $|t| < \epsilon$ for some $\epsilon > 0$, and $c(t) = (z_t,\lambda_t)$ is such that $\lambda_t > \lambda_0$ for all $t > \epsilon$ and $\lambda_t < \lambda_0$ for all $t < \epsilon$; and so λ_0 could not be an extrema of λ_j. As the tangent plane is horizontal, this implies that λ_j is differentiable (by the implicit function theorem) and that $\frac{\partial \lambda_j}{\partial z_i}(z_0,\lambda_0) = 0$ for $i = 1,\dots,d$. Thus, the gradients of λ and D at (z_0,λ_0) are linearly dependent, showing that it is a critical point. ◇

One can also prove that the extrema of the spectral edges are contained in the collection of critical points of the Bloch variety through Kato-Rellich perturbation theory (see [39, Proof of Theorem 2.5]).

4.2 Bounding the Number of Critical Points

Let $\mathbb{C}[z^{\pm}, \lambda]$ be the ring of Laurent polynomials in $z_1, \ldots, z_d, \lambda$ where λ occurs with only non-negative exponents. Note that $D(z, \lambda) \in \mathbb{C}[z^{\pm}, \lambda]$. Our goal is a mild extension of Kushnirenko's theorem to the system of critical point equations. In particular, in this section we will prove the following result.

Theorem 4.2.1. For a polynomial $\psi \in \mathbb{C}[z^{\pm}, \lambda]$, the critical point equations for ψ

$$\psi(z, \lambda) \;=\; z_1 \frac{\partial \psi}{\partial z_1} \;=\; \cdots \;=\; z_d \frac{\partial \psi}{\partial z_d} \;=\; 0 \tag{4.2}$$

have at most $(d{+}1)!\mathrm{vol}(\mathcal{N}(\psi))$ isolated solutions in $(\mathbb{C}^{\times})^d \times \mathbb{C}$, counted with multiplicity. When the bound is attained, all solutions are isolated.

As the Bloch variety is defined by the dispersion polynomial $D(z, \lambda) = \det(L(z) - \lambda I)$, we deduce the following from Theorem 4.2.1.

Corollary 4.2.2. The number of isolated critical points of the function λ on the Bloch variety for an operator L on a discrete periodic graph is at most $(d{+}1)!\mathrm{vol}(\mathcal{N}(D))$.

Example 4.2.3. We continue the example of the hexagonal lattice from Example 3.3.6. Writing ℓ for $\alpha{+}\beta{+}\gamma{-}\lambda$, the dispersion polynomial $D(x, y, \lambda)$, that is the characteristic polynomial of the matrix (3.3), is

$$(V(u) + \ell)(V(v) + \ell) \;-\; (-\alpha - \beta x^{-1} - \gamma y^{-1})(-\alpha - \beta x - \gamma y)\,. \tag{4.3}$$

In Figure 4.1 the monomials in $D(x, y, \lambda)$ label the columns of a 3×9 array which are their exponent vectors. Figure 4.1 also shows its Newton polytope, which has volume 2. By direct computation, one finds that for generic choices of $V(u), V(v), \alpha, \beta$, and γ there are exactly 12 isolated critical points, each with multiplicity 1. Moreover, as predicted by Theorem 4.2.1, all critical points are isolated. $\diamond$

x	xy^{-1}	y^{-1}	x^{-1}	$x^{-1}y$	y	1	λ	λ^2
1	1	0	−1	−1	0	0	0	0
0	−1	−1	0	1	1	0	0	0
0	0	0	0	0	0	0	1	2

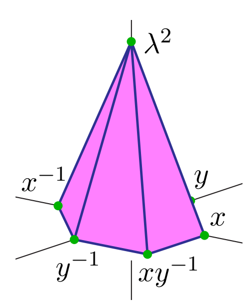


Figure 4.1: Support of the dispersion polynomial (4.3) and its Newton polytope.

We prove Theorem 4.2.1 and Corollary 4.2.2 after developing some preliminary results.

To extend Kushnirenko's theorem, we replace the nonlinear equations (4.2) on $(\mathbb{C}^\times)^d \times \mathbb{C}$ by linear equations on a projective variety. We follow the discussion of [49, Ch. 3]. Let $f \in \mathbb{C}[z^\pm, \lambda]$ be a polynomial with support $\mathcal{A} = \mathcal{A}(f)$. To simplify the presentation, we will at times assume that the origin $\mathbf{0}$ lies in $\mathcal{A}$. The results hold without this assumption, as explained in [49, Ch. 3].

Writing $\mathbb{C}^\mathcal{A}$ for the vector space with basis indexed by elements of $\mathcal{A}$, consider the map

$$\begin{array}{rcl}\varphi_\mathcal{A}:\ (\mathbb{C}^\times)^d \times \mathbb{C} & \longrightarrow & \mathbb{C}^\mathcal{A} \\ (z,\lambda) & \longmapsto & (z^a\lambda^j \mid (a,j) \in \mathcal{A})\,.\end{array}$$

This map linearizes nonlinear polynomials (*c.f.* the toric variety construction from Section 2.2). Indeed, write f as a sum of monomials,

$$f \;=\; \sum_{(a,j)\in\mathcal{A}} c_{(a,j)} z^a \lambda^j\ .$$

If $\{x_{(a,j)} \mid (a,j) \in \mathcal{A}\}$ are variables (coordinate functions) on $\mathbb{C}^\mathcal{A}$, then

$$\Lambda_f \;:=\; \sum_{(a,j)\in\mathcal{A}} c_{(a,j)} x_{(a,j)} \tag{4.4}$$

is a linear form on $\mathbb{C}^\mathcal{A}$, and we have $f(z,\lambda) = \Lambda_f(\varphi_\mathcal{A}(z,\lambda)) =: \varphi^*_\mathcal{A}(\Lambda_f)$, the pullback of Λ_f along φ_A.

Since $\mathbf{0} \in \mathcal{A}$, the corresponding coordinate $x_\mathbf{0}$ of $\varphi_\mathcal{A}$ is 1 and so the image of $\varphi_\mathcal{A}$ lies in the *principal affine open subset* $U_\mathbf{0}$ of the projective space $\mathbb{P}^\mathcal{A} := \mathbb{P}(\mathbb{C}^\mathcal{A}) = \mathbb{P}^{|\mathcal{A}|-1}$. This is the subset of $\mathbb{P}^\mathcal{A}$ where $x_\mathbf{0} \neq 0$, and it is isomorphic to the affine space $\mathbb{C}^{|\mathcal{A}|-1}$. We define $X_\mathcal{A}$ to be the closure of the image $\varphi_\mathcal{A}((\mathbb{C}^\times)^{d+1})$ in the projective space $\mathbb{P}^\mathcal{A}$, which is a projective toric variety. Because the map $\varphi_\mathcal{A}$ is continuous on $(\mathbb{C}^\times)^d \times \mathbb{C}$, $X_\mathcal{A}$ is also the closure of the image $\varphi_\mathcal{A}((\mathbb{C}^\times)^d \times \mathbb{C})$.

The map $\varphi_\mathcal{A}$ is not necessarily injective; we describe its fibers. Let $\mathbb{Z}\mathcal{A} \subset \mathbb{Z}^{d+1}$ be sublattice generated by all differences $\alpha-\beta$ for $\alpha, \beta \in \mathcal{A}$. As $\mathbf{0} \in \mathcal{A}$, this is the sublattice generated by $\mathcal{A}$, and it has full rank $d+1$ if and only if $\mathrm{conv}(\mathcal{A})$ has full dimension $d+1$. Let $G_\mathcal{A}$ be $\mathrm{Hom}(\mathbb{Z}^{d+1}/\mathbb{Z}\mathcal{A}, \mathbb{C}^\times) \subset (\mathbb{C}^\times)^{d+1}$, which acts on $(\mathbb{C}^\times)^d \times \mathbb{C}$. The fibers of $\varphi_\mathcal{A}$ are exactly the orbits of $G_\mathcal{A}$ on $(\mathbb{C}^\times)^d \times \mathbb{C}$. If $\mathrm{conv}(\mathcal{A})$ does not have full dimension, then $G_\mathcal{A}$ has positive dimension as do all fibers of $\varphi_\mathcal{A}$, otherwise $G_\mathcal{A}$ is a finite group and $\varphi_\mathcal{A}$ has finite fibers. On the torus $(\mathbb{C}^\times)^{d+1}$, $G_\mathcal{A}$ acts freely and $\varphi_\mathcal{A}((\mathbb{C}^\times)^{d+1})$ is identified with $(\mathbb{C}^\times)^{d+1}/G_\mathcal{A}$. To describe the fibers of $\varphi_\mathcal{A}$ on $(\mathbb{C}^\times)^d \times \{0\} = ((\mathbb{C}^\times)^d \times \mathbb{C}) \smallsetminus (\mathbb{C}^\times)^{d+1}$, note that $(\mathbb{C}^\times)^{d+1}$ acts on this through the homomorphism π that sends its last (λ) coordinate to $\{1\}$. Thus the fibers of $\varphi_\mathcal{A}$ on $(\mathbb{C}^\times)^d \times \{0\}$ are exactly the orbits of $\pi(G_\mathcal{A}) \subset (\mathbb{C}^\times)^d$.

Proposition 4.2.4. The dimension of $X_\mathcal{A}$ is the dimension of $\mathrm{conv}(\mathcal{A})$. The fibers of $\varphi_\mathcal{A}$ on $(\mathbb{C}^\times)^{d+1}$ are the orbits of $G_\mathcal{A}$ and its fibers on $(\mathbb{C}^\times)^d \times \{0\}$ are the orbits of $\pi(G_\mathcal{A})$.

We return to the situation of Theorem 4.2.1. Let $\psi \in \mathbb{C}[z^\pm, \lambda]$ be a polynomial with support $\mathcal{A}$. As each polynomial in (4.2) has support a subset of $\mathcal{A}$, each corresponds to a linear form on $\mathbb{P}^\mathcal{A}$ as in (4.4). The corresponding system of linear forms defines a linear subspace M_ψ of $\mathbb{P}^\mathcal{A}$. We have the following proposition (a version of [49, Lemma 3.5]).

Proposition 4.2.5. The solutions to (4.2) are the inverse images under $\varphi_{\mathcal{A}}$ of points in the linear section $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \mathbb{C}) \cap M_\psi$. When $\varphi_{\mathcal{A}}$ is an injection, it is a bijection between solutions to (4.2) on $(\mathbb{C}^\times)^d \times \mathbb{C}$ and points in $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \mathbb{C}) \cap M_\psi$.

Proof of Theorem 4.2.1. When $\mathrm{vol}(\mathcal{N}(\psi)) = 0$, so that $\mathcal{N}(\psi)$ does not have full dimension $d+1$, then each fiber of $\varphi_{\mathcal{A}}$ is positive-dimensional and so by Proposition 4.2.5 there are no isolated solutions to (4.2).

Suppose that $\mathrm{vol}(\mathcal{N}(\psi)) > 0$. Then every fiber of $\varphi_{\mathcal{A}}$ is an orbit of the finite group $G_{\mathcal{A}}$. Over points of $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^{d+1})$, each fiber consists of $|G_{\mathcal{A}}|$ points and over $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \{0\}$ each fiber consists of $|\pi(G_{\mathcal{A}})| \leq |G_{\mathcal{A}}|$ points. As $X_{\mathcal{A}}$ is the closure of $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \mathbb{C})$, the number of isolated points in $X_{\mathcal{A}} \cap M_\psi$ is at least the number of isolated points in $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \mathbb{C}) \cap M_\psi$, both counted with multiplicity. The degree of the projective variety $X_{\mathcal{A}}$ is an upper bound for the number of isolated points in $X_{\mathcal{A}} \cap M_\psi$, which is explained in [49, Ch. 3.3]. There, the product of $|G_{\mathcal{A}}|$ and the degree of $X_{\mathcal{A}}$ is shown to be $(d{+}1)!\mathrm{vol}(\mathcal{N}(\psi))$, the normalized volume of the Newton polytope of ψ. This gives the bound of Theorem 4.2.1. That all points are isolated when the bound of the degree is attained is Proposition 4.3.2 in the next section. □

4.3 General Bound

We now give conditions for when the upper bound of Corollary 4.2.2 is attained. By Proposition 4.1.1, the critical points of the function λ on the Bloch variety $\mathbf{V}(D)$ are the solutions in $(\mathbb{C}^\times)^d \times \mathbb{C}$ to the critical point equations (4.1). Let $\mathcal{A} = \mathcal{A}(D)$ be the support of the dispersion polynomial D. The critical points are $\varphi_{\mathcal{A}}^{-1}(X_{\mathcal{A}} \cap M_D)$, where $X_{\mathcal{A}} \subset \mathbb{P}^{\mathcal{A}}$ is the closure of $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \mathbb{C})$ and M_D is the subspace of $\mathbb{P}^{\mathcal{A}}$ defined by linear forms corresponding (as in (4.4)) to the polynomials in (4.1). For the bound of Theorem 4.2.1 and Corollary 4.2.2, note that the number of isolated points of $X_{\mathcal{A}} \cap M_D$ is at most the product of the degree of $X_{\mathcal{A}}$ with the cardinality of a fiber of $\varphi_{\mathcal{A}}$, which is $(d{+}1)!\mathrm{vol}(\mathcal{N}(D))$. This is because the degree of $X_{\mathcal{A}}$ gives an upper bound on the number of solutions to the linear form Λ_D, each of which may pullback up to the cardinality of a fiber of $\varphi_{\mathcal{A}}$ many solutions in $(\mathbb{C}^\times)^d \times \mathbb{C}$. We establish that the inequality of Theorem 4.2.1 holds and then interpret that for the critical point equations.

Remark 4.3.1. Let $Z \subset \mathbb{P}^n$ be a variety of dimension d and $M \subset \mathbb{P}^n$ a linear subspace of codimension d. When M is generic, the number of points in $X \cap M$ is the degree of X [45, p. 234]. When M is not generic, intersection theory gives a refinement [20, Ch. 6]. For each irreducible component Z of the intersection $X \cap M$, there is a positive integer—the intersection multiplicity along Z–such that the sum of these multiplicities is the degree of X. When Z is zero-dimensional (a point), it is the multiplicity [45, Ch. 4] as defined in Section 2.2, and when Z is positive-dimensional then the *multiplicity of X along Z* is given by the multiplicity of the generic point of Z [22, Definition 3.15]. ◇

A consequence of Remark 4.3.1 is the following.

Proposition 4.3.2. Let X, M be as in Remark 4.3.1. The number (counted with multiplicity) of isolated points of $X \cap M$ is strictly less than the degree of X if and only if the intersection has a positive-dimensional component.

Write $X_{\mathcal{A}}^\circ := \varphi_{\mathcal{A}}((\mathbb{C}^\times)^d \times \mathbb{C})$ for the image of $\varphi_{\mathcal{A}}$ and $\partial X_{\mathcal{A}} := X_{\mathcal{A}} \smallsetminus X_{\mathcal{A}}^\circ$, the points of $X_{\mathcal{A}}$ added to $X_{\mathcal{A}}^\circ$ when taking the closure. This is the *boundary* of $X_{\mathcal{A}}$.

Corollary 4.3.3. For a polynomial $\psi \in \mathbb{C}[z^{\pm}, \lambda]$, the inequality of Theorem 4.2.1 is strict if and only if $\partial X_{\mathcal{A}} \cap M_\psi \neq \emptyset$.

Proof. The inequality of Theorem 4.2.1 is strict if either of the following hold.

1. $X_{\mathcal{A}} \cap M_\psi$ has an isolated point not lying in $X^\circ_{\mathcal{A}}$.

2. $X_{\mathcal{A}} \cap M_\psi$ contains a positive-dimensional component Z.

In (1), $X_{\mathcal{A}} \cap M_\psi$ has isolated points in $\partial X_{\mathcal{A}} \cap M_\psi$, so the intersection is nonempty. In (2), Z is a projective variety of dimension at least one. The set $X^\circ_{\mathcal{A}}$ is an affine variety, and we cannot have $Z \subset X^\circ_{\mathcal{A}}$ as the only projective varieties that are also subvarieties of an affine variety are points. Thus $Z \cap \partial X_{\mathcal{A}} \neq \emptyset$, which completes the proof. □

With Corollary 4.3.3 established, we can now discuss conditions on the system of critical point equations that allow us to conclude when the bound of Corollary 4.2.2 is achieved.

4.3.1 Facial systems

We return to the general case of a toric variety. Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set of points with corresponding projective toric variety $X_{\mathcal{A}} \subset \mathbb{P}^{\mathcal{A}}$. We have the following description of the points of its boundary, $X_{\mathcal{A}} \smallsetminus \varphi_{\mathcal{A}}((\mathbb{C}^\times)^n)$.

Let $P := \text{conv}(\mathcal{A})$, the convex hull of $\mathcal{A}$. Let w be a non-zero vector $\mathbb{R}^n$, and let F be the face of P exposed by w, and write $\mathcal{F}$ for $F \cap \mathcal{A}$.

For each face F of P, there is a corresponding coordinate subspace $\mathbb{P}^{\mathcal{F}}$ of $\mathbb{P}^{\mathcal{A}}$—this is the set of points $z = [z_a \mid a \in \mathcal{A}] \in \mathbb{P}^{\mathcal{A}}$ such that $a \notin F$ implies that $z_a = 0$. The closure of the image of the map $\varphi_{\mathcal{F}} \colon (\mathbb{C}^\times)^n \to \mathbb{P}^{\mathcal{F}} \subset \mathbb{P}^{\mathcal{A}}$ is the toric variety $X_{\mathcal{F}}$. Its dimension is equal to the dimension of the face F. Write $X^\circ_{\mathcal{F}}$ for the image of $\varphi_{\mathcal{F}}$. This description and the following proposition is essentially [22, Prop. 5.1.9].

Proposition 4.3.4. The boundary of the toric variety $X_{\mathcal{A}}$ is the disjoint union of the sets $X^\circ_{\mathcal{F}}$ for all the proper faces F of $\text{conv}(\mathcal{A})$.

Let $f = \sum_{a\in\mathcal{A}} c_a x^a$ be a polynomial with support $\mathcal{A}$. We observed that if Λ_f is the corresponding linear form (4.4) on $\mathbb{P}^{\mathcal{A}}$, then the variety $\mathbf{V}(f) \subset (\mathbb{C}^\times)^n$ of f is the pullback along $\varphi_{\mathcal{A}}$ of $X^\circ_{\mathcal{A}} \cap M$, where $M := \mathbf{V}(\Lambda_f)$ is the hyperplane defined by Λ_f. Let F be a proper face of P. Then $X^\circ_{\mathcal{F}} \cap M$ pulls back along $\varphi_{\mathcal{F}}$ to the variety of

$$\varphi_{\mathcal{F}}^{-1}(\Lambda_f) \;=\; \sum_{a\in F} c_a x^a$$

in $(\mathbb{C}^\times)^n$. Recall from Section 2.1 that the sum of the terms of f whose exponents lie in F is a facial form of f and is written $f|_F$. Given a system $\Phi\colon f_1 = \cdots = f_n = 0$ involving Laurent polynomials with support $\mathcal{A}$, the system $f_1|_F = \cdots = f_n|_F = 0$ of their facial forms is the *facial system* $\Phi|_F$ of Φ.

Corollary 4.3.5. Let M be the intersection of the hyperplanes given by the polynomials in a system Φ of Laurent polynomials with support $\mathcal{A}$. For each face F of $\text{conv}(\mathcal{A})$, the points of $X^\circ_{\mathcal{F}} \cap M$ pull back under $\varphi_{\mathcal{F}}$ to the solutions of the facial system $\Phi|_F$.

If no facial system $\Phi|_F$ has a solution, then the number of solutions to $\Phi = 0$ on $(\mathbb{C}^\times)^n$ is $n!\text{vol}(\text{conv}(\mathcal{A}))$.

Proof. The first statement follows from the observation about a single polynomial f and its facial form $f|_F$, and the second is a consequence of a version of Corollary 4.3.3 for $X_{\mathcal{A}} \smallsetminus \varphi_{\mathcal{A}}((\mathbb{C}^\times)^n)$. □

The second statement is essentially [2, Thm. B] and is also explained in [49, Sect. 3.4].

4.3.2 Facial systems of the critical point equations

We now interpret the facial systems of the critical point equations.

Let $\psi \in \mathbb{C}[z^{\pm}, \lambda]$ have support $\mathcal{A} \subset \mathbb{Z}^d \times \mathbb{N}$ and Newton polytope $P := \operatorname{conv}(\mathcal{A})$. We will assume that P has dimension $d{+}1$, and also that $\mathcal{A} \cap \mathbb{Z}^d \times \{0\}$ is a facet of $\mathcal{A}$, called its ***base***. Let (4.2) be the critical point equations for λ on ψ and $M_\psi \subset \mathbb{P}^{\mathcal{A}}$ the corresponding linear subspace of codimension $d{+}1$.

Let $\mathbf{0} := 0^d$ in $\mathbb{Z}^d$ and $\mathbf{e} := (\mathbf{0}, 1)$. The base of $\mathcal{A}$ is exposed by $\mathbf{e}$ and it is the support of $\psi(z, 0)$. A main difference between the sparse equations of Section 4.3.1 and the critical point equations (4.1) is that the critical point equations allow solutions with $\lambda = 0$, which is the component of the boundary of the toric variety corresponding to the base of $\mathcal{A}$. A face F of P is ***vertical*** if it contains a vertical line segment, that is a segment parallel to $\mathbf{e}$.

Lemma 4.3.6. Suppose that F is a proper face of P that is not the base of P and is not vertical. Then the corresponding facial system of the critical point equations has a solution if and only if the hypersurface $\mathbf{V}(\psi|_F)$ defined by $\psi|_F$ in $(\mathbb{C}^\times)^{d+1}$ is singular.

Proof. Let $\mathbf{0} \neq w \in \mathbb{Z}^{d+1}$ be an integer vector that exposes the face F. As F is not vertical, we may assume that w_{d+1} is nonzero. Indeed, if w_{d+1} was 0, then, as F is not vertical, all points in F must have the same constant for their last coordinate; but then F has a quasi-homogeneity w' that is non-zero in w'_{d+1}. As F is not the base, it lies on an affine hyperplane that does not contain the origin, so that $\psi|_F$ is quasi-homogeneous with some quasi-homogeneity w. Write w_F for the constant $w \cdot a$ for $a \in F$. By Lemma 2.1.2 (2), we have

$$w_F\, \psi|_F \;=\; \sum_{i=1}^{d} w_i\, z_i \frac{\partial \psi|_F}{\partial z_i} \;+\; w_{d+1}\, \lambda \frac{\partial \psi|_F}{\partial \lambda}\,. \tag{4.5}$$

Suppose now that (z, λ) is a solution of the restriction of the critical point equations to the face F. That is, at (z, λ),

$$\psi|_{\mathcal{F}} \;=\; \left(z_1 \frac{\partial \psi}{\partial z_1} \right)\Big|_F \;=\; \cdots \;=\; \left(z_d \frac{\partial \psi}{\partial z_d} \right)\Big|_F \;=\; 0\,.$$

Observe that $(z_i \frac{\partial \psi}{\partial z_i})|_F = z_i \frac{\partial \psi|_F}{\partial z_i}$ (and the same for λ). Since $w_{d+1} \neq 0$, these equations and (4.5) together imply that $(\lambda \frac{\partial \psi}{\partial \lambda})|_F = 0$, which implies that (z, λ) is a singular point of the hypersurface $\mathbf{V}(\psi|_F) \subset (\mathbb{C}^\times)^{d+1}$ defined by $\psi|_F$. □

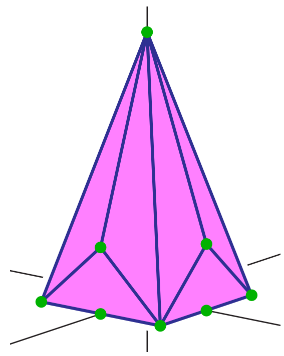
We deduce the following theorem.

Theorem 4.3.7. If the Newton polytope $\mathcal{N}(\psi)$ of ψ has no vertical faces and the restriction of ψ to each face that is not the base of $\mathcal{N}(\psi)$ defines a smooth variety, then the critical point equations have exactly $(d{+}1)!\mathrm{vol}(\mathcal{N}(\mathcal{A}))$ solutions in $(\mathbb{C}^\times)^d \times \mathbb{C}$.

We apply this when ψ is the dispersion polynomial $D(z, \lambda)$. Recall that the boundary of the variety $X_{\mathcal{A}}$ (X_D) corresponds to all proper faces of its Newton polytope $\mathcal{N}(D)$, except for its base.

Corollary 4.3.8. Let L be an operator on a periodic graph and set $D = \det(L(z) - \lambda I)$. If $\mathcal{N}(D)$ has no vertical faces and if for each face F that is not its base, $\mathbf{V}(D|_F)$ is smooth, then the Bloch variety has exactly $(d{+}1)!\mathrm{vol}(\mathrm{conv}(\mathcal{A}(D)))$ critical points.

Example 4.3.9. The restriction on vertical faces is necessary. General operators on the second graph in Figure 3.1 (an abelian cover of K_4) have the following Newton polytope:



It has base $[-1, 1]^2$, apex $(0, 0, 4)$, and the remaining vertices are at $(\pm 1, 0, 1)$ and $(0, \pm 1, 1)$. It has volume $20/3$, so we expect $40 = 3! \cdot 20/3$ critical points. However, there are at most 32 critical points, as direct computation shows that the critical point equations have two solutions on each of its four vertical faces. ⋄

4.4 Dense Periodic Graphs

The Newton polytope $\mathcal{N}(D)$ of the dispersion polynomial of an operator on a periodic graph is central to our results. In Section 4.4.1 we associate a polytope $\mathcal{N}(\Gamma)$ to any periodic graph Γ such that $\mathcal{N}(D) \subseteq \mathcal{N}(\Gamma)$ for any operator on Γ, and that we have equality for almost all parameter values. We call $\mathcal{N}(\Gamma)$ the *Newton polytope* of Γ.

Let Γ be a connected $\mathbb{Z}^d$-periodic graph. The graph Γ is *dense* if there exists a fundamental domain W such that for every $a \in \mathcal{A}(W)$, there is an edge in Γ between every pair of distinct vertices in the union of W and $a{+}W$. In particular, the restriction of Γ to W is the complete graph on W. When it is clear that we have designated a fixed fundamental domain W for Γ, we will sometimes refer to $\mathcal{A}(W)$ as the support of Γ; however, this distinction is necessary as, depending on our choice of W, $\mathcal{A}(W)$ can vary. Every periodic graph is a subgraph of a minimal dense periodic graph with the same fundamental domain W. If we assume that Γ is connected, then the integer span of $\mathcal{A}(W)$ is $\mathbb{Z}^d$. The graph of Figure 4.2 is dense, while no graph of Figure 3.1 is dense. We identify the Newton polytope of a dense periodic graph and show that when $d = 2$ or 3, a general operator on Γ satisfies Corollary 4.3.8.

The set of parameters (V, E) for operators on a periodic graph Γ with fundamental domain W is $Y = \mathbb{C}^W \times \mathbb{C}^{\mathbf{E}}$, where $\mathbf{E}$ is the set of orbits of edges. Given a labeling $c \in Y$, we write $L_c(z)$ for the Floquet matrix of the discrete periodic operator L on Γ with labeling c, and let $D_c := \det(L_c(z) - \lambda I)$. We observed that for any labeling $c \in Y$, each entry of $L_c(z)$ has support a subset of $\mathcal{A}(W)$. Consequently, each diagonal entry of $L_c(z) - \lambda I$ has support a subset of $\mathcal{A}(W) \cup \{\mathbf{e}\}$ and its Newton polytope is a subpolytope of $\mathrm{N} := \mathrm{conv}(\mathcal{A}(W) \cup \{\mathbf{e}\})$. Let $m := |W|$, the number of orbits of vertices.

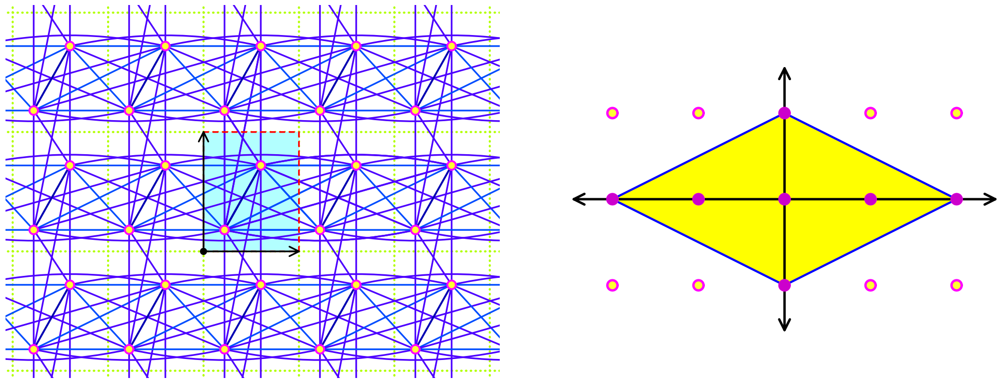


Figure 4.2: A dense graph Γ and its support $\mathcal{A}(W)$ with convex hull.

Lemma 4.4.1. The Newton polytope $\mathcal{N}(D_c)$ is a subpolytope of the dilation mN of N.

Proof. The dispersion polynomial D_c is a sum of products of m entries of the $m \times m$ matrix $L_c(z) - \lambda I$. Each such product has Newton polytope a subpolytope of mN as the Newton polytope of a product is the sum of Newton polytopes of the factors. □

Figure 4.3 shows $m\text{N} = 2\text{N}$ for the dense graph of Figure 4.2. Observe that mN is a pyramid

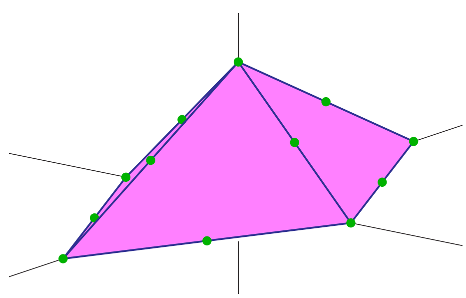


Figure 4.3: Newton polytope of a dense graph.

with base $m\text{conv}(\mathcal{A}(W))$ and apex $m\mathbf{e}$, and it has no vertical faces.

Theorem 4.4.2. Let Γ be a dense $\mathbb{Z}^d$-periodic graph with fundamental domain W. There is a nonempty Zariski open subset U of the parameter space Y such that for $c \in U$, the Newton polytope of $D_c(z, \lambda)$ is the pyramid mN. When $d = 2$ or 3, then we may choose U so that for every $c \in U$ and face F of mN that is not its base, $\mathbf{V}(D_c|_F)$ is smooth.

We prove Theorem 4.4.2 in the following two sections.

4.4.1 The Newton polytope of Γ

Treating parameters as indeterminates gives the *generic dispersion polynomial* $D(V, E, z, \lambda)$, which is a polynomial in z, λ whose coefficients are polynomials in the parameters V, E. The *Newton polytope* $\mathcal{N}(\Gamma)$ of Γ is the convex hull of the monomials in z, λ that appear in $D(V, E, z, \lambda)$. Notice that each of these monomials (in z, λ) that appear in $D(V, E, z, \lambda)$ has a coefficient that is a polynomial in V, E, and thus this will be the Newton polytope of $D(z, \lambda)$ for a generic choice of labeling. We now demonstrate this, along with several other facts about $\mathcal{N}(\Gamma)$.

Lemma 4.4.3. For $c \in Y$, $\mathcal{N}(D_c(z, \lambda))$ is a subpolytope of $\mathcal{N}(\Gamma)$. The set of $c \in Y$ such that $\mathcal{N}(D_c(z, \lambda)) = \mathcal{N}(\Gamma)$ is a dense open subset U. When Γ is a dense periodic graph, $\mathcal{N}(\Gamma) = m\mathrm{N}$.

Proof. For any $c = (V, E) \in Y$, $D_c(z, \lambda)$ is the evaluation of the generic dispersion polynomial $D(V, E, z, \lambda)$ at the point (V, E). Thus $\mathcal{N}(D_c) \subset \mathcal{N}(\Gamma)$.

The coefficient $C_{(a,j)}$ of a monomial $z^a\lambda^j$ in $D(V, E, z, \lambda)$ is a polynomial in (V, E). For any $c = (V, E) \in Y$, $z^a\lambda^j$ appears in D_c if and only if $C_{(a,j)}(V, E) \neq 0$. Thus, we have the equality $\mathcal{N}(D_c) = \mathcal{N}(\Gamma)$ of Newton polytopes if and only if $C_{(a,j)}(V, E) \neq 0$ for every vertex (a, j) of $\mathcal{N}(\Gamma)$, which defines a dense open subset $U \subset Y$.

When Γ is dense and c is vector of indeterminates, then every diagonal entry of $L_c(z) - \lambda I$ has support $\mathcal{A}(W) \cup \{\mathbf{e}\}$. It follows that for each vertex (a, j) of $m\mathrm{N}$ we have that $C_{(a,j)}$ has terms that come from the product $\prod_{i=1}^{m}(L_c(z) - \lambda I)_{i,i}$, and so $C_{(a,j)}$ is nonzero as a polynomial in (V, E). It follows by Lemma 4.4.1 that $\mathcal{N}(\Gamma) = m\mathrm{N}$. □

4.4.2 Smoothness of the Bloch variety at infinity

Let Γ be a connected dense periodic graph with $d = 2$ or 3. Let $U \subset Y$ be the subset of Lemma 4.4.3. We show that for each face F of $\mathcal{N}(\Gamma)$ that is not its base, there is a nonempty open subset U_F of U such that for $c \in U_F$, the restriction $D_c|_F$ to the monomials in F defines a smooth hypersurface. Then for parameters c in the intersection of the U_F, the operator satisfies the hypotheses of Corollary 4.3.8, which proves Theorem 4.4.2.

Let F be a face of $\mathcal{N}(\Gamma)$ that is not its base and let $c \in U$. We may assume that F is not a vertex, for then $D_c|_F$ is a single term and $\mathbf{V}(D_c|_F) = \emptyset$. Since $\mathcal{N}(\Gamma) = m\mathrm{N}$, there is a unique face M of N such that $F = m\mathrm{M}$. We have that

$$D_c(z, \lambda)|_F \;=\; \det\bigl((L_c(z) - \lambda I)|_{\mathrm{M}}\bigr) \;,$$

where each entry of the matrix $(L_c(z) - \lambda I)|_{\mathrm{M}}$ is the facial form $f|_{\mathrm{M}}$ of the corresponding entry f of $L_c(z) - \lambda I$.

Since M is not the base of N (and thus does not contain the origin), we make the following observation, which follows from the form of the operator L_c. If the apex $\mathbf{e} = (\mathbf{0}, 1)$ of N lies in M and f is a diagonal entry of $(L_c(z) - \lambda I)|_{\mathrm{M}}$, then f contains the term $-\lambda$. Any other integer point $a \in \mathrm{M}$ is not the origin and lies in the support $\mathcal{A}(W)$ of $W = \{\omega_1, \ldots, \omega_m\}$, and the coefficient of z^a in f is $-E(\omega_i, a + \omega_j)$, where f is the entry in row i and column j. Consequently, except possibly for terms $-\lambda$, all coefficients of entries in $(L_c(z) - \lambda I)|_{\mathrm{M}}$ are distinct parameters.

Let $Y' \subset Y$ be the set of parameters c where

$$E_{(\omega_i, a+\omega_j)} \;=\; 0 \qquad \text{if } a \in \mathrm{M} \text{ and } j \neq i, i{+}1.$$

(Here, $m{+}1$ is interpreted to be 1.) For $c \in Y'$, all entries of $L_c(z)|_{\mathrm{M}}$ are zero, except on the diagonal, the first super diagonal, and the lower left entry. The same arguments as in the proof of Lemma 4.4.3 show that there exist parameters $c \in Y'$ such that $D_c(z,\lambda)$ has Newton polytope $\mathcal{N}(\Gamma)$. Thus $Y' \cap U \neq \emptyset$, where $U \subset Y$ is the set of Lemma 4.4.3.

Theorem 4.4.4. There exists an open subset U' of Y' with $U' \subset U$ such that if $c \in U'$, then $\mathbf{V}(D_c(z,\lambda)|_F)$ is a smooth hypersurface in $(\mathbb{C}^\times)^{d+1}$.

Since smoothness of $\mathbf{V}(D_c(z,\lambda)|_F)$ is an open condition on the space Y of parameters, this will complete the proof of Theorem 4.4.2.

Proof. Let us write $\psi_c(z,\lambda)$ for the facial polynomial $D_c(z,\lambda)|_F$. We will show that the set of $c \in Y'$ such that $\mathbf{V}(\psi_c(z,\lambda))$ is singular is a finite union of proper algebraic subvarieties. As $c \in Y'$, the only nonzero entries in the matrix $(L_c(z){-}\lambda I)|_{\mathrm{M}}$ are its diagonal entries $f_1(z,\lambda),\ldots,f_m(z,\lambda)$ and the entries $g_1(z),\ldots,g_m(z)$ which are in positions $(1,2),\ldots,(m{-}1,m)$ and $(m,1)$, respectively. Thus

$$\psi_c(z,\lambda) \;=\; D_c(z,\lambda)|_F \;=\; \det((L_c(z)-\lambda I)|_{\mathrm{M}}) \;=\; \prod_{i=1}^m f_i(z,\lambda) \;-\; (-1)^m \prod_{i=1}^m g_i(z)\,.$$

For a polynomial f in the variables (z,λ), write $\nabla_{\mathbb{T}}$ for the toric gradient operator,

$$\nabla_{\mathbb{T}} f \;:=\; \Bigl(z_1\frac{\partial f}{\partial z_1},\ldots,z_d\frac{\partial f}{\partial z_d},\lambda\frac{\partial f}{\partial \lambda}\Bigr)\,.$$

Note that

$$\nabla_{\mathbb{T}}\psi_c \;=\; \sum_{i=1}^m (\nabla_{\mathbb{T}} f_i) f_1\cdots\widehat{f_i}\cdots f_m \;-\; (-1)^m \sum_{i=1}^m (\nabla_{\mathbb{T}} g_i) g_1\cdots\widehat{g_i}\cdots g_m\,. \tag{4.6}$$

Here $\widehat{f_i}$ indicates that f_i does not appear in the product, and the same for $\widehat{g_i}$.

Let $(z,\lambda) \in \mathbf{V}(\psi_c)$ be a singular point. Then $\psi_c(z,\lambda)=0$ and $\nabla_{\mathbb{T}}\psi_c(z,\lambda)=\mathbf{0}$. There are five cases that depend upon the number of polynomials f_i, g_j vanishing at (z,λ).

(i) At least two polynomials f_p and f_q and two polynomials g_r and g_s vanish at (z,λ). Thus $\psi(z,\lambda)=0$ and by (4.6) this implies that $\nabla_{\mathbb{T}}\psi_c(z,\lambda)=\mathbf{0}$.

(ii) At least two polynomials f_p and f_q and exactly one polynomial g_s vanish at (z,λ). Thus $\psi(z,\lambda)=0$ and by (4.6) if $\nabla_{\mathbb{T}}\psi_c(z,\lambda)=\mathbf{0}$, then $\nabla_{\mathbb{T}} g_s(z,\lambda)=\mathbf{0}$.

(iii) Exactly one polynomial f_p and at least two polynomials g_r and g_s vanish at (z,λ). Thus $\psi(z,\lambda)=0$ and by (4.6) if $\nabla_{\mathbb{T}}\psi_c(z,\lambda)=\mathbf{0}$, then $\nabla_{\mathbb{T}} f_p(z,\lambda)=\mathbf{0}$.

(iv) Exactly one polynomial f_p and one polynomial g_r vanish at (z,λ). Thus $\psi(z,\lambda)=0$ and by (4.6) if $\nabla_{\mathbb{T}}\psi_c(z,\lambda)=\mathbf{0}$, then, after reindexing so that $p=r=1$, we have

$$\nabla_{\mathbb{T}} f_1(z,\lambda)\cdot\prod_{i=2}^m f_i(z,\lambda) \;-\; (-1)^m \nabla_{\mathbb{T}} g_1(z,\lambda)\cdot\prod_{i=2}^m g_i(z,\lambda) \;=\; \mathbf{0}\,. \tag{4.7}$$

(v) No polynomials f_i or g_i vanish at (z, λ).

In each case, we will show that the set of parameters $c \in Y'$ such that there exist (z, λ) satisfying these conditions lies in a proper subvariety of Y'. Cases (i)—(iv) use arguments based on the dimension of fibers and figures of a map and are proven in the rest of this section. Case (v) is proven in Section 4.4.3 and it uses Bertini's Theorem. $\square$

Let us write X for the space $(\mathbb{C}^\times)^{d+1}$ and x for a point $(z, \lambda) \in X$. We first derive consequences of some vanishing statements. For a finite set $\mathcal{F} \subset \mathbb{Z}^{d+1}$, let $\mathbb{C}^\mathcal{F}$ be the space of coefficients of polynomials in $x \in X$ with support $\mathcal{F}$. This is the parameter space for polynomials with support $\mathcal{F}$. Recall that we write $\mathbf{0}$ for the vector with coordinates all 0 and m for $|W|$.

Lemma 4.4.5. We have the following.

1. For any $x \in X$, $f(x) = 0$ is a nonzero homogeneous linear equation on $\mathbb{C}^\mathcal{F}$.

2. For any $x \in X$, $\{\nabla_\mathbb{T} f(x) \mid f \in \mathbb{C}^\mathcal{F}\}$ is the linear span $\mathbb{C}\mathcal{F}$ of $\mathcal{F}$.

Suppose that the affine span of $\mathcal{F}$ does not contain the origin. Then

3. For any $f \in \mathbb{C}^\mathcal{F}$ and $x \in X$, $\nabla_\mathbb{T} f = \mathbf{0}$ implies that $f(x) = 0$.

4. For any $x \in X$, the equation $\nabla_\mathbb{T} f(x) = \mathbf{0}$ defines a linear subspace of $\mathbb{C}^\mathcal{F}$ of codimension $\dim \mathbb{C}\mathcal{F}$.

Proof. Writing $f = \sum_{a \in \mathcal{F}} c_a x^a$, the first statement is obvious. We have $\nabla_\mathbb{T} f = \sum a c_a x^a$. As the coefficients c_a are independent complex numbers and $x^a \neq 0$, Statement (2) is immediate. The hypothesis that the affine span of $\mathcal{F}$ does not contain the origin implies that any $f \in \mathbb{C}^\mathcal{F}$ is quasi-homogeneous. Statement (3) follows from Equation (4.5). The last statement follows from the observation that the set of f such that $\nabla_\mathbb{T} f = \mathbf{0}$ is the kernel of a surjective linear map $\mathbb{C}^\mathcal{F} \twoheadrightarrow \mathbb{C}\mathcal{F}$. $\square$

Let $\mathcal{F} := \mathrm{M} \cap (\mathcal{A}(W) \cup \{\mathbf{e}\})$, where $\mathbf{e} = (\mathbf{0}, 1)$, be the (common) support of the diagonal polynomials f_i and let $\mathcal{G} := \mathrm{M} \cap \mathcal{A}(W)$ be the (common) support of the polynomials g_j. We either have that $\mathcal{F} = \mathcal{G}$ or $\mathcal{F} = \mathcal{G} \cup \{\mathbf{e}\}$. Also, $|\mathcal{F}| > 1$ as M is not a vertex, and as M is a proper face of $Q = \mathrm{conv}(\mathcal{A}(W) \cup \{\mathbf{e}\})$, but not its base, the polynomials f_i, g_j are quasi-homogeneous with a common quasi-homogeneity.

The parameter space for the entries of $(L_c(z) - \lambda I)|_\mathrm{M}$ is

$$Z := (\mathbb{C}^\mathcal{F})^{\oplus m} \oplus (\mathbb{C}^\mathcal{G})^{\oplus m}.$$

We write $c = (f_\bullet, g_\bullet) = (f_1, \ldots, f_m, g_1, \ldots, g_m)$ for points of Z. This is a coordinate subspace of the parameter space Y'. As Z contains exactly those parameters that can appear in the facial polynomial $\psi_c(x)$, it suffices to show that the set of parameters $c = (f_\bullet, g_\bullet) \in Z$ such that $\mathbf{V}(\psi_c(x))$ is singular lies in a proper subvariety of Z. The same case distinctions (i)—(v) in the proof of Theorem 4.4.4 apply.

After reindexing, Case (i) in the proof of Theorem 4.4.4 follows from the next lemma.

Lemma 4.4.6. The set

$$\Theta \ := \ \{c \in Z \mid \exists x \in X \text{ with } f_1(x) = f_2(x) = g_1(x) = g_2(x) = 0\}$$

lies in a proper subvariety of Z.

Proof. Consider the incidence correspondence,

$$\Upsilon \ := \ \{(x, f_\bullet, g_\bullet) \in X \times Z \mid f_1(x) = f_2(x) = g_1(x) = g_2(x) = 0\}\,.$$

This has projections to X and to Z and its image in Z is the set Θ.

Consider the projection $\pi_X \colon \Upsilon \to X$. By Lemma 4.4.5(1), for $x \in X$, each condition $f_i(x) = 0$, $g_i(x) = 0$ for $i = 1, 2$ is a linear equation on $\mathbb{C}^{\mathcal{F}}$ or $\mathbb{C}^{\mathcal{G}}$. These are independent on Z as they involve different variables. Thus the fiber $\pi_X^{-1}(x)$ is a vector subspace of Z of codimension 4, and $\dim(\Upsilon) = \dim(Z) - 4 + \dim(X) = \dim(Z) + d - 3$.

Consider the projection π_Z to Z and let $(f_\bullet, g_\bullet) \in \pi_Z(\Upsilon)$. Then there is an $x \in X$ such that $f_1(x) = f_2(x) = g_1(x) = g_2(x) = 0$. Let $w \in \mathbb{Z}^{d+1}$ be a common quasi-homogeneity of the polynomials f_i, g_j. By Lemma 2.1.2 (1), for any $t \in \mathbb{C}^\times$, each of f_1, f_2, g_1, g_2 vanishes at $t^w \cdot x$. Thus the fiber $\pi_Z^{-1}(f_\bullet, g_\bullet)$ has dimension at least one. By the Theorem [45, Theorem 1.25] on the dimension of the image and fibers of a map, the image $\pi_Z(\Upsilon)$ has dimension at most $\dim(Z) + d - 4 < \dim(Z)$. It follows that $\pi_Z(\Upsilon) = \Theta$ must be a proper subvariety of Z, and thus the lemma is established. □

After reindexing and possibly interchanging f with g, Cases (ii) and (iii) in the proof of Theorem 4.4.4 follow from the next lemma.

Lemma 4.4.7. The set

$$\Theta \ := \ \{c \in Z \mid \exists x \in X \text{ with } f_1(x) = f_2(x) = g_1(x) = 0 \text{ and } \nabla_{\mathbb{T}} g_1(x) = \mathbf{0}\}$$

lies in a proper subvariety of Z.

Proof. Consider the incidence correspondence,

$$\Upsilon \ := \ \{(x, f_\bullet, g_\bullet) \in X \times Z \mid f_1(x) = f_2(x) = g_1(x) = 0 \text{ and } \nabla_{\mathbb{T}} g_1(x) = \mathbf{0}\}\,.$$

Let $x \in X$ and consider the fiber $\pi_X^{-1}(x)$. As in the proof of Lemma 4.4.6, the conditions $f_1(x) = f_2(x) = 0$ are two independent linear equations on Z. By Lemma 4.4.5 (3), $\nabla_{\mathbb{T}} g_1(x) = \mathbf{0}$ implies that $g_1(x) = 0$, and by Lemma 4.4.5 (4), the condition $\nabla_{\mathbb{T}} g_1(x) = \mathbf{0}$ is $\dim \mathbb{C}\mathcal{G}$ further independent linear equations on Z.

If $|\mathcal{G}| = 1$, so that $g_1 = c_a x^a$ is a single term, then $g(x) = 0$ implies that $c_a = 0$. Consequently, the image Θ of Υ in Z lies in a proper subvariety (particularly one that has codimension at least 1). Otherwise, $|\mathcal{G}| > 1$ which implies that $\dim \mathbb{C}\mathcal{G} \geq 2$, and thus the fiber $\pi_X^{-1}(x)$ has codimension at least 4 (as the codimension is $2 + \dim \mathbb{C}\mathcal{G}$). As in the proof of Lemma 4.4.6, this implies that Θ lies in a proper subvariety of Z. □

Case (iv) in the proof of Theorem 4.4.4 is more involved.

Lemma 4.4.8. The set

$$\Theta \;:=\; \{c \in Z \mid \exists x \in X \text{ with } f_1(x) = g_1(x) = 0 \text{ and } \nabla_{\mathbb{T}}\psi_c(x) = \mathbf{0}\}$$

lies in a proper subvariety of Z.

Proof. The set Θ includes the sets of Lemmas 4.4.6 and 4.4.7. Let $\Theta^\circ \subset \Theta$ be the set of $c = (f_\bullet, g_\bullet)$ that have a witness $x \in X$ (that is a point x such that $f_1(x) = g_1(x) = 0$ and $\nabla_{\mathbb{T}}\psi_c(x) = \mathbf{0}$) such that none of $\nabla_{\mathbb{T}} f_1(x)$, $\nabla_{\mathbb{T}} g_1(x)$, or $f_i(x)g_i(x)$ for $i > 1$ vanish. As Θ is exactly the union of Θ° and the sets from Lemmas 4.4.6 and 4.4.7 (which we know are proper subvarieties of Z), it will suffice to show that Θ° lies in a proper subvariety of Z.

For this, we use the incidence correspondence,

$$\begin{aligned}\Upsilon \;:=\; \{(y, x, f_\bullet, g_\bullet) \in \mathbb{C}^\times \times X \times Z \mid f_1(x) = g_1(x) = 0\,,\\ y\prod_{i=2}^{m} f_i(x) \;-\; (-1)^m \prod_{i=2}^{m} g_i(x) = 0\,,\ \text{and } \nabla_{\mathbb{T}} f_1(x) \;-\; (-1)^m y \nabla_{\mathbb{T}} g_1(x) = \mathbf{0}\}\,.\end{aligned}$$

We show that $\Theta^\circ \subset \pi_Z(\Upsilon)$. Let $c = (f_\bullet, g_\bullet) \in \Theta^\circ$ with witness $x \in X$ in that $f_1(x) = g_1(x) = 0$ and $\nabla_{\mathbb{T}}\psi_c(x) = \mathbf{0}$, but none of $\nabla_{\mathbb{T}} f_1(x)$, $\nabla_{\mathbb{T}} g_1(x)$, or $f_i(x)g_i(x)$ for $i > 1$ vanish. There is a unique $y \in \mathbb{C}^\times$ satisfying

$$y\prod_{i=2}^{m} f_i(x) \;-\; (-1)^m \prod_{i=2}^{m} g_i(x) \;=\; 0\,.$$

Dividing (4.7) by $\prod_{i=2}^{m} f_i(x)$ (which is nonzero by our assumptions) gives

$$\nabla_{\mathbb{T}} f_1(x) - (-1)^m y \nabla_{\mathbb{T}} g_1(x) = \mathbf{0},$$

and thus $(y, x, f_\bullet, g_\bullet) \in \Upsilon$.

We now determine the dimension of Υ. Let $(y, x) \in \mathbb{C}^\times \times X$ and consider the fiber $\pi^{-1}(y, x) \subset Z$ above it in Υ. The two linear and one nonlinear equations

$$f_1(x) \;=\; g_1(x) \;=\; y\prod_{i=2}^{m} f_i(x) - (-1)^m \prod_{i=2}^{m} g_i(x) \;=\; 0 \tag{4.8}$$

are independent on Z as they involve disjoint sets of variables, and thus define a subvariety $T \subset Z$ of codimension 3. Consider the remaining equation, $\nabla_{\mathbb{T}} f_1(x) - (-1)^m y \nabla_{\mathbb{T}} g_1(x) = \mathbf{0}$.

Note that if $\mathbf{e} = (\mathbf{0}, 1)$ lies in the support $\mathcal{F}$ of f_1, so that $\mathcal{F} = \mathcal{G} \cup \{\mathbf{e}\}$, then $\nabla_{\mathbb{T}} f_1(x)$ contains the term $-\mathbf{e}$ and thus cannot lie in the span $\mathbb{C}\mathcal{G}$ of $\mathcal{G}$, which contains $\nabla_{\mathbb{T}} g_1(x)$ by Lemma 4.4.5(2). In this case the fiber is empty and $\Theta^\circ = \emptyset$.

Suppose that $\mathcal{F} = \mathcal{G}$ and $(f_\bullet, g_\bullet) \in \Theta^\circ$. Let $w \in \mathbb{Z}^{d+1}$ be any homogeneity for f_1 (or g_1). Then there exists $w_{\mathcal{F}} \neq 0$ such that $w \cdot a = w_{\mathcal{F}}$ for all $a \in \mathcal{F}$. Equation (4.5) implies that

$$w \cdot \nabla_{\mathbb{T}} f_1(x) \;=\; w_{\mathcal{F}}\, f_1(x) \;=\; 0\,,$$

and the same for g_1. Thus $\nabla_{\mathbb{T}} f_1(x)$ and $\nabla_{\mathbb{T}} g_1(x)$ are annihilated by all homogeneities and so these vectors lie in the affine span of $\mathcal{F}$—the linear span of differences $a - b$ for $a, b \in \mathcal{F}$. This has dimension $\dim \mathbb{C}\mathcal{F} - 1$. Consequently, $\nabla_{\mathbb{T}} f_1(x) - (-1)^m y \nabla_{\mathbb{T}} g_1(x) = \mathbf{0}$ consists of $\dim \mathbb{C}\mathcal{F} - 1$ independent linear equations on the subset of $\mathbb{C}^{\mathcal{F}} \oplus \mathbb{C}^{\mathcal{F}}$ consisting of pairs f_1, g_1 such that $f_1(x) = g_1(x) = 0$. These are independent of the third equation in (4.8). Thus the fiber $\pi^{-1}(y, x) \subset Z$ has codimension $3 + \dim \mathbb{C}\mathcal{F} - 1 = 2 + \dim \mathbb{C}\mathcal{F}$ and so

$$\dim \Upsilon \;=\; \dim(\mathbb{C}^\times \times X) + \dim Z - \dim \mathbb{C}\mathcal{F} - 2 \;=\; \dim Z + d - \dim \mathbb{C}\mathcal{F}\,.$$

Let $(f_\bullet, g_\bullet) \in \pi_Z(\Upsilon)$ have witness (y, x). That is, the equations (4.8) hold, as well as $\nabla_{\mathbb{T}} f_1(x) - (-1)^m y \nabla_{\mathbb{T}} g_1(x) = \mathbf{0}$. As in the proof of Lemma 4.4.6, if $w \in \mathbb{Z}^{d+1}$ is a quasi-homogeneity for polynomials of support $\mathcal{F}$, then $(y, t^w \cdot x)$ also satisfies these equations.

We have $\mathcal{F} = \mathcal{G} = \mathrm{M} \cap \mathcal{A}(W)$, so that M is a face of the base of N. Thus there are at least two (in fact the codimension of M in N) independent homogeneities, which implies that the fiber $\pi_Z^{-1}(f_\bullet, g_\bullet)$ has dimension at least two. This implies that the image Θ° has dimension at most $\dim Z + d - \dim \mathbb{C}\mathcal{F} - 2$. Since M is not a vertex, $\dim \mathbb{C}\mathcal{F} \geq 2$, which shows that $\dim \Theta^\circ < \dim Z$ and completes the proof. □

4.4.3 Case (v)

For $\alpha \in \mathbb{C}^\times$, define $\Psi(\alpha, f_\bullet, g_\bullet) \subset X$ to be the set

$$\Big\{ x \in X \;\Big|\; \text{none of } f_i(x) g_i(x) \text{ for } i \geq 1 \text{ vanish and } \prod_{i=1}^m f_i(x) \;-\; (-1)^m \alpha \prod_{i=1}^m g_i(x) \;=\; 0 \Big\}\,.$$

Case (v) in the proof of Theorem 4.4.4 follows from the next lemma.

Lemma 4.4.9. There is a dense open subset $U_1 \subset Z$ such that if $(f_\bullet, g_\bullet) \in U_1$, then $\Psi(1, f_\bullet, g_\bullet)$ is smooth.

We will deduce this from a weaker lemma.

Lemma 4.4.10. There is a dense open subset $U \subset \mathbb{C}^\times \times Z$ such that if $(\alpha, f_\bullet, g_\bullet) \in U$, then $\Psi(\alpha, f_\bullet, g_\bullet)$ is smooth.

Proof. Let $T \subset X \times Z$ be the set of $(x, f_\bullet, g_\bullet)$ such that none of $f_i(x) g_i(x)$ for $i \geq 1$ vanish. Define $\varphi\colon T \to \mathbb{C}^\times \times Z$ by

$$\varphi(x, f_\bullet, g_\bullet) \;=\; \big((-1)^m \textstyle\prod_{i=1}^m f_i(x) / \prod_{i=1}^m g_i(x)\;,\; f_\bullet\;,\; g_\bullet\big)\,.$$

Notice that $\varphi^{-1}(\alpha, f_\bullet, g_\bullet) = \Psi(\alpha, f_\bullet, g_\bullet)$ for $(\alpha, f_\bullet, g_\bullet) \in \mathbb{C}^\times \times Z$.

We claim that $\varphi(T)$ is dense in $\mathbb{C}^\times \times Z$. For this, recall that the polynomials f_i have support $\mathcal{F}$, which is $\mathrm{M} \cap (\mathcal{A}(W) \cup \{\mathbf{e}\})$ for some face M of $Q = \mathrm{conv}(\mathcal{A}(W) \cup \{\mathbf{e}\})$ that is neither its base nor a vertex, and the polynomials g_i have support $\mathcal{G} = \mathrm{M} \cap \mathcal{A}(W)$. Since M is not a vertex, there are $a, b \in \mathcal{F}$ with $a \neq b$ and $b \in \mathcal{A}(W)$.

Let $f_i := x^a$ and $g_i := x^b$ for $i = 1, \ldots, m$. Then $X \times \{(f_\bullet, g_\bullet)\} \subset T$ and for $x \in X$ $\varphi(x, f_\bullet, g_\bullet) = (x^{ma} - (-1)^m x^{mb}, f_\bullet, g_\bullet)$. The map $X = (\mathbb{C}^\times)^{d+1} \to \mathbb{C}^\times$ given by $x \mapsto x^{ma} - (-1)^m x^{mb}$ is surjective as $ma - mb \neq 0$. This implies that the differential $d\varphi$ is surjective at any point of $X \times \{(f_\bullet, g_\bullet)\}$, and therefore $\varphi(T)$ is dense in $\mathbb{C}^\times \times Z$.

Since T is an open subset of the smooth variety $X \times Z$, it is smooth. Then Bertini's Theorem [45, Thm. 2.27, p. 139] implies that there is a dense open subset $U \subset \mathbb{C}^\times \times Z$ such that for $(\alpha, f_\bullet, g_\bullet) \in U$, $\varphi^{-1}(\alpha, f_\bullet, g_\bullet) = \Psi(\alpha, f_\bullet, g_\bullet)$ is smooth. □

We now can deduce Lemma 4.4.9.

Proof of Lemma 4.4.9. If we knew that the set U of Lemma 4.4.10 contained a point $(1, f_\bullet, g_\bullet)$, then $U_1 := U \cap (\{1\} \times Z)$ would be a dense open subset of Z, which would complete the proof. As we do not know this, we must instead argue indirectly.

Suppose that there is no such open set U_1 as in Lemma 4.4.9. Then the set $\Xi \subset Z$ consisting of $(f_\bullet, g_\bullet)$ such that $\Psi(1, f_\bullet, g_\bullet)$ is singular is dense in Z.

For $\alpha \in \mathbb{C}^\times$ and $(f_\bullet, g_\bullet) \in Z$, define $\alpha.(f_\bullet, g_\bullet)$ to be $(f_\bullet, \alpha.g_\bullet)$ where

$$\alpha.(g_1, g_2, \dots, g_m) \;=\; (\alpha g_1, g_2, \dots, g_m)\,.$$

This is a $\mathbb{C}^\times$-action on Z. Consequently, $\alpha.\Xi$ is dense in Z for all $\alpha \in \mathbb{C}^\times$.

Let $U \subset \mathbb{C}^\times \times Z$ be the set of Lemma 4.4.10. As it is nonempty, let $(\alpha, f'_\bullet, g'_\bullet) \in U$. Then $U_\alpha := U \cap (\{\alpha\} \times Z)$ is nonempty and open in $\{\alpha\} \times Z$. As $\alpha.\Xi$ is dense, we have

$$U_\alpha \bigcap \big(\{\alpha\} \times \alpha.\Xi\big) \;\neq\; \emptyset\,.$$

This is a contradiction, for if $(\alpha, f_\bullet, g_\bullet) \in U_\alpha$, then $\Psi(\alpha, f_\bullet, g_\bullet)$ is smooth, but if $(f_\bullet, g_\bullet) \in \alpha.\Xi$, then $(f_\bullet, \alpha^{-1}g_\bullet) \in \Xi$ and $\Psi(1, f_\bullet, \alpha^{-1}g_\bullet)$ is singular. The contradiction follows from the equality of sets $\Psi(\alpha, f_\bullet, g_\bullet) = \Psi(1, f_\bullet, \alpha^{-1}g_\bullet)$. □

4.5 Critical Points Property

We illustrate our results, using them to establish the critical points property (and thus the spectral edges nondegeneracy conjecture) for three periodic graphs. Through Theorem 4.5.2, we will see that a single calculation is sufficient for proving that the critical points property holds for the examples to come. We first state this property.

Let Γ be a connected $\mathbb{Z}^d$-periodic graph with parameter space $Y = \mathbb{C}^E \times \mathbb{C}^W$ for discrete operators on Γ. We say that Γ has the *critical points property* if there is a dense open subset $U \subset Y$ such that if $c \in U$, then every critical point of the function λ on the Bloch variety $\mathbf{V}(D_c(z, \lambda))$ is nondegenerate in that the Hessian determinant

$$\det\left(\left(\frac{\partial^2 \lambda}{\partial z_i \partial z_j}\right)_{i,j=1}^{d}\right) \tag{4.9}$$

is nonzero at that critical point. Here, the derivatives are implicit, using that $D(z, \lambda) = 0$.

4.5.1 Reformulation of Hessian condition

Let $D = \det(L_c(z) - \lambda I)$ be the dispersion polynomial for an operator L_c on a periodic graph Γ. In Section 4.1.2 we derived the equations for the critical points of the function λ on the Bloch variety $\mathbf{V}(D(z, \lambda))$,

$$D(z, \lambda) \;=\; 0 \qquad \text{and} \qquad \frac{\partial D}{\partial z_i} \;=\; 0 \qquad \text{for } i = 1, \dots, d\,. \tag{4.10}$$

Implicit differentiation of $D = 0$ gives $\frac{\partial D}{\partial z_j} + \frac{\partial D}{\partial \lambda} \cdot \frac{\partial \lambda}{\partial z_j} = 0$. If $\frac{\partial D}{\partial \lambda} \neq 0$, then $\frac{\partial \lambda}{\partial z_j} = 0$. If $\frac{\partial D}{\partial \lambda} = 0$, then (z, λ) is a singular point hence is also a critical point of the function λ and so we again have $\frac{\partial \lambda}{\partial z_j} = 0$. Differentiating again we obtain,

$$0 \;=\; \frac{\partial}{\partial z_i}\left(\frac{\partial D}{\partial z_j} + \frac{\partial D}{\partial \lambda} \cdot \frac{\partial \lambda}{\partial z_j}\right) \;=\; \frac{\partial^2 D}{\partial z_i \partial z_j} + \frac{\partial^2 D}{\partial z_i \partial \lambda} \cdot \frac{\partial \lambda}{\partial z_j} + \frac{\partial D}{\partial \lambda} \cdot \frac{\partial^2 \lambda}{\partial z_i \partial z_j}\,.$$

At a critical point (so that $\frac{\partial\lambda}{\partial z_j}=0$), we have

$$\frac{\partial^2 D}{\partial z_i \partial z_j} \;=\; -\frac{\partial D}{\partial \lambda}\cdot\frac{\partial^2 \lambda}{\partial z_i \partial z_j}\,.$$

Thus

$$\det\left(\left(\frac{\partial^2 D}{\partial z_i \partial z_j}\right)_{i,j=1}^{d}\right) \;=\; \left(-\frac{\partial D}{\partial \lambda}\right)^{d}\cdot\det\left(\left(\frac{\partial^2 \lambda}{\partial z_i \partial z_j}\right)_{i,j=1}^{d}\right)\,.$$

Consider now the Jacobian matrix of the critical point equations (4.10),

$$J \;=\; \begin{pmatrix} \frac{\partial D}{\partial z_1} & \cdots & \frac{\partial D}{\partial z_d} & \frac{\partial D}{\partial \lambda} \\ \frac{\partial^2 D}{\partial z_1^2} & \cdots & \frac{\partial^2 D}{\partial z_d \partial z_1} & \frac{\partial^2 D}{\partial \lambda \partial z_1} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 D}{\partial z_1 \partial z_d} & \cdots & \frac{\partial^2 D}{\partial z_d^2} & \frac{\partial^2 D}{\partial \lambda \partial z_d} \end{pmatrix}\,.$$

At a critical point, the first row is $(0\ \cdots\ 0\ \frac{\partial D}{\partial\lambda})$, and thus

$$\det(J) \;=\; \frac{\partial D}{\partial \lambda}\cdot\det\left(\left(\frac{\partial^2 D}{\partial z_i \partial z_j}\right)_{i,j=1}^{d}\right) \;=\; (-1)^d\left(\frac{\partial D}{\partial \lambda}\right)^{d+1}\cdot\det\left(\left(\frac{\partial^2 \lambda}{\partial z_i \partial z_j}\right)_{i,j=1}^{d}\right)\,.$$

We deduce the following lemma.

Lemma 4.5.1. A nonsingular critical point (z,λ) on $\mathbf{V}(D_c(z,\lambda))$ is nondegenerate if and only if it is a regular solution of the critical point equations (4.10).

The following theorem is adapted from arguments in [10, Sect. 5.4].

Theorem 4.5.2. Let Γ be a $\mathbb{Z}^d$-periodic graph. If there is a parameter value $c\in Y$ such that the critical point equations have $(d{+}1)!\mathrm{vol}(\mathcal{N}(\Gamma))$ regular solutions, then the critical points property holds for Γ.

Proof. Let Y be the parameter space for operators L on Γ. Consider the variety

$$CP \;:=\; \{(c,z,\lambda)\in Y\times(\mathbb{C}^\times)^d\times\mathbb{C} \mid \text{ the critical point equations (4.1) hold}\}\,,$$

which is the incidence variety of critical points on all Bloch varieties for operators on Γ. Let π be its projection to Y. For any $c\in Y$, the fiber $\pi^{-1}(c)$ is the set of critical points of the function λ on the corresponding Bloch variety for D_c. By Corollary 4.2.2, there are at most $(d{+}1)!\mathrm{vol}(\mathcal{N}(D_c))$ isolated points in the fiber.

Let $c\in Y$ be a point such that the critical point equations have $(d{+}1)!\mathrm{vol}(\mathcal{N}(\Gamma))$ regular solutions. Then $(d{+}1)!\mathrm{vol}(\mathcal{N}(\Gamma))\leq(d{+}1)!\mathrm{vol}(\mathcal{N}(D_c))$. By Lemma 4.4.3, $\mathcal{N}(D_c)$ is a subpolytope of $\mathcal{N}(\Gamma)$, so that $\mathrm{vol}(\mathcal{N}(D_c))\leq\mathrm{vol}(\mathcal{N}(\Gamma))$. We conclude that both polytopes have the same volume and are therefore equal. In particular, the corresponding Bloch variety has the maximum number of critical points, and each is a regular solution of the critical point equations (4.1). Because they are regular solutions, the implicit function theorem implies that there is a neighborhood U_c of c in the classical topology on Y such that the map $\pi^{-1}(U_c) \to U_c$ is proper (it is a $(d+1)!\mathrm{vol}(\mathcal{N}(\Gamma))$-sheeted cover).

The set DC of degenerate critical points is the closed subset of CP given by the vanishing of the Hessian determinant (4.9). Since π is proper over U_c, if $DP = \pi(DC)$ is the image of DC in Y, then $DP \cap U_c$ is closed in U_c. As the points of $\pi^{-1}(c)$ are regular solutions, Lemma 4.5.1 implies they are all nondegenerate and thus $c \notin DP$, so that $U_c \smallsetminus DP$ is a nonempty classically open subset of Y consisting of parameter values c' with the property that all critical points on the corresponding Bloch variety are nondegenerate.

This implies that there is a nonempty Zariski open subset of Y consisting of parameters such that all critical points on the corresponding Bloch variety are nondegenerate, which completes the proof. □

By Theorem 4.5.2, it suffices to find a single Bloch variety with the maximum number of isolated critical points to establish the critical points property for a periodic graph. The following examples use such a computation to establish the critical points property for $2^{19} + 2$ graphs Γ. Computer code and output are available at the github repository[1].

Example 4.5.3. Let us consider the dense $\mathbb{Z}^2$-periodic graph Γ of Figure 4.2. It has $m = 2$ points in its fundamental domain, the highlighted region, and the convex hull of the support $\mathcal{A}(W)$ has area 4. By Theorem C, a general operator on Γ has $2! \cdot 2^{2+1} \cdot 4 = 64$ critical points. There are

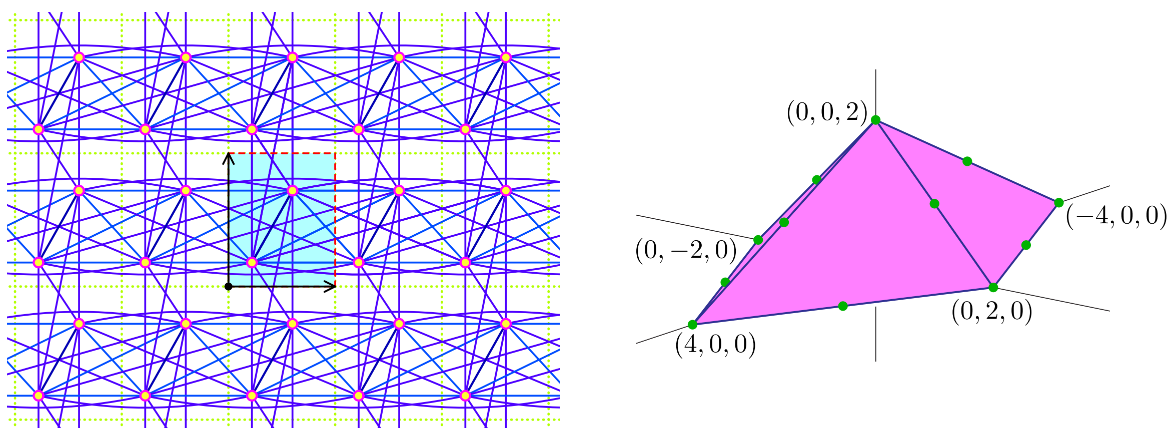


Figure 4.4: Dense periodic graph and its polytope from Figure 4.2.

13 edges and two vertices in W, and independent computations in the computer algebra systems Macaulay2 [24] and Singular [9] find a point $c \in Y = \mathbb{C}^{15}$ such that the critical point equations have 64 regular solutions on $(\mathbb{C}^\times)^2 \times \mathbb{C}$. By Theorem 4.5.2, the critical points property holds for Γ. ◇

Example 4.5.4. The graph Γ in Figure 4.5 is not dense. Its restriction to the highlighted fundamental domain is not the complete graph on 3 vertices and there are three and not nine edges between

[1] https://mattfaust.github.io/CPODPO.

any two adjacent fundamental domains. Altogether, it has $3 \cdot 6 + 1 = 19$ fewer edges than the corresponding dense graph. Its support $\mathcal{A}(W)$ forms the columns of the matrix $\left(\begin{smallmatrix} 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 \end{smallmatrix}\right)$ whose convex hull is a hexagon of area 3.

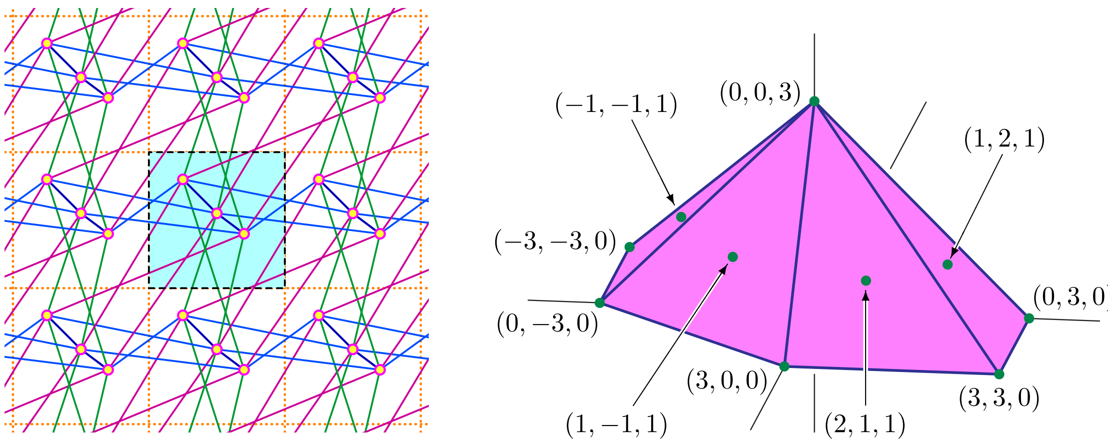


Figure 4.5: Sparse graph with the same Newton polytope as the corresponding dense graph.

Despite Γ not being dense, its Newton polytope $\mathcal{N}(\Gamma)$ is equal to the Newton polytope of the dense graph with the same parameters, $\mathcal{A}(W)$ and W. Figure 4.5 displays the Newton polytope, along with elements of the support of the dispersion polynomial that are visible. Observe that on each triangular face, there are four and not ten monomials.

By Theorem A (Corollary 4.2.2), there are at most $2! \cdot 3^{2+1} \cdot 3 = 162$ critical points. There are eleven edges and three vertices in W, and independent computations in Macaulay2 and Singular find a point $c \in Y = \mathbb{C}^{14}$ such that the critical point equations have 162 regular solutions on $(\mathbb{C}^{\times})^2 \times \mathbb{C}$. By Theorem 4.5.2, the critical points property holds for Γ.

Let Γ' be a graph that has the same vertex set and support as Γ, and contains all the edges of Γ—then [10, Thm. 22] implies that the critical points property also holds for Γ'. This establishes the critical points property for an additional $2^{19} - 1$ periodic graphs. $\diamond$

Example 4.5.5. The graph Γ of Figure 4.6 has only ten edges but the same highlighted fundamental domain W and support $\mathcal{A}(W)$ as the the graph of Figure 4.5, which had eleven edges. Its Newton polytope is smaller, as it is missing the vertices $(3, 3, 0)$ and $(-3, -3, 0)$.

It has volume $70/3$ and normalized volume $3! \cdot 70/3 = 140$. Independent computations in Macaulay2 and Singular find a point $c \in Y = \mathbb{C}^{13}$ such that the critical point equations have 140 regular solutions on $(\mathbb{C}^{\times})^2 \times \mathbb{C}$. Thus there are no critical points at infinity, and Theorem B implies that the Bloch variety is smooth at infinity.

As before, achieving the bound of Corollary 4.2.2 with regular solutions implies that all critical points are nondegenerate and the critical points property holds for Γ. $\diamond$

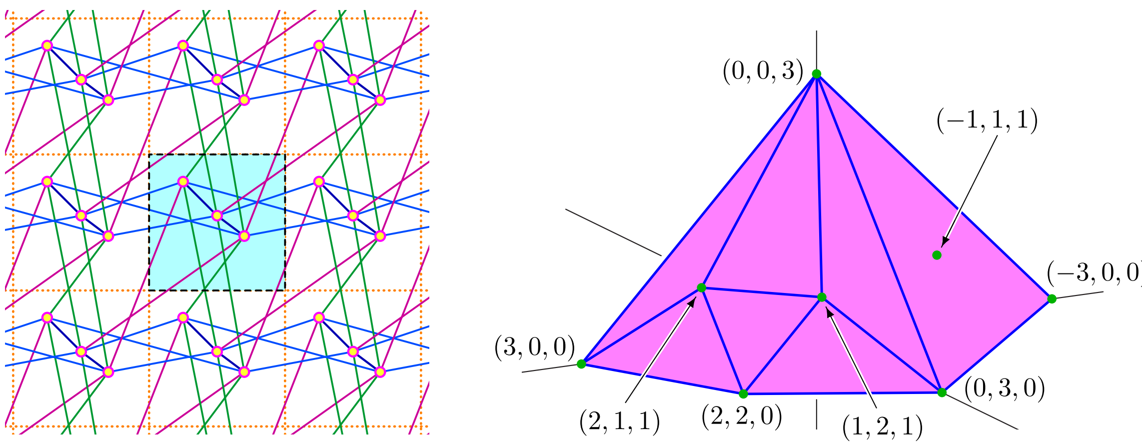


Figure 4.6: A periodic graph and its Newton polytope.

5. Irreducibility of Bloch and Fermi Varieties

Much of this chapter is adapted from, and based on, [12].

5.1 History and Overview

Studying whether Bloch varieties or Fermi varieties are irreducible has important consequences. Indeed, irreducibility of Fermi varieties implies the absence of embedded eigenvalues [33–35]. This result has several applications, see [31,37,39]. Recently, it was shown in [38] that irreducibility of the Bloch variety implies quantum ergodicity when, additionally, the dispersion polynomial satisfies $D(z,\lambda) \neq D(\mu z, \lambda)$ for all $\mu \in \mathbb{T}^d \smallsetminus \{(1,1,\dots,1)\}$. In [1,15,23,39], these varieties were shown to be irreducible for operators defined for a class of finite-range graphs, such as the square lattice. Irreducibility of Fermi varieties was studied with respect to planarity of the graph in [36]. Reducibility of Bloch variety has also been studied in [18,44,47,48]. For a more detailed history, see [31,32,40].

Studying the irreducibility of Bloch and Fermi varieties upon changing the period of the potential dates back to the 1980s, with a focus on the discrete periodic Schrödinger operator for the square lattice. In this case, irreducibility of the Bloch variety was proven in [3] for $d = 2$. For Fermi varieties, [23] showed irreducibility for $d = 2$ for all but finitely many values of λ, and [1] showed that every Fermi variety is irreducible for $d = 3$. In [39] this result was extended to higher dimensions, Fermi varieties are irreducible for all λ if the potential V is $Q\mathbb{Z}$-periodic where Q is primitive. Most recently, [15] proved irreducibility of Bloch varieties for a large class of graphs, which includes the triangular lattice and Harper lattice, and [16] proved irreducibility of all but finitely many Fermi varieties for a large class of graphs, including the Lieb lattice.

As irreducibility of the dispersion polynomial implies irreducibility of its Bloch variety, we focus on examining the effect that changing the period lattice has on the irreducibility of this polynomial. That is, following the notation of Section 3.3.3, if we know that $D(z,\lambda)$ is irreducible for a $\mathbb{Z}^d$-periodic potential, then can we use that knowledge to conclude that $D_Q(z,\lambda)$ is irreducible for a $Q\mathbb{Z}$-periodic potential? We will apply our observations towards discussing the irreducibility of the Bloch varieties associated to various families of periodic graphs, including the diamond lattice (Example 5.3.4), the dice lattice (Example 5.3.9), and dense periodic graphs. We will also discuss the irreducibility of Fermi varieties associated to various families of dense periodic graphs (Example 5.3.7).

To begin, we fix a $\mathbb{Z}^d$-periodic potential and study the reducibility of the dispersion polynomial $D(z,\lambda)$ after replacing the period lattice with the sublattice $Q\mathbb{Z} = \bigoplus_{i=1}^d q_i\mathbb{Z}$, where $Q \in \mathbb{N}^d$; that is, we will study whether $D_Q(z,\lambda)$ is irreducible for the same $\mathbb{Z}^d$-periodic potential. We will show that the irreducibility of the corresponding dispersion polynomial $D_Q(z,\lambda)$ depends on the irreducibility of $D(z,\lambda)$ and a relationship between the support of $D(z,\lambda)$ and the tuple Q (Theorem 5.2.4 and Lemma 5.2.9).

We will then use the theory developed in Section 2.1.3 to study when irreducibility is preserved for a potential that is periodic with respect to the sublattice $Q\mathbb{Z}$. Roughly, for a $Q\mathbb{Z}$-periodic potential, we show that if enough of the facial polynomials of $D_Q(z,\lambda)$ are also facial polynomials for some $\mathbb{Z}^d$-periodic potential and the corresponding facial polynomials of $D(z,\lambda)$ are irreducible, then the polynomial factors "only homothetically" (Corollary 5.2.12). If this condition is met, then $D_Q(z,\lambda)$ is irreducible if it has an irreducible facial polynomial (Corollary 2.1.8). We conclude with various applications of these results.

5.2 On Irreducibility of the Dispersion Polynomial

Let Γ be a $\mathbb{Z}^d$-periodic graph with fundamental domain W, let $L(z)$ be the Floquet matrix of a discrete periodic operator L with a $\mathbb{Z}^d$-periodic labeling (V, E) (that is E is a $\mathbb{Z}^d$-periodic edge labeling and V is a $\mathbb{Z}^d$-periodic potential), and let $D(z,\lambda) := \det(L(z) - \lambda I)$ be the dispersion polynomial. Fix $Q = (q_1, \dots, q_d) \in \mathbb{N}^d$, and consider Γ as a $Q\mathbb{Z}$-periodic graph with fundamental domain W_Q. For a $Q\mathbb{Z}$-periodic potential V_Q, let $L_Q(z)$ be the Floquet matrix of L acting on the $Q\mathbb{Z}$-periodic graph Γ with fundamental domain W_Q and with edge labeling E and potential V_Q. Let $\hat{L}_Q(z)$ be the matrix obtained from the Floquet matrix $L_Q(z)$ after the change of basis (3.8), and let $\hat{V}$ be V_Q after change of basis (3.8) (see Section 3.3.3).

Recall that $D_Q(z,\lambda) = \det(L_Q(z) - \lambda I)$, $\hat{D}_Q(z,\lambda) = \det(\hat{L}_Q(z) - \lambda I)$, $D_Q(z^Q,\lambda) = \hat{D}_Q(z,\lambda)$, $|Q| := \prod_{i=1}^d q_i$, and $\mathcal{U}_Q := \prod_{i=1}^d \mathcal{U}_{q_i}$, where $\mathcal{U}_{q_i}$ is the multiplicative group of q_ith roots of unity. We will often write D, D_Q, and $\hat{D}_Q$ in place of $D(z,\lambda), D_Q(z,\lambda)$, and $\hat{D}_Q(z,\lambda)$ respectively.

We seek conditions on D and Q which imply that if $V_Q = V$ then D_Q is irreducible. In this case, the $Q\mathbb{Z}$-periodic potential V_Q is also $\mathbb{Z}^d$-periodic.

Suppose that $V_Q = V$. By Remark 3.3.7, $\hat{V}$ is given by a diagonal matrix when the potential V_Q is $\mathbb{Z}^d$-periodic. It follows that $\hat{D}_Q(z,\lambda)$ may be expressed in terms of $D(z,\lambda)$ as

$$\hat{D}_Q(z,\lambda) := \det(\hat{L}_Q(z) - \lambda I) = \prod_{\mu \in \mathcal{U}_Q} \det(L(\mu z) - \lambda I) = \prod_{\mu \in \mathcal{U}_Q} D(\mu z, \lambda). \tag{5.1}$$

Due to this expression, we have that $\mathcal{N}(\hat{D}_Q) = |Q|\mathcal{N}(D)$. As $D_Q(z^Q,\lambda) = \hat{D}_Q(z,\lambda)$, $\mathcal{N}(D_Q)$ is the polytope obtained after multiplying the ith coordinate of each point of $\mathcal{N}(D)$ by $\frac{|Q|}{q_i}$. That is, $(a_1, \dots, a_d, a_{d+1})$ is a vertex of $\mathcal{N}(D)$ if and only if $(\frac{|Q|a_1}{q_1}, \dots, \frac{|Q|a_d}{q_d}, |Q|a_{d+1})$ is a vertex of $\mathcal{N}(D_Q)$. Therefore, $w = (w_1, \dots, w_{d+1}) \in \mathbb{Z}^d$ exposes a face of $\mathcal{N}(D)$ if and only if $w' = (q_1 w_1, \dots, q_d w_d, w_{d+1})$ exposes a face of $\mathcal{N}(D_Q)$. We call $\mathcal{N}(D_Q)$ a *contracted Q-dilation* of $\mathcal{N}(D)$ (a contracted Q-dilation is a $(\frac{|Q|}{q_1}, \dots, \frac{|Q|}{q_d}, |Q|)$-dilation in the sense of Section 5.3). We will often write $(D_Q)|_w$ for $(D_Q)|_{w'}$; similarly, if F is the face of $\mathcal{N}(D_Q)$ exposed by w', then we will write $D|_F$ for the corresponding facial polynomial of D and vice versa.

The following lemma is an immediate consequence of (5.1), we include a proof for the reader's convenience.

Lemma 5.2.1. Let V_Q be the $\mathbb{Z}^d$-periodic potential V. Suppose that D is only homothetically reducible, then D_Q is only homothetically reducible.

Proof. Suppose D is only homothetically reducible and $D_Q = g(z,\lambda)h(z,\lambda)$, where $g(z,\lambda)$ is not a monomial. As $D_Q(z^Q,\lambda) = \hat{D}_Q(z,\lambda)$, it suffices to show that $\mathcal{N}(g(z^Q,\lambda))$ is homothetic to $\mathcal{N}(\hat{D}_Q)$.

By Remark 2.1.5, as D is only homothetically reducible, if $f_1, \dots, f_l$ are its irreducible factors, then there exist $r_1, \dots, r_l \in \mathbb{Q}$ such that $\mathcal{N}(f_i) = r_i \mathcal{N}(D)$. As V_Q is $\mathbb{Z}^d$-periodic, it follows that $\mathcal{N}(f_i) = \frac{r_i}{|Q|}\mathcal{N}(\hat{D}_Q)$.

By (5.1), $g(z^Q,\lambda)h(z^Q,\lambda) = \hat{D}_Q = \prod_{\mu\in\mathcal{U}_Q} D(\mu z,\lambda)$. Thus for some integer s, such that $0 < s \le l|Q|$, $g(z^Q,\lambda) = \prod_{i=1}^{s} \kappa_i(z,\lambda)$, where each $\kappa_i(z,\lambda) = f_j(\mu z,\lambda)$ for some $j \in [l]$ and $\mu \in \mathcal{U}_Q$ (noting that each $f_j(\mu z,\lambda)$ is an irreducible factor of $D(z^Q,\lambda)$). If $\kappa_i(z,\lambda) = f_j(\mu z,\lambda)$ then let $\chi_i = r_j$. We conclude that $\mathcal{N}(g(z^Q,\lambda)) = \frac{\sum_{i=1}^{s}\chi_i}{|Q|}\mathcal{N}(\hat{D}_Q)$, and thus we have $\mathcal{N}(g(z,\lambda)) = \frac{\sum_{i=1}^{s}\chi_i}{|Q|}\mathcal{N}(D_Q)$. □

Remark 5.2.2. Lemma 5.2.1 extends to facial polynomials and specializations (such as the specializations that define Fermi varieties). That is, for a $\mathbb{Z}^d$-periodic potential, if $D|_w$ is only homothetically reducible then so is $(D_Q)|_w$. Let $\lambda_0 \in \mathbb{C}$. If $D(z,\lambda_0)$ is only homothetically reducible then so is $D_Q(z,\lambda_0)$. Finally, if $D|_w(z,\lambda_0)$ is only homothetically reducible then so is $(D_Q)|_w(z,\lambda_0)$. Thus the results of this section extend to $D_Q(z,\lambda_0)$, as well as the facial forms $(D_Q)|_w(z,\lambda_0)$ and $(D_Q)|_w$. ◇

The following lemma is considered folklore, and will provide us motivation. We include a proof for the reader's convenience. For $A \in \mathbb{N}^d$, let $Q/A := (\frac{q_1}{a_1},\ldots,\frac{q_d}{a_d})$, and write $A \mid Q$ if $a_i \mid q_i$ for all $i = 1,\ldots,d$.

Lemma 5.2.3. Suppose $A = (a_1,\ldots,a_d) \in \mathbb{N}^d$ such that $A \mid Q$ and let V_Q be an $A\mathbb{Z}$-periodic potential. If D_Q is irreducible, then D_A is irreducible.

Proof. By way of contradiction, suppose that D_A is reducible, that is, $D_A = f(z,\lambda)g(z,\lambda)$. The fundamental domain W_Q is a Q/A-expansion of W_A, and so by Section 3.3.3,

$$D_Q(z_1^{\frac{q_1}{a_1}},\ldots,z_d^{\frac{q_d}{a_d}},\lambda) = \prod_{\mu\in\mathcal{U}_{Q/A}} D_A(\mu z,\lambda) = \prod_{\mu\in\mathcal{U}_{Q/A}} f(\mu z,\lambda)g(\mu z,\lambda).$$

By Lemma 3.1 of [15], there exist f' and g' such that

$$f'(z_1^{\frac{q_1}{a_1}},\ldots,z_d^{\frac{q_d}{a_d}},\lambda) = \prod_{\mu\in\mathcal{U}_{Q/A}} f(\mu z,\lambda) \quad\text{and}\quad g'(z_1^{\frac{q_1}{a_1}},\ldots,z_d^{\frac{q_d}{a_d}},\lambda) = \prod_{\mu\in\mathcal{U}_{Q/A}} g(\mu z,\lambda).$$

Therefore $D_Q(z_1,\ldots,z_d,\lambda) = f'(z_1,\ldots,z_d,\lambda)g'(z_1,\ldots,z_d,\lambda)$. □

By Lemma 5.2.3, if D_Q is irreducible and $A \mid Q$, then D_A is irreducible. For the remaining section, we assume D is irreducible for the $\mathbb{Z}^d$-periodic potential V and that $V_Q = V$. Let $\sigma = \{\sigma_1,\ldots,\sigma_k\} \in \binom{[d]}{k}$ be a k-element subset of the set $[d] := \{1,2,\ldots,d\}$. Let $\bar{\sigma} = [d] \smallsetminus \sigma$ be the complement of σ in $[d]$. Define $\sigma \odot Q = (\sigma \odot q_1, \sigma \odot q_2,\ldots,\sigma \odot q_d)$, where $\sigma \odot q_i = q_i$ if $i \in \sigma$, and $\sigma \odot q_j = 1$ if $j \notin \sigma$. Let $D_{\sigma\odot Q}$ be the dispersion polynomial given by the discrete periodic operator L, with the $\mathbb{Z}^d$-periodic (and therefore $(\sigma \odot Q)\mathbb{Z}$-periodic) labeling (V_Q, E) associated to the $(\sigma \odot Q)\mathbb{Z}$-periodic graph Γ with fundamental domain given by the expansion $W_{\sigma\odot Q}$ of W. This notation will allows us study irreducibility of the dispersion polynomial as we incrementally expand coordinate-wise from $D = D_{(1,\ldots,1)}$ to D_Q. Lemma 5.2.3 suggests this approach, as we know that if D_Q is irreducible, then for any $k < d$ we must have that any $D_{\sigma\odot Q}$ is irreducible for all $\sigma \in \binom{[d]}{k}$. Each time we are assuming that $D_{\sigma\odot Q}$ is the dispersion polynomial given by the discrete periodic operator L, with the $\mathbb{Z}^d$-periodic (and therefore $(\sigma \odot Q)\mathbb{Z}$-periodic) labeling (V_Q, E). Indeed, this thread of thought leads us to the following theorem.

Theorem 5.2.4. Fix a positive integer $k < d$ and suppose that $D_{\sigma\odot Q}$ is irreducible for all $\sigma \in \binom{[d]}{k}$. If no $k+1$ coordinates of Q share a common factor, then D_Q is irreducible.

Proof. Assume that no $k+1$ coordinates of Q share a common factor. Suppose there exist polynomials g, h, with g not a monomial, such that

$$D_Q = g(z,\lambda)h(z,\lambda).$$

Reordering, if necessary, we may assume that $\sigma = [k]$. As W_Q is an expansion of $W_{\sigma\odot Q}$,

$$D_Q(z^{\bar{\sigma}\odot Q},\lambda) = \prod_{\gamma\in\mathcal{U}_{\bar{\sigma}\odot Q}} D_{\sigma\odot Q}(z_1,\ldots,z_k,\gamma_1 z_{k+1},\ldots,\gamma_{d-k}z_d,\lambda).$$

As $D_{\sigma\odot Q}$ is irreducible, there exist $\gamma^1,\ldots,\gamma^s \in \mathcal{U}_{\bar{\sigma}\odot Q}$ for some $s \geq 1$ such that

$$g(z^{\bar{\sigma}\odot Q},\lambda) = \prod_{i=1}^{s} D_{\sigma\odot Q}(z_1,\ldots,z_k,\gamma_1^i z_{k+1},\ldots,\gamma_{d-k}^i z_d,\lambda).$$

Expand this so that

$$g(z^{Q},\lambda) = \prod_{i=1}^{s}\prod_{\mu\in\mathcal{U}_Q} D(\mu_1 z_1,\ldots,\mu_k z_k,\gamma_1^i z_{k+1},\ldots,\gamma_{d-k}^i z_d,\lambda)$$

can be written as a product of $S := s\prod_{i=1}^k q_i$ irreducible polynomials. As σ is arbitrary (that is, the same argument holds for any $\sigma \in \binom{[d]}{k}$ after reordering coordinates), the product $q_{\sigma_1}\cdots q_{\sigma_k}$ divides S for all $\sigma \in \binom{[d]}{k}$. By our assumption, no $k+1$ coordinates of Q share a common factor. Therefore, if p^a is a prime power that divides $|Q|$, there exists $\sigma \in \binom{[d]}{k}$ such that $p^a \mid q_{\sigma_1}\cdots q_{\sigma_k}$, and thus $p^a \mid S$. As S is at most $|Q|$, it follows that $S = |Q|$, and so h must be a monomial. $\square$

To apply Theorem 5.2.4, we need to find conditions that imply $D_{\sigma\odot Q}$ is irreducible for all $|\sigma| \geq 1$. Rather than depending strictly on Q, these conditions examine the reducibility of D_Q in relation to the interplay between Q and the support of D. We begin this discussion with a remark.

Remark 5.2.5. In what follows, we study how $D(z,\lambda)$ relates to $D(\mu z,\lambda)$ for $\mu \in \mathcal{U}_Q$. In particular, we consider if there exists a $\mu \in \mathcal{U}_Q$ such that $D(\mu z,\lambda)$ is given by $D(z,\lambda)$ up multiplication by some constant. As $D(z,\lambda)$ has a term that is constant as a polynomial in z (that is, a term that is constant or a power of λ), we may always assume that if such a μ exists then $D(\mu z,\lambda) = D(z,\lambda)$.

Much of what we discuss will also apply to $D(z,\lambda_0)$, $D|_F$, and $D|_F(z,\lambda_0)$ (that is, the polynomials defining the Fermi varieties and facial polynomials), however, there may be a $\mu \in \mathbb{T}$ such that $D(\mu z,\lambda_0) = cD(z,\lambda_0)$ for some $c \in \mathbb{T}\smallsetminus\{1\}$. In this case we may multiply by a monomial unit to obtain a new polynomial with a constant term or a power of λ, enabling us to assume that if such a μ exists, then $c = 1$. This will allow us to extend the arguments to come to to these cases. $\diamond$

Before stating these conditions in generality, we begin by building some intuition by studying the case when $d = 1$. Suppose that $\sigma = \{1\}$, $z = z_1$, and that $q = q_1 > 1$. As $D(z,\lambda)$ is irreducible, we have that

$$D_q(z^q,\lambda) = \prod_{\mu\in\mathcal{U}_q} D(\mu z,\lambda).$$

If $D_q(z,\lambda) = gh$, where g and h are not monomials, then there exist $\mu_1, \ldots, \mu_s \in \mathcal{U}_q$, where $1 \leq s < q$, such that

$$g(z^q, \lambda) = \prod_{i=1}^{s} D(\mu_i z, \lambda).$$

As $s < q$, there exists $\mu' \in \mathcal{U}_{q_1}$ such that $\mu'\mu_1 \notin \{\mu_1, \ldots, \mu_s\}$; indeed, such a μ' must exist otherwise $s = q$ and $D_q(z,\lambda)$ is irreducible as then h must be a monomial. As multiplying z by elements of $\mathcal{U}_q$ does not change $g(z^q, \lambda)$, we have

$$g(z^q, \lambda) = g((\mu' z)^q, \lambda) = \prod_{i=1}^{s} D(\mu'\mu_i z, \lambda).$$

As each $D(\mu z, \lambda)$ is irreducible, there is a $j \in \{1, \ldots, s\}$ such that $D(\mu'\mu_1 z, \lambda) = D(\mu_j z, \lambda)$ (see Remark 5.2.5). As $\mu'\mu_1 \neq \mu_j$, we have that $\hat{\mu} = \mu'\mu_1(\mu_j)^{-1}$ is not 1, and thus we have a nontrivial element $\hat{\mu} \in \mathcal{U}_q$ satisfying $D(\hat{\mu} z, \lambda) = D(z, \lambda)$. Since $D(\hat{\mu} z, \lambda) = D(z, \lambda)$, if $\upsilon(z,\lambda)$ is a monomial term of $D(z,\lambda)$, then $\upsilon(\hat{\mu} z, \lambda) = \upsilon(z, \lambda)$. Thus if $D_q(z,\lambda)$ is reducible, then $\text{ord}(\hat{\mu})$, the order of $\hat{\mu}$, must divide the exponent of z in any term $\upsilon(z,\lambda)$ of $D(z,\lambda)$.

Let b' be the greatest common divisor of the finite set of integers $\{r \mid (r,t) \in \mathcal{A}(D(z,\lambda)\}$. As $\hat{\mu}$ fixes all terms of $D(z,\lambda)$, then we must have that $\text{ord}(\hat{\mu})$ divides b'. As $\text{ord}(\hat{\mu})$ divides $q = |\mathcal{U}_q|$, we see that $\gcd(q, b') \neq 1$. It follows that if $D(z,\lambda)$ is irreducible and $\gcd(q, b') = 1$, then we have a contradiction and can conclude that $D_q(z,\lambda)$ is irreducible (as our assumption that $D_q(z,\lambda)$ is reducible implies that $\gcd(q, b') \neq 1$).

Indeed, if $\gcd(q, b') = 1$, then we have that $\gcd(\text{ord}(\hat{\mu}), b') = 1$. By Euclid's algorithm and the definition of b', there must exist $z^{r_1}\lambda^{t_1}, \ldots, z^{r_l}\lambda^{t_l}$ as monomials or their inverses that appear as a term with nonzero coefficient in $D(z,\lambda)$ with $\sum r_i = b'$. Therefore, as $\hat{\mu}^{b'} \neq 1$, we see that

$$\hat{\mu}^{b'} z^{r_1+\cdots+r_l} \lambda^{t_1 \ldots t_l} \neq z^{r_1+\cdots+r_l} \lambda^{t_1 \ldots t_l}, \tag{5.2}$$

and so we cannot have $(\hat{\mu} z)^{r_i}\lambda^{t_i} = z^{r_i}\lambda^{t_i}$ for all $i \in [l]$. This contradicts the assumption that $D_q(z,\lambda)$ is reducible, as if it were then, as discussed, $\hat{\mu}$ would fix all terms of $D(z,\lambda)$.

To state the more general case, we first need to introduce a definition that will allows us to identify the values $\text{ord}(\hat{\mu})$ can take for $D_{\sigma \odot Q}$ to be reducible.

Definition 5.2.6. Let $\sigma \in \binom{[n]}{k}$ for some $k \in [n]$ and let $j \in \sigma$. Let B be the collection of b such that there is a vector in the integer span of $\mathcal{A}(D)$ that is b in the jth coordinate and 0 for every other coordinate $i \in \sigma$. B forms an ideal of $\mathbb{Z}$ and is therefore principal. Define $\text{Div}_{j,\sigma}(D)$ to be the principal generator of B (equivalently, the greatest common divisor of the elements in B). $\diamond$

If $Q = q_1$, then $\text{Div}_{1,\{1\}}(D) = b'$ (where b' is from the discussion of the one-dimensional case). In general, $D_{\sigma \odot Q}$ can factor only if $\text{ord}(\hat{\mu})$ divides $\text{Div}_{1,\sigma}(D)$ (as otherwise the same situation as (5.2) arises). We now present some examples.

Example 5.2.7. Consider the polynomial $f(z_1, z_2, \lambda) = z_1^2 z_2^2 + \lambda z_1^4 + \lambda^3$. Suppose there is a $\mu_1 \in \mathbb{T}$ such that $f(\mu_1 z_1, z_2, \lambda) = cf(z_1, z_2, \lambda)$ for some $c \in \mathbb{C}$. As every term must be fixed under $z_1 \to \mu_1 z_1$, $c = 1$ because λ^3 is invariant with respect to this change of variables. By definition, $\text{Div}_{1,\{1\}}(f(z_1, z_2, \lambda)) = 2$. Thus $\mu_1^2 = 1$, that is $\mu_1 = \pm 1$. This agrees with the fact that $\mu_1^2 z_1^2 z_2^2 = z_1^2 z_2^2$.

In this same case, $\mathrm{Div}_{1,\{1,2\}}(f(z_1, z_2, \lambda)) = 4$. Given μ_1 and μ_2 in $\mathbb{T}$, where $f(\mu_1 z_1, \mu_2 z_2, \lambda) = cf(z_1, z_2, \lambda)$, then $c = 1$. As λz_1^4 is independent of μ_2, the order of μ_1 must divide 4.

Finally consider $h(z_1, z_2, \lambda) = z_1^3 z_2^2 + z_1^2 z_2^{-1} + \lambda$. Assume μ_1 and μ_2 are in $\mathbb{T}$ such that $h(\mu_1 z_1, \mu_2 z_2, \lambda) = ch(z_1, z_2, \lambda)$. Again, $c = 1$. We have $(z_1^3 z_2^2)(z_1^2 z_2^{-1})^2 = z_1$. Therefore, we find that $\mathrm{Div}_{1,\{1,2\}}(h(z_1, z_2, \lambda)) = 1$. As $\mu_1^3 z_1^3 \mu_2^2 z_2^2 = z_1^3 z_2^2$ and $\mu_1^2 z_1^2 \mu_2^{-1} z_2^{-1} = z_1^2 z_2^{-1}$, it follows $z_1 = (\mu_1^3 z_1^3 \mu_2^2 z_2^2)(\mu_1^2 z_1^2 \mu_2^{-1} z_2^{-1})^2 = \mu_1 z_1$; we conclude that $\mu_1 = 1$. $\diamond$

Remark 5.2.8. If $\sigma' \subseteq \sigma$ then $\mathrm{Div}_{j,\sigma'}(D)$ divides $\mathrm{Div}_{j,\sigma}(D)$. $\diamond$

We now state the general case. Recall Remark 5.2.5; that is, we assume that if there exists a $\mu \in \mathcal{U}_Q$ such that $D(\mu z, \lambda) = cD(z, \lambda)$, then $c = 1$.

Lemma 5.2.9. Let V_Q be a $\mathbb{Z}^d$-periodic potential. Suppose that there exists $\sigma' \in \binom{[d]}{k-1}$, where $1 \leq k \leq d$, such that $D_{\sigma' \odot Q}$ is irreducible. Let $\sigma = i \cup \sigma'$ for some $i \notin \sigma'$. If q_i is coprime to $b = \mathrm{Div}_{i,\sigma}(D)$, then $D_{\sigma \odot Q}$ is irreducible.

Proof. After reordering we may assume that $i = 1$ and $\sigma = \{1, 2, \ldots, k\}$. By way of contradiction, suppose that $D_{\sigma \odot Q}$ is reducible with factor g that is not a monomial, but $\gcd(q_1, b) = 1$. Let $\sigma' = \sigma \smallsetminus \{1\}$. Then

$$D_{\sigma \odot Q}(z_1^{q_1}, z_2, \ldots, z_d, \lambda) = \prod_{\mu \in \mathcal{U}_{q_1}} D_{\sigma' \odot Q}(\mu z_1, z_2, \ldots, z_d, \lambda).$$

As each $D_{\sigma' \odot Q}(\mu z_1, z_2, \ldots, z_d, \lambda)$ is irreducible, g must have the following factorization,

$$g(z_1^{q_1}, z_2, \ldots, z_d, \lambda) = \prod_{i=1}^{s} D_{\sigma' \odot Q}(\mu_i z_1, z_2, \ldots, z_d, \lambda),$$

where $\mu_i \in U_{q_1}$ and $1 \leq s < q_1$; that is, a nonempty proper subset of the irreducible factors of $D_{\sigma \odot Q}(z_1^{q_1}, z_2, \ldots, z_d, \lambda)$ must appear as the irreducible factors of $g(z_1^{q_1}, z_2, \ldots, z_d, \lambda)$. As $s < q_1$, there exists $\mu' \in \mathcal{U}_{q_1}$ such that $\mu' \mu_1 = \hat{\mu} \notin \{\mu_1, \mu_2, \ldots, \mu_s\}$. Notice we have the following two factorizations,

$$g(z_1^{q_1}, \ldots, z_k^{q_k}, z_{k+1}, \ldots, z_d, \lambda) = \prod_{i=1}^{s} \prod_{\gamma \in \mathcal{U}_{(1,q_2,\ldots,q_k)}} D(\mu_i z_1, \gamma_2 z_2, \ldots, \gamma_k z_k, z_{k+1}, \ldots, z_d, \lambda),$$

$$g((\mu' z_1)^{q_1}, \ldots, z_k^{q_k}, z_{k+1}, \ldots, z_d, \lambda) = \prod_{i=1}^{s} \prod_{\gamma \in \mathcal{U}_{(1,q_2,\ldots,q_k)}} D(\mu' \mu_i z_1, \gamma_2 z_2, \ldots, \gamma_k z_k, z_{k+1}, \ldots, z_d, \lambda).$$

As each $D(\mu z, \lambda)$ is irreducible and

$$g(z_1^{q_1}, \ldots, z_k^{q_k}, z_{k+1}, \ldots, z_d, \lambda) = g((\mu' z_1)^{q_1}, \ldots, z_k^{q_k}, z_{k+1}, \ldots, z_d, \lambda),$$

these two factorizations are the same. For $\hat{\mu}$ and a given $\gamma \in \mathcal{U}_{(1,q_2,\ldots,q_k)}$, there exists $\mu_l \in \{\mu_1, \mu_2, \ldots \mu_s\}$ and $\gamma' \in \mathcal{U}_{(1,q_2,\ldots,q_k)}$ with

$$D(\hat{\mu} z_1, \gamma_2 z_2, \ldots, \gamma_k z_k, z_{k+1}, \ldots, z_d, \lambda) = D(\mu_l z_1, \gamma_2' z_2, \ldots, \gamma_k' z_k, z_{k+1}, \ldots, z_d, \lambda).$$

Let $\tilde{\mu} = \mu_l(\hat{\mu})^{-1}$. Notice that, as $\hat{\mu} \notin \{\mu_1, \ldots, \mu_s\}$, $\tilde{\mu} \neq 1$. In particular, $\mathrm{ord}(\tilde{\mu})$ is an integer greater than 1 that divides q_1. If $z_2, \ldots, z_k$ do not appear in a monomial $v(z_1, z_{k+1}, \ldots z_d, \lambda)$ with support in the integral span of $\mathcal{A}(D)$ then $v(\tilde{\mu} z_1, z_{k+1}, \ldots z_d, \lambda) = v(z_1, z_{k+1}, \ldots z_d, \lambda)$. Since $\gcd(q_1, b) = 1$, by the definition of $\mathrm{Div}_{1,\{1,2,\ldots,k\}}(D)$ $(= b)$, there exists a term $v(z_1, z_{k+1}, \ldots z_d, \lambda)$ of D that is not fixed by $\tilde{\mu}$, a contradiction. □

Remark 5.2.10. From the proof of Lemma 5.2.9 we can recover a version of [15, Lemma 3.4]. In particular, suppose that for all $\mu \in \mathcal{U}_Q$ one has $D(\mu z, \lambda) \neq D(z, \lambda)$ (using the assumption that D has a term that is constant as a polynomial in z). This condition essentially encapsulate condition (A2) of [15], which is an assumption of [15, Lemma 3.4]. Notice that if $g | D_Q$ and g is not a monomial, then we must have that $D(\mu z, \lambda) | g(z^Q, \lambda)$ for every $\mu \in \mathcal{U}_Q$; but then $g(z^Q, \lambda) = \prod_{\mu \in \mathcal{U}_Q} D(\mu z, \lambda) = D_Q(z^Q, \lambda)$. Thus Lemma 5.2.9 essentially gives conditions for when (A2) holds when expanding the fundamental domain along a single coordinate axis (allowing us to apply the argument of [15, Lemma 3.4]).

More generally, we have the following. Suppose that there exists $\sigma' \subsetneq \sigma$ such that $D_{\sigma' \odot Q}$ is irreducible and contains a constant term as a polynomial in z. Without loss of generality, let $\sigma = \{1, \ldots, l\} \cup \sigma'$ where $\{1, \ldots, l\} \subseteq \overline{\sigma'}$, and let $\mathcal{U}_{Q'} = \mathcal{U}_{\{1,\ldots,l\} \odot Q}$. If

$$D_{\sigma' \odot Q}(z_1, \ldots, z_d, \lambda) \neq D_{\sigma' \odot Q}(\mu_1 z_1, \ldots, \mu_l z_l, z_{l+1}, \ldots, z_d, \lambda) \text{ for any } \mu \in \mathcal{U}_{Q'},$$

then $D_{\sigma \odot Q}$ is irreducible.

We avoid further discussions of this general criteria, as our goal is to present practically verifiable conditions on D that enable us to conclude irreducibility for D_Q. ◇

We will use the following corollary often in the examples of Section 5.3.

Corollary 5.2.11. Let V_Q be the $\mathbb{Z}^d$-periodic potential V, and suppose that D is irreducible. If there exist terms $z_1^{a_1}, \ldots, z_d^{a_d}$ with nonzero coefficients in D, then D_Q is irreducible for all Q such that $\gcd(q_i, a_i) = 1$ for all i.

Proof. Notice that no matter our choice of i and $\sigma \subseteq [d]$, we have that $\mathrm{Div}_{i,\sigma}(D) | a_i$ and thus $\gcd(q_i, \mathrm{Div}_{i,\sigma}(D)) = 1$. Starting with the fact that $D = D_{\{\} \odot Q}$ is irreducible and then consecutively applying Lemma 5.2.9; we see that at each step, as $\gcd(q_i, \mathrm{Div}_{i,\sigma}(D)) = 1$, we have that $D_{\sigma \odot Q}$ is irreducible for each $\sigma \subseteq [d]$. □

We say that a facial polynomial $(D_Q)|_F$ is *$\mathbb{Z}^d$-periodic* if there exists a $\mathbb{Z}^d$-periodic potential V' corresponding to the dispersion polynomial D'_Q such that $(D_Q)|_F = (D'_Q)|_F$. Suppose that for a $Q\mathbb{Z}$-periodic potential V_Q the facial polynomial $(D_Q)|_F$ is $\mathbb{Z}^d$-periodic due to the existence of a $\mathbb{Z}^d$-periodic V'. By Remark 5.2.2, if $D|_F$ is only homothetically reducible for V', then $(D_Q)|_F$ is only homothetically reducible. By Theorem 2.1.7, we obtain a corollary.

Corollary 5.2.12. Suppose that for every facet F of $\mathcal{N}(D_Q)$, except possibly one, $(D_Q)|_F$ is $\mathbb{Z}^d$-periodic via the existence of a $\mathbb{Z}^d$-periodic potential V_F. If, for each F, $D|_F$ is only homothetically reducible for V_F, then D_Q is only homothetically reducible.

5.3 Applications

We conclude with examples of discrete periodic operators associated to various families of periodic graphs which have irreducible Bloch varieties, irreducible Fermi varieties, or Bloch varieties that are the union of an irreducible hypersurface and flat bands. We assume that all edge labels are nonzero. We begin by introducing useful definitions and notation to be used in the examples to come.

For a Laurent polynomial f, a facial polynomial $f|_F$ for a face F of $\mathcal{N}(f)$ is *potential-independent* if the potential, treated as a finite vector of indeterminates, does not appear in the coefficients of $f|_F$. If a facial polynomial $(D_Q)|_F$ is potential-independent, then $(D_Q)|_F$ is $\mathbb{Z}^d$-periodic via the zero potential. A face F of $\mathcal{N}(D)$ (and its facial polynomial $D|_F$) is *apical* if F contains the apex of $\mathcal{N}(D)$, $(0, \ldots, 0, m)$.

Let S_m be the symmetric group on m elements, $L(z, \lambda) := L(z) - \lambda I$, and let $L_{i,j}(z, \lambda)$ be the (i, j) entry of $L(z, \lambda)$. For $w \in \mathbb{Z}^{d+1}$, we call $w \cdot (a, l)$ the *weight* of the term $z^a \lambda^l$ with respect to w. The monomial terms in $\tau L(z, \lambda) := \prod_{i=1}^{m}(L_{i,\tau(i)}(z, \lambda))$ are said to be terms *produced* by τ. Notice that in this way, $D(z, \lambda) = \sum_{\tau \in S_m} \operatorname{sgn}(\tau)\tau L(z, \lambda)$. We say a permutation $\tau \in S_m$ *contributes* to terms of $D|_w$ if $\mathcal{A}(\tau L(z, \lambda)) \cap \mathcal{A}(D|_w) \neq \emptyset$. We say τ is *nonzero* if $\tau L(z, \lambda) \neq 0$.

5.3.1 Bloch Varieties

We say a $\mathbb{Z}^d$-periodic graph is a *1-vertex graph* if it has a single vertex orbit with respect to its $\mathbb{Z}^d$-action. For $d \geq 1$, let Γ be a 1-vertex $\mathbb{Z}^d$-periodic graph. Let L be a discrete periodic operator associated to Γ. By [21], D is irreducible as $\mathcal{N}(D)$ is a pyramid of height 1. Any apical facet of $\mathcal{N}(D)$ is also a pyramid of height 1 and thus has an irreducible facial polynomial.

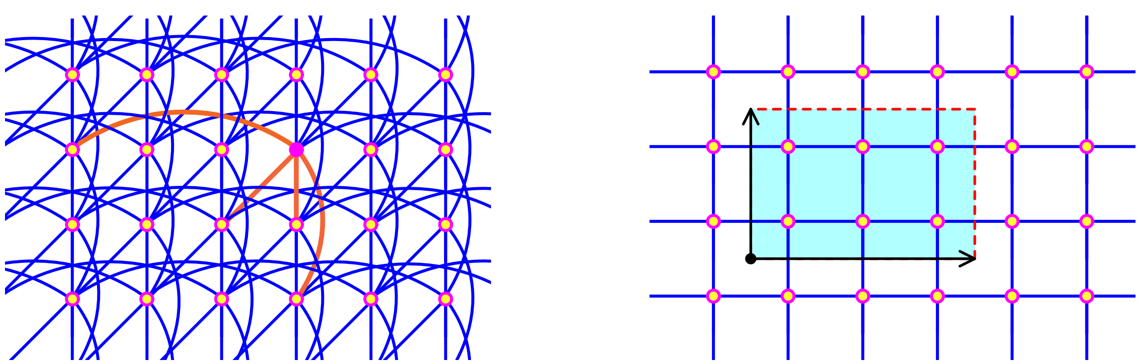


Figure 5.1: Two 2-dimensional 1-vertex graphs. On the left, the orange edges are representatives of the edge orbits. On the right, the square lattice with a highlighted $(3, 2)$-expansion of the fundamental domain is depicted.

Suppose $Q \in \mathbb{N}^d$ and let V be a $Q\mathbb{Z}$-periodic potential. For any nonbase face F, $\hat{L}_Q(z, \lambda)$ has one contributing permutation (given by the identity permutation) through its diagonal, each entry contributing terms of the same negative weight. We conclude that for every nonbase facial polynomial $\hat{D}_Q|_F$, and therefore $D_Q|_F$, is potential-independent. As the faces are potential-independent (see the discussion before Lemma 5.2.1), $\mathcal{N}(D_Q)$ is a contracted Q-dilation of $\mathcal{N}(D)$, and therefore a pyramid; that is, D_Q is only homothetically reducible. To conclude that D_Q is irreducible, we must show that one facial polynomial is irreducible.

Example 5.3.1. Consider a $\mathbb{Z}^d$-periodic 1-vertex graph Γ, where $d \geq 1$, such that D has a facial polynomial $D|_F$ with the extreme monomials $z_1^{a_1}, z_2^{a_2}, \ldots, z_d^{a_d}$. Two 2-dimensional examples of 1-vertex graphs with this property are shown in Figure 5.1. If q_i is coprime to a_i for each i, then by Corollary 5.2.11 $(D_Q)|_F$ is irreducible. Thus, by Corollary 2.1.8, it follows that $D_Q(z, \lambda)$ is irreducible for all potentials.

For example, consider the left-hand graph of Figure 5.1. We have that $\mathcal{N}(D)$ has a face with the extreme monomials z_1^3 and z_2^2; thus if q_1 is coprime to 3 and q_2 is coprime to 2 then $D_Q(z, \lambda)$ is irreducible. ◇

Example 5.3.2. Let $d \geq 1$. Take any 1-vertex $\mathbb{Z}^d$-periodic graph with at least one edge. Pick an apical facet F of $\mathcal{N}(D)$. Notice that there must be some monomial z^a occurring as a term of $D|_F$ with a non-zero coefficient, for some a $(\neq \mathbf{0}) \in \mathbb{Z}^d$. Due to this, the collection of $Q = (q_1, \ldots, q_d) \in \mathbb{N}^d$ such that $D_{\{i\} \odot Q}$ is irreducible for all $i \in [d]$ is infinite. In particular, this set contains the $Q \in \mathbb{N}^d$ such that $\gcd(a_i, q_i) = 1$; as $\mathrm{Div}_{i,\{i\}}(D|_F)$ must divide a_i, we have that $D_{\{i\} \odot Q}$ is irreducible by Lemma 5.2.9. Moreover, we consider the infinite set of $Q \in \mathbb{N}^d$ such that $\gcd(a_i, q_i) = 1$ and the coordinates of Q are pairwise coprime. Given a Q in this infinite subset, we see that $D_Q|_F$ is irreducible by Theorem 5.2.4. Thus D_Q is irreducible for all potentials. ◇

Let us briefly compare our methods and results with those of [15].

Remark 5.3.3. The results of these last two examples overlap with the results and methods of [15]. In particular, if F is a facet that is not the base, then D_Q is irreducible if $D|_F(\mu z, \lambda) \neq D|_F(z, \lambda)$ for all $\mu \in \mathcal{U}_Q$ (this is what they refer to as condition (A2), see Remark 5.2.10). In [15] 1-vertex graphs were considered, and thus checking whether (A2) is satisfied is sufficient to conclude irreducibility of D_Q; as this condition implies irreducibility of the facial form $D_Q|_F$ and only homothetic reducibility is immediately from the fact that the Newton polytope are pyramids (this is essentially [15, Lemma 3.6]).

A distinct difference between these methods is that we only require that some facial form be irreducible, where as in [15] they always fix the face given by $w = (1, \ldots, 1, -1)$ which limits its applications (this is mainly an artifact of choice rather than a bottleneck in their arguments). For example, [15] concluded that the dispersion polynomial obtained from the Schrödinger operator over the Harper lattice is irreducible for all $(q_1, q_2)\mathbb{Z}$-periodic potentials when q_1 and q_2 are coprime, but choosing another facial form (for example, corresponding to $w = (-1, 0, -1)$) reveals that q_1 and q_2 do not need to be coprime.

Example 5.3.4. The dth member of the *hexagonal-diamond family* is a $\mathbb{Z}^d$-periodic graph Γ with fundamental domain W given by two vertices u and v and edges (u, v), $(u, v - e_i)$, and $(e_i + u, v)$, where for each $i = 1, \ldots, d$ the vector e_i has 1 in its ith component and is 0 elsewhere. Due to periodicity, $E((u, v - e_i)) = E((e_i + u, v))$. The $d = 2$ and $d = 3$ members are shown in

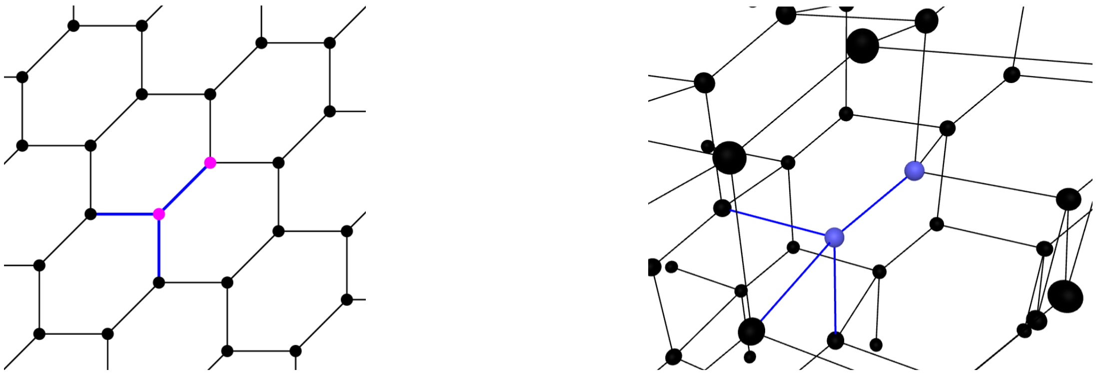


Figure 5.2: Local realizations of the hexagonal (left) and the diamond lattice (right). The purple vertices and blue edges are representatives of vertex and edge orbits of the graphs, respectively.

Figure 5.2. Let γ_i and α be $\mathbb{Z}^d$-periodic edge labels of Γ. For a $\mathbb{Z}^d$-periodic potential, a discrete periodic operator associated to Γ has the Floquet matrix

$$L(z,\lambda) = \begin{pmatrix} \alpha + \sum_{i=1}^d \gamma_i + V(u) - \lambda & -\alpha - \gamma_1 z_1^{-1} - \cdots - \gamma_d z_d^{-1} \\ -\alpha - \gamma_1 z_1 - \cdots - \gamma_d z_d & \alpha + \sum_{i=1}^d \gamma_i + V(v) - \lambda \end{pmatrix}.$$

The apical facial polynomials of D are potential-independent and irreducible. Let $Q \in \mathbb{N}^d$. We show that the apical facial polynomials of $\hat{D}_Q$, and thus D_Q, are potential-independent. Consider $\hat{L}_Q(z,\lambda)$ for a $Q\mathbb{Z}$-periodic potential V:

$$\hat{L}_Q(z,\lambda) = \begin{pmatrix} L(\mu_1 z,\lambda) & \begin{matrix} \hat{V}_{12,1} & 0 \\ 0 & \hat{V}_{12,2} \end{matrix} & \cdots & \begin{matrix} \hat{V}_{1|Q|,1} & 0 \\ 0 & \hat{V}_{1|Q|,2} \end{matrix} \\ \begin{matrix} \hat{V}_{21,1} & 0 \\ 0 & \hat{V}_{21,2} \end{matrix} & L(\mu_2 z,\lambda) & \cdots & \vdots \\ \vdots & & \ddots & \\ \begin{matrix} \hat{V}_{|Q|1,1} & 0 \\ 0 & \hat{V}_{|Q|1,2} \end{matrix} & \cdots & & L(\mu_{|Q|} z,\lambda) \end{pmatrix},$$

where $\hat{V}_{ij,k} = (\hat{V}_{\mu_i,\mu_j})_{k,k}$ and $L(z,\lambda)$ has $\tilde{V}(u) = \frac{1}{|Q|}\sum_{\omega|\omega+u\in W_Q} V(\omega+u)$ as its $\mathbb{Z}^d$-periodic potential. Viewing a nonzero permutation τ as a collection of paths (or directed cycles) through the matrix, we say that τ *leaves* the main block-diagonal if there exists an $i \in [2Q]$ such that $(i,\tau(i))$ is an entry not belonging to the block-diagonal, i.e. for some i we have that

$$(i,\tau(i)) \notin \{(2n-1,2n-1),(2n-1,2n),(2n,2n-1),(2n,2n) \mid 1 \le n \le |Q|\}.$$

This perspective will be expanded upon in Example 5.3.9.

Without loss of generality, let F be the apical facet exposed by $(-2,\ldots,-2,-1)$. Every diagonal $L(\mu_k z,\lambda)$ can contribute at most λ^2 or z_i to a term of $\tau\hat{L}_Q(z,\lambda)$, and so for τ to contribute to $\hat{D}_Q|_F$, every $L(\mu_k z,\lambda)$ must contribute either λ^2 or z_i, but not both. Notice that as τ leaves the diagonal there exists an $L(\mu_k z,\lambda)$ that cannot contribute either λ^2 or z_i to $\tau\hat{L}_Q(z,\lambda)$. This is because τ leaves the main block-diagonal, and thus for some $k \in [|Q|]$ the permutation of τ cannot involve entries in the bottom row or left-hand column of $L(\mu_k z, \lambda)$; but then we must have that τ does not contribute to $\hat{D}_Q|_F$. It follows that $\hat{D}_Q|_F = \prod_{i=1}^{|Q|} \det(L(\mu_i z, \lambda))|_F$.

Thus D_Q has only potential-independent, and thus only homothetically reducible, apical facial polynomials. As $D|_F$ is irreducible and has extreme monomials $z_1, z_2, \ldots, z_d$, and λ; it follows from Corollary 5.2.11 that $D_Q|_F$ is irreducible. Thus, by Corollary 2.1.8, D_Q is irreducible. $\diamond$

Recall from Section 4.4 that a $\mathbb{Z}^d$-periodic graph Γ is dense if there is a fundamental domain W such that whenever $a \in \mathcal{A}(W) \neq \emptyset$, the union of W and $a + W$ induces a complete graph.

Example 5.3.5. Consider a $\mathbb{Z}^d$-periodic dense graph Γ. As Γ is dense, by Lemma 4.4.3, $\mathcal{N}(D)$ and its apical facets are pyramids for a generic labeling. For $Q \in \mathbb{N}^d$, it is straightforward to deduce that any nonbase facial polynomial D_Q is potential-independent and thus only homothetically reducible (such as in the first two examples; in fact, any connected 1-vertex graph is dense). By Theorem 2.1.7, D_Q is only homothetically reducible. To show D_Q is irreducible for all potentials, it suffices to show that $(D_Q)|_F$ is irreducible for some face F.

In dimensions 2 and 3, we have already proven that each $D|_F$ is irreducible when F is not a vertex. By Theorem 4.4.2, for a generically labeled dense $\mathbb{Z}^2$- or $\mathbb{Z}^3$-periodic graph, the zero-set of $D|_F$ is smooth (and $D|_F$ is square-free) for any nonbase face F. In particular, this implies that for every facet F, $D|_F$ is irreducible (see [22]). It follows that $D|_F$ is irreducible for any nonbase face F. As the nonbase facial polynomials are potential-independent, it follows that D_Q is irreducible for all potentials for infinitely many choices of Q (as in Example 5.3.2).

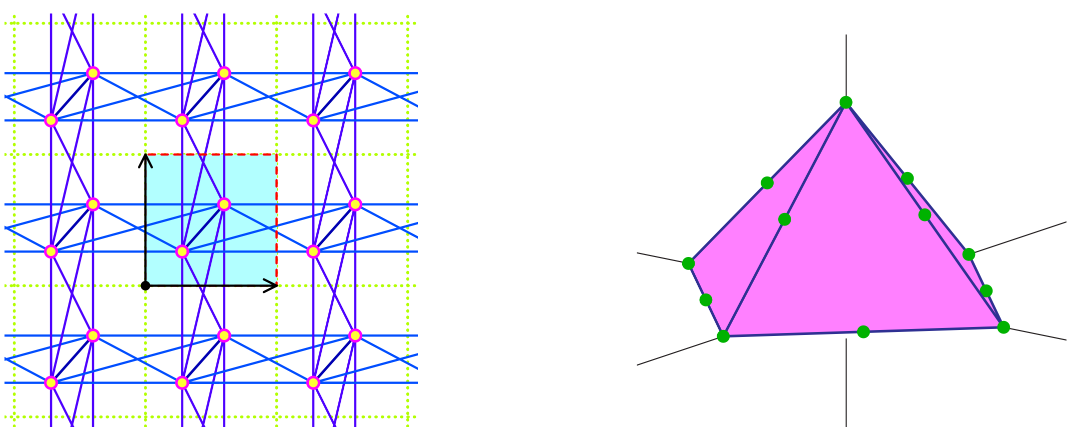


Figure 5.3: A $\mathbb{Z}^2$-periodic dense graph and corresponding Newton polytope.

Consider the $\mathbb{Z}^2$-periodic dense graph from [10] shown in Figure 5.3. The Floquet matrix of the discrete periodic operator is:

$$L_{1,1}(z,\lambda) = \alpha + \beta_1(2 - z_1 - z_1^{-1}) + \beta_2 + \beta_3 + \gamma_1(2 - z_2 - z_2^{-1}) + \gamma_2 + \gamma_3 + V_1 - \lambda$$
$$L_{1,2}(z,\lambda) = -\alpha - \beta_2 z_1 - \beta_3 z_1^{-1} - \gamma_2 z_2 - \gamma_3 z_2^{-1}$$
$$L_{2,1}(z,\lambda) = -\alpha - \beta_2 z_1^{-1} - \beta_3 z_1 - \gamma_2 z_2^{-1} - \gamma_3 z_2$$
$$L_{2,2}(z,\lambda) = \alpha + \beta_4(2 - z_1 - z_1^{-1}) + \beta_2 + \beta_3 + \gamma_4(2 - z_2 - z_2^{-1}) + \gamma_2 + \gamma_3 + V_2 - \lambda.$$

Where $\alpha, \beta_i, \gamma_j$ are edge labels. Let F be the facet of $\mathcal{N}(D)$ exposed by $(-1,-1,-1)$, then $D|_F$ has exactly the monomials terms $\lambda^2, z_1^2, z_2^2, \lambda z_1, \lambda z_2$, and $z_1 z_2$. By Theorem 4.4.2, $D|_F$ is irreducible. As λz_1, λz_2, and λ^2 are terms of $D|_F$, $\mathrm{Div}_{1,\sigma}(D|_F)$ and $\mathrm{Div}_{2,\sigma}(D|_F)$ both equal 1 for $\sigma = \{1,2\}$. Thus by Corollary 5.2.11, $D_Q|_F$ is irreducible for any choice of q_1 and q_2. By Corollary 2.1.8, D_Q is irreducible for all potentials. $\diamond$

5.3.2 Fermi Varieties

The d-dimensional *cross-polytope*, CP_d, is the polytope with vertices given by the $2d$ d-dimensional vectors that are either 1 or -1 in one coordinate, and 0 elsewhere. Letting $A = (a_1, \dots, a_d) \in \mathbb{N}^d$, we define $\psi_A : \mathbb{R}^d \to \mathbb{R}^d$ to be the linear map given by $(v_1, v_2, \dots, v_d) \mapsto (a_1 v_1, a_2 v_2, \dots, a_d v_d)$. A d-dimensional *A-dilated cross-polytope* is the image $\psi_A(CP_d)$. The image of a polytope P under the map ψ_A is called the *A-dilation* of P.

By Remarks 5.2.2 and 5.2.5, we may use our methods to discuss the irreducibility of Fermi varieties for $\mathbb{Z}^d$-periodic graphs when $d > 2$. We note that when $d = 2$, the Newton polytope of the polynomial defining a Fermi variety is 2-dimensional, and thus we cannot apply the theory of only homothetic reducible polynomials that was developed in Section 2.1.3 (as the faces in a strong chain must share at least an edge). As with Bloch varieties, we begin with 1-vertex graphs.

Example 5.3.6. Let d be some integer greater than 2. Fix $\lambda_0 \in \mathbb{C}$. Consider a 1-vertex $\mathbb{Z}^d$-periodic graph with fundamental domain W, such that the convex hull of $\mathcal{A}(W)$ is an A-dilated cross-polytope for some $A = (a_1, \dots, a_d) \in \mathbb{N}^d$ such that $\gcd(a_1, \dots, a_d) = 1$. For example, the 3-dimensional square lattice is a 1-vertex $\mathbb{Z}^3$-periodic graph, and the convex hull of its support, $\mathcal{A}(W)$, is the cross-polytope. Another example (although $d < 3$) is the left-hand graph of Figure 5.1, the convex hull of its support, $\mathcal{A}(W)$, is the $(3,2)$-dialated cross-polytope.

As 1-vertex graphs are dense periodic graphs, by Lemma 4.4.3, we have that $\mathcal{N}(D(z,\lambda_0)) = \mathrm{conv}(\mathcal{A}(W))$. It follows that any facet of $\mathcal{N}(D(z,\lambda_0))$ is a pyramid with apex $(a_1, 0, \dots, 0)$ or $(-a_1, 0 \dots, 0)$ and thus is only homothetically decomposable. As $d > 2$, we have that any two vertices can be connected by a strong chain of only homothetically decomposable facets, and thus, by [46], $\mathcal{N}(D(z,\lambda_0))$ is only homothetically decomposable. By [21], as $\gcd(a_1, \dots, a_d) = 1$, if F is a facet then $D|_F(z,\lambda_0)$ is irreducible. Let $Q \in \mathbb{N}^d$ be such that $\gcd(q_i, a_i) = 1$ for each i and $\gcd(q_1, \dots, q_d) = 1$. By the arguments given in Section 5.3.1 on 1-vertex graphs, the facial polynomials of $D_Q(z,\lambda_0)$ are independent of the potential and of λ_0.

Let F be a facet of $\mathcal{N}(D_Q(z,\lambda_0))$. As $\gcd(q_i, a_i) = 1$ for each i, $D_{\sigma \odot Q}|_F(z,\lambda_0)$ is irreducible for each $\sigma \in \binom{[d]}{d-1}$ by Lemma 5.2.9 (here, a power of z_i, for $i \in \bar{\sigma}$, acts as the constant mentioned in Remark 5.2.5). Since $\gcd(q_1, \dots, q_d) = 1$, $D_Q|_F(z,\lambda_0)$ is irreducible by Theorem 5.2.4. By Corollary 2.1.8, $D_Q(z,\lambda_0)$ is irreducible for all potentials. $\diamond$

Example 5.3.7. Let d be some integer greater than 2 and let $\lambda_0 \in \mathbb{C}$. Consider a $\mathbb{Z}^d$-periodic dense graph Γ with generic edge labels. As in Example 5.3.6, each facial polynomial of $D_Q(z,\lambda_0)$ is independent of the potential and of λ_0. For a given edge label, if one can show all facial polynomials of $D(z,\lambda_0)$ are only homothetically reducible, then all facial polynomials of $D_Q(z,\lambda_0)$ are only homothetically reducible for any $Q \in \mathbb{N}^d$. If there exists an irreducible facial polynomial $D|_F(z,\lambda_0)$, one can use Section 5.2 to find $Q \in \mathbb{N}^d$ where $D_Q|_F(z,\lambda_0)$ is irreducible. Finally, we can apply Theorem 2.1.7 to conclude irreducibility of $D_Q(z,\lambda_0)$. As in Example 5.3.5, by [14, Theorem 4.2] when $d = 3$, $D|_F(z,\lambda_0)$ is irreducible. It follows that there are infinitely many Q such that $(D_Q)|_F(z,\lambda_0)$ is irreducible.

Consider the 3-dimensional dense graph shown in Figure 5.4. This is the 3-dimensional analog of the dense graph from Example 5.3.5. The associated discrete periodic operator L has a Floquet matrix with entries:

$$\begin{aligned}
L_{1,1}(z,\lambda) &= \alpha + \beta_1(2 - z_1 - z_1^{-1}) + \beta_2 + \beta_3 + \gamma_1(2 - z_2 - z_2^{-1}) \\
&\quad + \gamma_2 + \gamma_3 + \epsilon_1(2 - z_3 - z_3^{-1}) + \epsilon_2 + \epsilon_3 + V_1 - \lambda \\
L_{1,2}(z,\lambda) &= -\alpha - \beta_2 z_1 - \beta_3 z_1^{-1} - \gamma_2 z_2 - \gamma_3 z_2^{-1} - \epsilon_2 z_3 - \epsilon_3 z_3^{-1} \\
L_{2,1}(z,\lambda) &= -\alpha - \beta_2 z_1^{-1} - \beta_3 z_1 - \gamma_2 z_2^{-1} - \gamma_3 z_2 - \epsilon_2 z_3^{-1} - \epsilon_3 z_3 \\
L_{2,2}(z,\lambda) &= \alpha + \beta_4(2 - z_1 - z_1^{-1}) + \beta_2 + \beta_3 + \gamma_4(2 - z_2 - z_2^{-1}) \\
&\quad + \gamma_2 + \gamma_3 + \epsilon_4(2 - z_3 - z_3^{-1}) + \epsilon_2 + \epsilon_3 + V_2 - \lambda.
\end{aligned}$$

Here, $\alpha, \beta_i, \gamma_j$, and ϵ_k are edge labels.

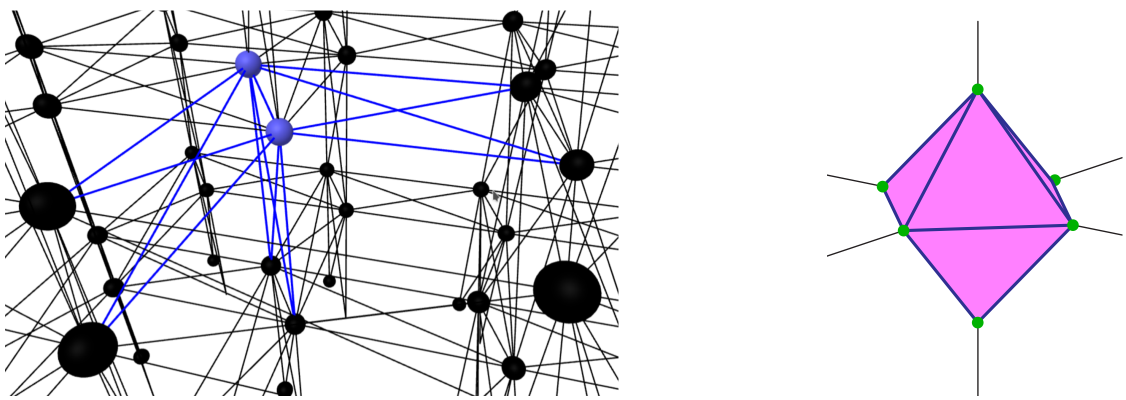


Figure 5.4: A 3-dimensional dense graph (left). The polytope $\mathcal{N}(D(z,\lambda_0))$ is a 3-dimensional dilated cross-polytope (right).

As shown in Figure 5.4, $\mathcal{N}(D(z,\lambda_0))$ is a $(2,2,2)$-dilated cross-polytope, hence its facial polynomials are only homothetically reducible. Consider the face F of $\mathcal{N}(D(z,\lambda_0))$ exposed by $w = (-1,-1,-1)$. By Theorem 4.4.2, $D|_F(z,\lambda_0)$ is irreducible. By Lemma 5.2.9, $(D_{\sigma\odot Q})|_F(z,\lambda_0)$ is irreducible for each $\sigma \in \binom{[3]}{2}$ for all $Q \in \mathbb{N}^d$ because $1 = \mathrm{Div}_{i,\sigma}(D|_F(z,\lambda_0))$ for each $i \in \sigma \in \binom{[3]}{2}$. Therefore, if Q is chosen so that $\gcd(q_1,q_2,q_3) = 1$, then $(D_Q)|_F(z,\lambda_0)$ is irreducible by Theorem 5.2.4. By Corollary 2.1.8, $D_Q(z,\lambda_0)$ is irreducible for all potentials. $\diamond$

Notice that in the case of both of these examples, the choice of λ_0 did not affect the reducibility of $D_Q(z,\lambda_0)$. This is not always the case, as demonstrated in [16] through the Lieb lattice (a $\mathbb{Z}^2$-periodic square lattice with an additional vertex between each edge).

5.3.3 Flat Bands

For a $\mathbb{Z}^d$-periodic graph Γ, a *flat band* in the Bloch variety $B(\Gamma)$ is the zero-set of a linear factor $\lambda - r$ of $D(z, \lambda)$, for some $r \in \mathbb{C}$. Flat bands are studied for their interesting spectral properties [29]; in particular, a flat band corresponds to an eigenvalue in the spectrum $\sigma(L)$.

Let $Q \in \mathbb{N}^d$. As λ is unaffected by the covering map from Section 3.3.3; when the potential is $\mathbb{Z}^d$-periodic, if $(\lambda - r)$ divides D, then $(\lambda - r)^{|Q|}$ divides D_Q.

Example 5.3.8 (Trivial Flat Bands)**.** Let Ω be a 1-vertex $\mathbb{Z}^d$-periodic graph with no edges, and let Γ be a $\mathbb{Z}^d$-periodic graph. Fix fundamental domains W and W' of Ω and Γ, respectively. Consider the $\mathbb{Z}^d$-periodic graph $H := \Omega \coprod \Gamma$; that is, the disjoint union of Ω and Γ, with fundamental domain $W \coprod W'$. Then

$$L_H(z) := \begin{pmatrix} L_\Gamma(z) & 0 \\ 0 & L_\Omega(z) \end{pmatrix},$$

where $L_\Omega(z)$ and $L_\Gamma(z)$ are the Floquet matrices of the discrete periodic operators associated with Ω and Γ, respectively. Denote by D_Ω, D_Γ, and D_H the respective dispersion polynomials. Then $D_H = D_\Gamma D_\Omega$. It follows that D_H has the linear factor $D_\Omega(z, \lambda) = V_\Omega - \lambda$, as $L_\Omega(z)$ is just a potential operator, and the remaining factors of D_H depend on D_Γ. Moreover, $(D_H)_Q = (D_\Gamma)_Q(D_\Omega)_Q$, and $(D_\Omega)_Q$ has $|Q|$ flat bands counted with multiplicity. The remaining factors of $(D_H)_Q$ depend only on $(D_\Gamma)_Q$. $\diamond$

Example 5.3.9. The dice lattice is obtained by identifying vertices of two separate hexagonal lattices, as shown in Figure 5.5 (the edges of one hexagonal lattice are blue and those of the other in orange). In particular, let Γ_1 and Γ_2 be two copies of the hexagonal lattice with fundamental domains $\{u_1, v_1\}$ and $\{u_2, v_2\}$, respectively. If we take the union of Γ_1 and Γ_2 and identify the vertices $v_1 + a$ and $u_2 + a$ for all $a \in \mathbb{Z}^2$, then we obtain the dice lattice. With respect to this construction, the dice lattice has a fundamental domain given by $\{u_1, v_1, v_2\}$. We consider the d-dimensional analog, that is, the $\mathbb{Z}^d$-periodic graph obtained by identifying vertices of two separate d-dimensional members of the hexagonal-diamond family introduced in Example 5.3.4.

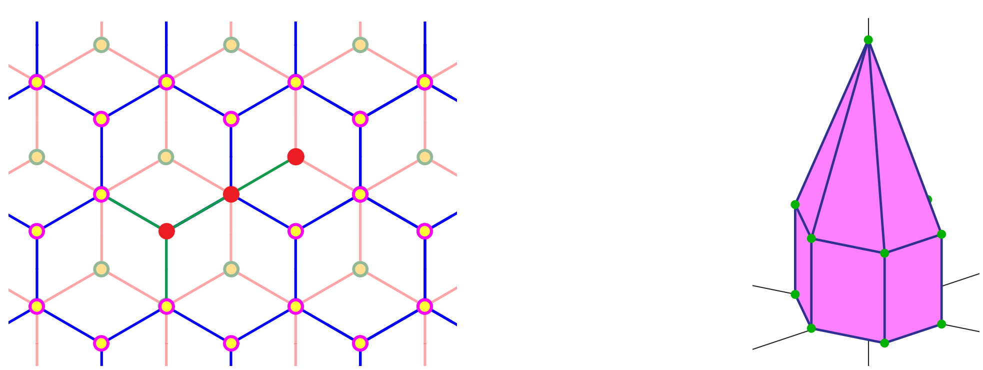


Figure 5.5: The $\mathbb{Z}^2$-periodic dice lattice, and its corresponding Newton polytope.

Let Γ be a *$\mathbb{Z}^d$-periodic dice lattice*, where $d > 1$ and let $Q \in \mathbb{N}^d$. The discrete periodic operator

L has the Floquet matrix $L(z, \lambda)$ given by

$$\begin{pmatrix} \sum_{i=0}^{d} \gamma_i + V(u_1) - \lambda & -\gamma_0 - \gamma_1 z_1^{-1} - \cdots - \gamma_d z_d^{-1} & 0 \\ -\gamma_0 - \gamma_1 z_1 - \cdots - \gamma_d z_d & \sum_{i=0}^{d} \gamma_i + \sum_{i=0}^{d} \beta_i + V(u_2) - \lambda & -\beta_0 - \beta_1 z_1^{-1} - \cdots - \beta_d z_d^{-1} \\ 0 & -\beta_0 - \beta_1 z_1 - \cdots - \beta_d z_d & \sum_{i=0}^{d} \beta_i + V(u_3) - \lambda \end{pmatrix}.$$

The edge labels (γ, β) are assumed generic, in particular $\beta_i \beta_j \neq -\gamma_i \gamma_j$ for any $i \neq j$, so that all vertices of the apical facets of $\mathcal{N}(D)$ appear (that is, $\mathcal{N}(D)$ is the Newton polytope shown in Figure 5.5). Let $Q \in \mathbb{N}^d$. For a $Q\mathbb{Z}$-periodic potential V, we claim the zero set of D_Q in $(\mathbb{C}^\times)^d \times \mathbb{C}$ is the union of an irreducible hypersurface and up to $|Q|$ flat bands. To prove this, we will first show that $\mathcal{N}(D_Q)$ is contained in the contracted Q-dilation of $\mathcal{N}(D)$, and that the apical facial polynomials of D_Q are potential-independent.

Recall that $\hat{L}_Q(z, \lambda)$ is given by

$$\begin{pmatrix} \multicolumn{3}{c}{L(\mu_1 z, \lambda)} & \begin{matrix} \hat{V}_{12,1} & 0 & 0 \\ 0 & \hat{V}_{12,2} & 0 \\ 0 & 0 & \hat{V}_{12,3} \end{matrix} & \cdots & \begin{matrix} \hat{V}_{1|Q|,1} & 0 & 0 \\ 0 & \hat{V}_{1|Q|,2} & 0 \\ 0 & 0 & \hat{V}_{1|Q|,3} \end{matrix} \\ \begin{matrix} \hat{V}_{21,1} & 0 & 0 \\ 0 & \hat{V}_{21,2} & 0 \\ 0 & 0 & \hat{V}_{21,3} \end{matrix} & L(\mu_2 z, \lambda) & \cdots & \vdots \\ \vdots & & \ddots & \\ \begin{matrix} \hat{V}_{|Q|1,1} & 0 & 0 \\ 0 & \hat{V}_{|Q|1,2} & 0 \\ 0 & 0 & \hat{V}_{|Q|1,3} \end{matrix} & \cdots & & L(\mu_{|Q|} z, \lambda) \end{pmatrix}.$$

By the Floquet matrix $L(z, \lambda)$, it is easy to see that $D(\mu_i z, \lambda)$ $(= \det(L(\mu_i z, \lambda)))$ has extreme monomials $z_i^{\pm}$, $z_i^{\pm} z_j^{\mp}$, $\lambda z_i^{\pm}$, $\lambda z_i^{\pm} z_j^{\mp}$, and λ^3. Therefore, any permutation τ that does not leave the diagonal of the block matrix has at most such a monomial contribution from each diagonal matrix entry $L(\mu_i z, \lambda)$.

If $\mathcal{N}(D_Q)$ is not contained in the contracted Q-dilation of $\mathcal{N}(D)$, then there exists a permutation through off-diagonal matrix entries of $\hat{L}_Q(z, \lambda)$ that produces a monomial with support outside of $|Q|\mathcal{N}(D)$. This follows immediately as permutations that only go through the block-diagonal of $\hat{L}_Q(z, \lambda)$ have support contained in $\mathcal{N}(\prod_{\mu \in \mathcal{U}_Q} D(\mu z, \lambda)) = |Q|\mathcal{N}(D)$.

Similarly, if an apical facial polynomial $D_Q|_F$ is not potential-independent, then there is a contributing permutation of $\hat{D}_Q|_F$ that involves the off block-diagonal matrix entries. This is because the apical facial polynomials of D are potential-independent. Thus, if no contributing permutation leaves the block-diagonal, potential-independence is inherited.

To show that the apical facial polynomials are potential-independent and that $\mathcal{N}(D_Q)$ is not contained in the contracted Q-dilation of $\mathcal{N}(D)$, we will rely on some graph theoretic arguments by viewing nonzero permutations as vertex disjoint cycle covers on a digraph.

The *digraph* of an $n \times n$ matrix M is a weighted directed graph with n vertices labeled by $[n]$, and with edges (i, j) of weight $M_{i,j}$ for each $i, j \in [n] \times [n]$ such that $M_{i,j} \neq 0$. Consider the digraph C of $\hat{L}_Q(z, \lambda)$ [26, 50]. To describe the graph with ease, we construct C as follows: let $C_1, \dots, C_{|Q|}$ be vertex disjoint subgraphs of C, where each C_s is a 3-vertex subgraph of C with vertices $v_{1,s}, v_{2,s}, v_{3,s}$. Each C_s has edges $(v_{1,s}, v_{2,s}), (v_{2,s}, v_{1,s}), (v_{2,s}, v_{3,s}), (v_{3,s}, v_{2,s})$, and loops $(v_{l,s}, v_{l,s})$ for $l = 1, 2, 3$. In this way, C_s is the digraph of $L(\mu_s z, \lambda)$. The union of these C_i along with the edges $(v_{l,s}, v_{l,r})$ and $(v_{l,r}, v_{l,s})$ for each $l = 1, 2, 3$ which connect C_r and C_s for all $r \neq s$, yield the graph C.

Suppose that τ is a nonzero permutation that leaves the block-diagonal of the matrix. Then a cycle of τ corresponds to a directed cycle of C in which every vertex appears at most once (and τ itself corresponds to a vertex disjoint cycle cover of C, see [13, 27] for more on this construction and correspondence). As τ leaves the block-diagonal, there exists a cycle of τ corresponding to a directed cycle η containing an edge $(v_{l,r}, v_{l,s})$ for some l, r, and s. Starting at the edge $(v_{l,r}, v_{l,s})$, the only way η can return to the subgraph C_r is if η has an edge $(v_{l',s'}, v_{l',r})$ for some s' and l'. Consider a coarsening of C, denoted C', where C' has one vertex for each subgraph C_r and these vertices inherit the edges of C that connect distinct subgraphs C_r and C_s. That is, the C'_r and C'_s are vertices of C' connected via the directed edges $(C'_r, C'_s)_l$ and $(C'_s, C'_r)_l$ for each $l = 1, 2, 3$, see Figure 5.6. Clearly the directed cycle η of C induces a directed cycle η' of C' by identifying $(v_{l,s}, v_{l,r}) \in \eta$ and $(C'_s, C'_r)_l \in \eta'$.

If τ produces a polynomial with support not contained in $|Q|\mathcal{N}(D)$, then, for some i, $L(\mu_i z, \lambda)$ contributes terms with exponent vectors lying outside of $\mathcal{N}(D)$ to τ, the only possibilities being $z_j z_k$ or $z_j^{-1} z_k^{-1}$. Without loss of generality, assume $L(\mu_1 z, \lambda)$ contributes z_1^2 to τ.

Let η' be a directed cycle of C', corresponding to a cycle of τ, with at least two vertices, one of which is C'_1. If C'_s in η' has edges $(C'_p, C'_s)_i$ and $(C'_s, C'_r)_j$, then the remaining accessible entries of $L(\mu_s z, \lambda)$ that can contribute to τ are at most the entries of the matrix obtained after deleting the ith column and the jth row of $L(\mu_s z, \lambda)$. To find a permutation that contributes a monomial whose exponent of z_1 is maximal, we only need to consider the maximal potential contributions of $L(\mu_s z, \lambda)$ with respect to z_1 if $C'_s \in \eta'$ (we also include the monomials minimized by the vector $w = (-2, -2, \dots, -2, -1)$ to be used later). We write each case $i \to j$ to mean η' has edges $(C'_p, C'_s)_i$ and $(C'_s, C'_r)_j$. In the table below we abbreviate maximal monomial contributions by MMC.

$i \to j$ for C'_s	MMC of of $L(\mu_s z, \lambda)$ to τ
$1 \to 1$	z_1 or λ^2
$1 \to 2$	1 or λ
$1 \to 3$	1
$2 \to 1$	z_1 or $z_1\lambda$
$2 \to 2$	1 or λ^2
$2 \to 3$	1 or λ
$3 \to 1$	z_1^2
$3 \to 2$	z_1 or $z_1\lambda$
$3 \to 3$	z_1 or λ^2

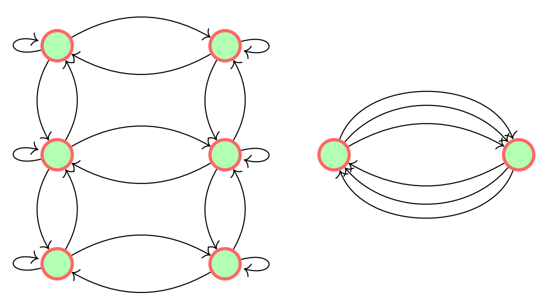


Figure 5.6: C and C' of the $(2, 1)\mathbb{Z}$-periodic dice-lattice.

As $L(\mu_1 z, \lambda)$ contributes z_1^2, we must have $3 \to 1$ for C'_1. After this, we must eventually have a vertex in η with a $1 \to 3$ or two vertices with a $1 \to 2$ and $2 \to 3$, respectively. Thus, either one block-diagonal matrix contributes a constant or two diagonal matrices contribute a constant or a λ. Thus, if $C'_{s_1}, \dots, C'_{s_l}$ are the vertices of η', then each $L(\mu_{s_i} z, \lambda)$ contributes at most z_1 on average. We conclude that $\mathcal{N}(D_Q(z, \lambda))$ is indeed contained in the contracted Q-dilation of $\mathcal{N}(D)$.

Similarly, the apical facets of $\mathcal{N}(D_Q)$ are potential-independent. Without loss of generality, consider a permutation with a cycle that leaves the block-diagonal, but still contributes a term of weight less than or equal to $-3|Q|$ with respect to the inner normal $w = (-2, -2, \ldots, -2, -1)$. Such a cycle corresponds to a cycle η' of C' such that either we have a $3 \to 1$ in η' contributing z_1^2, or $3 \to 2$ or $2 \to 1$ contributing $z_1\lambda$. In either case, on average the matrices corresponding to vertices in this cycle will have to contribute terms of greater weight than -3.

As each apical facial polynomial $D_Q|_F$ is potential-independent and each $\mathcal{N}(D_F)$ is a d-dimensional pyramid, it follows that each $D_Q|_F$ is only homothetically reducible by Corollary 5.2.12. Furthermore, there is an apical facet F' such that $D_{F'}$ has the monomials λ^3 and $z_i\lambda$ for each i as terms, and so $D_Q|_{F'}$ is irreducible for all $Q \in \mathbb{N}^d$ by Corollary 5.2.11. Taking a strong chain of all the apical facets of $\mathcal{N}(D_Q)$ and using the arguments of Theorem 2.1.7 and Corollary 2.1.8; we see that if $D_Q = gh$, then for each apical facet F, $D_Q|_F = g|_F h|_F$, where $h|_F$ is a monomial, which we may assume to be λ^l, for some $l \geq 0$. Note that h cannot have any monomial term $z^a\lambda^b$ for $a \neq 0$. Otherwise, if some $a_i \neq 0$, then one of $z^a z_i^{\pm|Q|}\lambda^{|Q|-l+b}$ is a term of $g(z^Q, \lambda)h(z^Q, \lambda)$ with a monomial exponent lying is outside of $|Q|\mathcal{N}(D)$; which is impossible. Therefore, it must be the case that h is a degree-l polynomial in λ. This implies that $\mathcal{N}(g) + \mathcal{N}(\lambda^l + 1) = \mathcal{N}(D_Q)$, where $0 \leq l \leq |Q|$. We conclude that D_Q is the product of at most $|Q|$ linear terms of λ and one irreducible polynomial for any $Q\mathbb{Z}$-periodic potential. ⋄

6. Summary

Using tools from algebraic geometry, we are able to effectively answer questions that arise from the spectral theory of discrete periodic operators. In particular, we demonstrated the efficacy of these tools for answering questions regarding the spectral edges of the dispersion relation by studying the critical points of the Bloch variety. We also utilize methods of algebra and discrete geometry to study when the Bloch and Fermi varieties are irreducible as algebraic varieties.

Through an extended version of Kuchnirenko's theorem, we provide an effective upper bound on the number of isolated critical points of the Bloch variety. We then used this criterion to identify a family of discrete periodic operators for which this bound is exact. We concluded by demonstrating that one can use this bound to determine when the spectral edges nondegeneracy conjecture holds, in some cases, through a single computation.

By methods from algebra and discrete geometry we introduced several methods for checking when irreducibility of the dispersion polynomial is preserved upon changing the period lattice. We concluded by proving irreducibility of the Bloch and Fermi varieties for various families of discrete periodic operators.

REFERENCES

[1] D. Bättig, A toroidal compactification of the Fermi surface for the discrete Schrödinger operator, Comment. Math. Helv. **67** (1992), no. 1, 1–16. MR 1144610

[2] David N. Bernstein, The number of roots of a system of equations, Functional Anal. Appl. **9** (1975), no. 3, 183–185.

[3] Daniel Bättig, A toroidal compactification of the two dimensional bloch-manifold, Doctoral thesis, ETH Zurich, Zürich, 1988, Diss. Math. ETH Zürich, Nr. 8656, 1988. Ref.: E. Trubowitz ; Korref.: H. Knörrer.

[4] Yves Colin de Verdière, Sur les singularités de van Hove génériques, Analyse globale et physique mathématique (Lyon, 1989), no. 46, Soc. Math. France, 1991, pp. 99–110.

[5] D. Cox, J. Little, and D. O'Shea, Ideals, varieties, and algorithms, third ed., Undergraduate Texts in Mathematics, Springer, New York, 2007.

[6] David Cox, What is a toric variety?, Topics in algebraic geometry and geometric modeling, Contemp. Math., vol. 334, Amer. Math. Soc., Providence, RI, 2003, pp. 203–223. MR 2039974

[7] David A. Cox, John B. Little, and Henry K. Schenck, Toric varieties, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322

[8] David Damanik and Jake Fillman, One-dimensional ergodic Schrödinger operators—I. General theory, Graduate Studies in Mathematics, vol. 221, American Mathematical Society, Providence, RI, [2022] ©2022. MR 4567742

[9] W. Decker, G-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR 4-3-0 — A computer algebra system for polynomial computations, `http://www.singular.uni-kl.de`, 2022.

[10] Ngoc Do, Peter Kuchment, and Frank Sottile, Generic properties of dispersion relations for discrete periodic operators, J. Math. Phys. **61** (2020), no. 10, 103502, 19.

[11] Günter Ewald, Combinatorial convexity and algebraic geometry, Graduate Texts in Mathematics, vol. 168, Springer-Verlag, New York, 1996. MR 1418400

[12] Matthew Faust and Jordy Lopez Garcia, Irreducibility of the dispersion relation for periodic graphs, 2023.

[13] Matthew Faust, Wencai Liu, Rodrigo Matos, Jenna Plute, Jonah Robinson, Yichen Tao, Ethan Tran, and Cindy Zhuang, Floquet isospectrality of the zero potential for discrete periodic schrödinger operators, 2024.

[14] Matthew Faust and Frank Sottile, Critical points of discrete periodic operators, 2022.

[15] Jake Fillman, Wencai Liu, and Rodrigo Matos, Irreducibility of the Bloch variety for finite-range Schrödinger operators, J. Funct. Anal. **283** (2022), no. 10, Paper No. 109670, 22. MR 4474842

[16] Jake Fillman, Wencai Liu, and Rodrigo Matos, Algebraic properties of the fermi variety for periodic graph operators, 2023.

[17] Nikolay Filonov and Ilya Kachkovskiy, On the structure of band edges of 2-dimensional periodic elliptic operators, Acta Math. **221** (2018), no. 1, 59–80. MR 3877018

[18] Lee Fisher, Wei Li, and Stephen P. Shipman, Reducible Fermi surface for multi-layer quantum graphs including stacked graphene, Comm. Math. Phys. **385** (2021), no. 3, 1499–1534. MR 4283995

[19] William Fulton, Introduction to toric varieties, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037

[20] ———, Intersection theory, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 2, Springer-Verlag, Berlin, 1998.

[21] Shuhong Gao, Absolute irreducibility of polynomials via Newton polytopes, Journal of Algebra **237** (2001), no. 2, 501–520.

[22] I. M. Gel′ fand, M. M. Kapranov, and A. V. Zelevinsky, Discriminants, resultants, and multidimensional determinants, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1994.

[23] D. Gieseker, H. Knörrer, and E. Trubowitz, The geometry of algebraic Fermi curves, Perspectives in Mathematics, vol. 14, Academic Press, Inc., Boston, MA, 1993. MR 1184395

[24] D.R. Grayson and M.E. Stillman, Macaulay2, a software system for research in algebraic geometry, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[25] Branko Grünbaum, Convex polytopes, second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003, Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler. MR 1976856

[26] Thomas Kappeler, On isospectral periodic potentials on a discrete lattice. I, Duke Math. J. **57** (1988), no. 1, 135–150. MR 952228

[27] ———, Isospectral potentials on a discrete lattice. III, Trans. Amer. Math. Soc. **314** (1989), no. 2, 815–824. MR 961624

[28] Frédéric Klopp and James Ralston, Endpoints of the spectrum of periodic operators are generically simple, vol. 7, 2000, Cathleen Morawetz: a great mathematician, pp. 459–463. MR 1869296

[29] P. A. Kuchment, On the Floquet theory of periodic difference equations, Geometrical and algebraical aspects in several complex variables (Cetraro, 1989), Sem. Conf., vol. 8, EditEl, Rende, 1991, pp. 201–209. MR 1222215

[30] Peter Kuchment, Floquet theory for partial differential equations, Operator Theory: Advances and Applications, vol. 60, Birkhäuser Verlag, Basel, 1993.

[31] ———, An overview of periodic elliptic operators, Bull. Amer. Math. Soc. (N.S.) **53** (2016), no. 3, 343–414. MR 3501794

[32] ———, Analytic and algebraic properties of dispersion relations (Bloch varieties) and Fermi surfaces. What is known and unknown, J. Math. Phys. **64** (2023), no. 11, Paper No. 113504, 11. MR 4667406

[33] Peter Kuchment and Boris Vainberg, On embedded eigenvalues of perturbed periodic Schrödinger operators, pp. 67–75, Springer US, Boston, MA, 1998.

[34] ———, On absence of embedded eigenvalues for Schrödinger operators with perturbed periodic potentials, Communications in Partial Differential Equations **25** (2000), no. 9-10, 1809–1826.

[35] ———, On the structure of eigenfunctions corresponding to embedded eigenvalues of locally perturbed periodic graph operators, Communications in Mathematical Physics **268** (2006), no. 3, 673–686.

[36] Wei Li and Stephen P. Shipman, Irreducibility of the Fermi surface for planar periodic graph operators, Lett. Math. Phys. **110** (2020), no. 9, 2543–2572. MR 4139163

[37] Wencai Liu, Criteria for embedded eigenvalues for discrete Schrödinger operators, Int. Math. Res. Not. IMRN (2021), no. 20, 15803–15832. MR 4329883

[38] ———, Bloch varieties and quantum ergodicity for periodic graph operators, (2022).

[39] ———, Irreducibility of the Fermi variety for discrete periodic Schrödinger operators and embedded eigenvalues, Geom. Funct. Anal. **32** (2022), no. 1, 1–30. MR 4388759

[40] ———, Topics on Fermi varieties of discrete periodic Schrödinger operators, J. Math. Phys. **63** (2022), no. 2, Paper No. 023503, 13. MR 4376993

[41] Peter McMullen, Indecomposable convex polytopes, Israel J. Math. **58** (1987), no. 3, 321–323. MR 917362

[42] S. P. Novikov, Bloch functions in the magnetic field and vector bundles. Typical dispersion relations and their quantum numbers, Dokl. Akad. Nauk SSSR **257** (1981), no. 3, 538–543.

[43] ———, Two-dimensional Schrödinger operators in periodic fields, Current problems in mathematics, Vol. 23, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983, pp. 3–32.

[44] Mostafa Sabri and Pierre Youssef, Flat bands of periodic graphs, J. Math. Phys. **64** (2023), no. 9, Paper No. 092101, 23. MR 4635298

[45] Igor R. Shafarevich, Basic algebraic geometry. 1, translated from the 2007 third russian edition ed., Springer, Heidelberg, 2013, Varieties in projective space.

[46] G. C. Shephard, Decomposable convex polyhedra, Mathematika **10** (1963), 89–95. MR 172176

[47] Stephen P. Shipman, Eigenfunctions of unbounded support for embedded eigenvalues of locally perturbed periodic graph operators, Comm. Math. Phys. **332** (2014), no. 2, 605–626. MR 3257657

[48] ———, Reducible Fermi surfaces for non-symmetric bilayer quantum-graph operators, J. Spectr. Theory **10** (2020), no. 1, 33–72. MR 4071332

[49] Frank Sottile, Real solutions to equations from geometry, University Lecture Series, vol. 57, American Mathematical Society, Providence, RI, 2011.

[50] Richard P. Stanley, Enumerative combinatorics. Volume 1, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR 2868112

[51] Toshikazu Sunada, Topological crystallography, Surveys and Tutorials in the Applied Mathematical Sciences, vol. 6, Springer, Tokyo, 2013, With a view towards discrete geometric analysis. MR 3014418

[52] Gerald Teschl, Mathematical methods in quantum mechanics, Graduate Studies in Mathematics, vol. 99, American Mathematical Society, Providence, RI, 2009, With applications to Schrödinger operators. MR 2499016

[53] Günter M. Ziegler, Lectures on polytopes, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR 1311028