February 26, 2008 Moscow, Russia

[In response to your letter dated January 30, 2008: A question about a conjecture]

Dear Professor Sottile,

I am sorry for delaying my answer.

The preffered transcription of my names is "Anatoli Kushnirenko" (unfortunally, my Social Security Number and some other U.S. legal stuff still be linked to the old french style transcription "Kouchnirenko" in my passport, issued by USSR authority in 1989).

Now about your questions.

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> In the interests of good scholarship,
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I have spent some time to prepare a comprehensive report.

1.

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> a conjecture which many people have
> attributed to you concerning the maximal number of real
> solutions to a system of polynomial equations.
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They did it right, with the only exception that my Conjecture nas been broader than that you cited in your letter. See Sections 3 and 6 below.

2. Thanks again for your letter. It has been a pleasure for me to remember the period of my life when the Fewnomials story starts and I wish to share with You some detail.

I remember the moment, (the last vacation week before the school year of 1977-78, August of 1977) when the notion of "fewnomial" has been invited.

That day I started to write a paper for the "Kvant journal" (popular math and physics journal for high school students of USSR - http://www.nsta.org/quantum/backgrnd.asp) .

I planed to title my paper "How to find the number of roots of a polynomial" [1]

The very first paragraph of the paper was as follows

"A **poly**nomial is the sum of a lot of terms called monomials, like

$1 + 2x + 3x^2 + 4x^3 + 5x^4 + ... + 101x^{100}$ "

In Russian

polynomial == *mnogo_chlen* (*mnogo* means *many*, *chlen* means *term*)

monomial == *odno_chlen* (*odno* means *single*)

several, or *a few* == *malo*

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[1] This paper never has been finished. Instead it has been gradually transformed to the paper of A. Kushnirenko and E. Korkina "One another proof of the Tarski -Seidenberg theorem".

In Russian *mnogo-malo* is a standard pair of antonyms, like *hot-cold,* or *up-down* in English, so

when rereading aloud the first lines of my future paper I has been striked by pure linguistic feeling that there is a room in the Russian language to add a new world for the antonym of "mnogo-chlen", namely "malo-chlen" (eng. fewnomial). I am using here "THE antonym" not "An antonym" because for any Russian speaking person there is the only solution - "*malo*"- to the puzzle

Replace dots by a missed word in *"mnogo-chlen"-"...-chlen"*

Without any problems I successfully translated the new word to English: "poly-nomilas" - "few-nomials" (and I was happy with the results because "few-nomials" sounds in English as good as "malo-chlen" in Russian). After all these linguistic exercises I switched my mind from linguistics to math.

My fuzzy mathematical idea was as follows:

"We know that polynomials (**mnogo**-chleny) have a lot of (i.e. **mnogo**) roots, so probably few-nomials (**malo**-chleny) have only a few (i.e. **malo**) roots".

But what is the number of roots of square system - nothing but a characteristic of topological complexity of the set of real solutions of the system whereas the total number of non-zero terms is a characteristic of algebraic complexity of symbolic description of the same system"

That is how I came up with a very broad idea

3. Kushnirenko Hypothesis:

I. "Topological complexity of an objects, defined by real-valuated polynomials, can be controlled by the complexity of definition of these polynomials (say by number of non-zero terms, or by lenght of straight line program etc) rather than by degrees or by some characteristics of Newton polyhedra of equations"

The successfull experience with Newton polyhedra taught me that the topological complexity of a object defined by system of equation usually can be bounded by the number of root of an appropriate square system, so to be more specific I specialized the broad Hypothesis I to a more concrete

II. "The number of real root of an GENERAL system of n polynomial equations with n unknowns can be bounded from above by the total number of non-zero terms of these equations"

Being a former student of Vladimir Arnold and active participant of his seminar I remember well the question "What is the right multivariate generalisation of Descartes' rule of signs" Arnold repeated each semester and I didn't hesitated to refine II and state a combination of Descartes' rule of signs and Bezout's theorems, exactly in the form you stated in your letter:

III. Given polynomials with real coefficients f_1,...,f_n in
n variables, where polynomial f_i has 1 + m_i monomials.
Then the system

f_1(x_1,...,x_n) = ... = f_n(x_1,...,x_n) = 0

has at most the product $m_1 m_2 \dots m_n$ number of
non degenerate positive real solutions.

The next day I discussed all three parts of my Hypothesis with Vladimir Arnold (that summer we rented two dacha with a common backyard and contacted very frequently) . Arnold's reaction on all three parts of Hypotesis was rather optimistic but he said me "Right now You have no evidences to support your promissing conjecture so try to state and prove some simple partial case". And I followed this peace of advice, with the help of K. Sevostyanov, 21, that time a junior of The Department of Mathematics of The Moscow State University. Formally, Sevosyanov wasn't my student. As far as I remember his official adviser was Professor Arnold. But it happens, that during the school year of 1976-77 I regularly discussed with Kostya Sevostyanov some problems related to my work on Newton polyhedra theory. For example Kostya provided me with the compact and elegant formal proof of integral geometry formulae, used in my paper [1].

Kostya was a brillaint person sometimes proposing non-trivial and not-expected ideas. Once in his life he was a Number 1 Winner of a super-prestigious All Union (USSR) Mathematics Olympiad for High School Students.

So a few days after begining of the Fall semester of 1977 I discussed with Kostya

Hypothesis IV - A Polynomials-Fewnomial Conjecture:

Given two polynomials with real coefficients $f_1,\dots,f_2$ in
2 variables, where polynomial f_1 has $1 + m_1$ monomials and

f_2 has degree d_2
Then the system

$$f_1(x_1,x_2) = f_2(x_1,x_2) = 0$$

has at most $N(m_1, d_2)$ number of
non degenerate positive real solutions, where N - is some universal functions.

And Kostya proved that Conjecture a half a year later. (I don't remember in detail when the proof appears , probably in Winter of 1978). About at that time Kostya also constructed a counter-example you asked about (see below). While the Sevostyanov's proof of Polynomials-Fewnomial Conjecture happens to be 1-dimentional and not expandable on multi-dimensional situation, nevertheless, in my opinion, when Kostya proved the Polynomials-Fewnomials Conjecture he did made the crucial first step in fewnomials theory. (I have heard that about a years ago Vladimir Arnold in a private conversation with his former students expressed a similar opinion during his visit to California.)

It would be probably appropriate to dissemenate in real algebraic geometry community (for example during the future Lausanne Workshop) the idea to name this result Sevostyanov's Theorem and ask Askold Hovansky to write down a short paper containing the proof of the theorem and some comments.

BTW, may be it is more productive to consider some recent results, say Martin Avendano's upper bound, as a refinement of a (simple) Sevostyanov's Theorem not a Hovansky' Theorem. Chanses are that the study of Polynomials-Fewnomials style examples give us a hint for the good form of Fewnomials bound. Avendano's Lemma 2.2 actually is a problem number 32 in "Problems and Theorems in Analysis 2" by **George Polya and Gabor Szego** [2]. I am sure that for different Polynomials-Fewnomials examples sharp bounds (at least reasonably sharp) can be found.

4. Now let me parse your first question in three parts and answer them separately

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First, did you make this conjecture?      (a)

If so, about when and (how - AGK) did it appear

in print,                     (b)

or in a lecture,              (c)

or just in conversaions?(d)
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a) **Yes!**

d) Well, legally speaking, I'm ready to swear that **in August 1977** I coined the term "malo-chlen" and his english version "fewnomial" and stated the sections I, II, and III of Kushnirenko Hypothesis in a private conversation with Professor Vladimir Arnold **in an private informal conversations** (entre quatre yeux) .

c) **In a lecture**, this Hypothesis first appears in Arnold's talk on the introductory meeting of Arnold's seminar **in September of 1977**. Traditionally, during such a meeting Arnold discussed the list of problems (worth to be worked out by participant of the seminar during the new school year) . So, the Kushnirenko Hypothesis was first stated in public by V. Arnold on his own seminar in September of 1977 (just for the record: Seminar of The Department of Mathematics, Moscow State University, Moscow, USSR :-)

To be more precise with dates and events I have found an old box under by bed, a folder inside the box and a stack of personal notes related to my talks on different seminars and meetings during the period of 1974-1979. In accordance with my notes [3], I have mentioned "malo-chleny" in the final section ("Other properties of sums of exponent") of my talk on the meeting of The Moscow Mathematical Society on December 6 1977 . The abstract of my talk (titled "Zeros of exponential sums and Newton polyledra") has been published in "Uspehi Matemeticheskih Nauk" [4], but the word "fewnomials" appeared in my talk but not in these short abstract. (This is very understandable because at that moment the whole "fewnomial" Hypothesis was a pure speculation – The Sevostyanov Theorem wasn't proved yet).

b) **I never published a word** concerning this Kushnirenko Hypothesis. The first printed reference to the Part III of my Hypothesis I can refer to is an abstract of Hovansky talk on the meeting of The Moscow Mathematical Society on October 9, 1979 [5, look at my translation to english of a sentence, extracted from the abstract: "This upper bound has been proved in an attempt to prove a conjecture by A. Kushnirenko who proposed multivariate generalization of Descartes upper bound of number of positive roots of a polynomial".

5. Because of clearness and attractiveness of the part III of Kushnirenko Hypothesis (as well as by some other reasons) only part III usually mentioned in papers and attributed to me. But this part III happens to be wrong if interpreted literally, whereas Part II happens to be 100% right and has been proved by Hovansky, as a part of bigger pfaffian theory. That is why I think that an appropriate source of information on my role in Fewnomial Story is a book of Hovansky, namely the russian version of "Fewnomials" ("Malochleny"), Moscow, Phasis, 1997. ISBN 5-7036-0023-5, pp 175-176. [6]

6. In Chapter 9, Section 1 of this book Hovansky gives a short description of the Fewnomials Theory History. In a foreword to "Malochleny" Hovansky explain that The Chapter 9 has been revised during preparation of Russian translation of earlier English version of "Fewnomials". So when preparing this letter I don't know whether or not the material of section 9.1 has been published in English version of Hovansky book (or published in English elsewere). That's why I am incliding here my non-professional translation to English of Section 9.1 of the above Hovansky book ("Malochleny"), [7].

"9.1. Systems of fewnomial equations

Let us start with history of our subject. As a phenomen, fewnomials effect first appears in Decartes estimate. Problem to bound from above the number of isolated roots of polynomial system of the form P_1=...=P_k=0, lying in positive ortant of R^k via number of monoms of the system actively advertized by A.G. Kushnirenko at the end of '70s The term "fewnomial" also was coined by A.G.Kushnirenko. The first result in this theory was proved by young moscow mathematician K.A. Sevostyanov, who later tragically died in an accident. Sevastyanov proved the upper bound of number of roots of polynomial system where one equation contains a few (no more than a given number) monomials, and othe equations have a bounded degrees. His proof based on a version of Descartes reasoning applyed to 1-dimentional algebraic curve and cannot be generalized to multidimentional settings. Multidimentional bound has been proved by author in 1979. Other results of this book has been proved between 1979 and 1987.

Let me point out that in general settings problem posed by Kushnirenko still unsolved: the known bound 2^{q/(q-1)/2}(k+1)^q (see 6.3 Corolary 2) seems to be overestimatated, the exact bound still be unknown. Moreever Descartes Theorem estimates maximal number of real root of a polynomial by number of signs changes in the sequence of coefficients not by the number of not-zero coefficients. It is also known a generalization of Descartes theorem on polynomials in one complex variable.

Problem 1. Find a multivariate generalization of Descartes Rule of Signs.

Problem 2. Improve the upper bound of number of roots of system of fewnomial equations.

Problem 3. Find examples of systems of fewnomial equations with possibly many roots."

---- The End of Section 9.1 ----

7. A Remarks on the exact meaning of the word polynomial in Kushnirenko Hypothesis.

In I, II and III by polynomial I actually mean not only classical polynomials but also polynomilas with real exponents having positive ortant as the domain. Let me explain why that time I prefered to work in this settings..

First, in complex theory is technically more convinient to work with Newton polyhedra of Loran's polynomials even if you interested in classical polynomials only.

Second, The Lagger's generalization of Descartes rule of signs on exponential sums (Problem 77) and even on integral (continuous spectrum in Problem 80) seems to be promissing. [2]

Third. At the end of 1975 I have found an analytic proof of my theorem on the number of complex solution of unmixed system of polynomial (Loran) equations. In this proof I used the integral geometry to calculate the mean number of complex solutions of a polynomil system as the volume of some complex Kahler manifold. I prepared paper in English [1] but wasn't allowed by USSR authority to publish this paper abroad (In Inventiones of Mathematics) [2]. The proof in the paper works well for the exponential sums with real exponents (Probably Boris Kazarnovsky was the first who actively advertized such a generalization.) So by that time I used to works with polynomials with real exponent and I hoped to use deformation of both coefficients and exponents of equations to do different things.

8. `Whether you did or not make the conjecture, was a counter example known previously? (Did you know of a counter example?)`

Yes, a counter example was known no later than Winter of 1978, but in 90's I failed to restore it.

I remember well the moment when Kostya meet me somewhere on the level "-1" of enourmus building of Moscow State University. He informed me that my upper bound in the simplest case of 2 trinomials in 2 variables. As far as I remember staying near a wall Kostya draw on a free half-page on his appointment book a one-parameter family of system with real (not integer) exponents. He demonstrated, that when parameter (actually the exponent of one term of equation) tends to zero a fifth root appears. Concluding remarks was "Now let us choose a rational value of a parameter and we are done". The whole conversation took no more than 10 minutes. It seems to me now, that the proof of Sevostyanov theorem and the counter example appeared practically in parallel, as two parts of one package. That is why nobody has considered at that time the counter example as a serious result.

About a decade ago Bernd Sturmfels contacted me semi-officially and asked whether or not I can write down a counter example presumably invited by Sevostyanov two decades ago. I have spent a weekend trying to restore a construction but finally gave up.

So it's fair to attribute this example to Haas, mentioning Sevostyanov too.

9. Kostya Sevast'jnov was born in January 1956. He died in result of car-pedestrian accident (unidentified driver leaved the scene of the accident) on December 9th, 1981 or 1982. It was a tragic accident not an act of suicide.

10. In conclusion, I would like to remark that reaching my today's age [8] [3] I realized well that not enought credits has been awarded to Vladimir Arnold by me personally for his explicit and

[2] I has been upset and never tryed to publish that paper again at home or abroad.
[3] "When I grow older, losing my hair, ..."

implicit support of my work on Newton polyhedra theory and fewnomials theory. In my talk on Moscow International Workshop in Algebra, Moscow State University, November 2005, I have explained my feeling in the following words [9 slide 5]:

"In Seventies, V. Arnold and his disciples in Moscow developed two methods to control complexity of algebraic objects defined by multivariate polynomials with real or complex coefficients in linear spaces over real or complex numbers.

Fewnomials theory deals with objects in real spaces and complexity of an object bounded by some function depending only on number of non-zero terms of equations.

Newton Polyhedra theory deals with object in complex spaces and complexity of an object bounded by some metric and combinatorial characteristics of Newnon polyhedra of equations."

Best regards from Moscow,

Anatoli Kushnirenko,
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List of attached files

1. `How to guess.doc` – My unpublished paper "How to Guess The Number of Roots ..."

2. `Polya Szego.doc` – Frontpages of "Problems and Theorems in Analysis 2" by George Polya and Gabor Szego in English and in German, problem 32, 77, 80 in Russian.

3. `Notes MMS-6-12-1977.doc` – Notes prepared for my talk on The Moscow Mathematical Society Meeting dated December 12, 1977

4. `Abstract MMS-6-12-1977` **`Kushnirenko.jpg`** - Abstract of my talk on The Moscow Mathematical Society Meeting dated December 12, 1977

5. `Abstract MMS-9-10-1979 Hovansky.jpg` - Abstract of Hovansky's talk on The Moscow Mathematical Society Meeting dated October 9, 1979

6. `Malochleny 1997 Frontpage.jpg` – Frontpage of Hovansky's book in Russian

7. `Malochleny 1977 Section 9.1.jpg` – Section 9.1 of Hovansky's book in Russian

8. `Anatoli Kushnirenko during his MSU talk November 2005.jpg`

9. `Anatoli Kushnirenko MSU talk November 2005.ppt` – Presentation in Russian (formulae still be written in Greek :-)

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